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Solution of Time-Varying Index-2 Linear Differential Algebraic Control Systems Via A Variational Formulation Technique

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Abstract

This paper deals with finding an approximate solution to the index-2 time-varying linear differential algebraic control system based on the theory of variational formulation. The solution of index-2 time-varying differential algebraic equations (DAEs) is the critical point of the equivalent variational formulation. In addition, the variational problem is transformed from the indirect into direct method by using a generalized Ritz bases approach. The approximate solution is found by solving an explicit linear algebraic equation, which makes the proposed technique reliable and efficient for many physical problems. From the numerical results, it can be implied that very good efficiency, accuracy, and simplicity of the present approach are obtained.

Keywords: Control problems, Direct method of calculus of variation, Generalized Ritz method, Index-two time-varying linear differential algebraic equations, Variational formulation.

حل نظم السيطرة الخطية التفاضلية-الجبرية ذو الزمن المتغير والدليل التفاضلي الثنائي بأسلوب الصياغة التغايرية

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الخلاصة

هذا البحث يّعنى بإيجاد الحل التقريبي للأنظمة الخطية التفاضلية الجبرية ذات الزمن المتغير ومن الدليل التفاضلي الثنائي باستخدام اسلوب الصياغة التغايرية. حيث ان الحل لمسائل السيطرة التفاضلية الجبرية ذات الدليل التفاضلي الثنائي هو النقاط الحرجة للدالة التغايرية. بالإضافة الى ان المسائل التغايرية يتم تحويلها من الاسلوب الغير مباشر الى الاسلوب المباشر باستخدام قواعد ريتز المعممة. ان هذا الحل التقريبي الذي ينتج عن طريق حل المعادلات الجبرية الخطية والتي تقدم طريقة كفؤه للعديد من المسائل التطبيقية الفيزيائية. كذلك امثله توضيحيه وضعت لتوضيح الطريقة المعطاة ونتائج توضح الكفاءة والدقة العالية للطريقة.

1. Introduction

Many real life problems can be modelled as a differential algebraic (control) system. Finding a novel reliable and efficient technique for solving differential algebraic equations

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has become an interesting aim for mathematicians and engineers. Numerical methods that solve higher index differential algebraic equations can be found in literature [1-7]. Many of these methods were based on the index reduction technique to avoid the difficulties in the higher index differential algebraic equations. Time- varying linear differential algebraic equations is a subject of many real life problems and has been the subject of many researchers in recent years [8, 9, 10]. An efficient and easily implemented technique to solve some classes of DAEs (index-1), approximately using non-classical variational formulation approach, was developed [11, 12, 13]. The aim of this work is to extend and develop the results of the latter three studies to solve higher index time-varying linear differential algebraic control equations, especially for index-2 problems, without using the reducing technique which is not applicable for many real life problems. Since the proposed DAEs problem has the non-symmetrical time-derivative linear operator with respect to the classical bilinear form, a new bilinear form, based on the old one, is taken to ensure the necessary requirements for the existences of the variational problem corresponding to the given constrained problem.

2. Basic Concepts

Let X and Y be linear spaces and $: \mathbb{D}(L) \subset X \to \mathbb{R}(L)$ in Y, then L is called symmetric with respect to the bilinear form $\langle a, b\rangle$ if

$$\langle La, b\rangle = \langle Lb, a\rangle \text{ satisfied for } a, b \in \mathbb{D}(L).$$

Moreover, a bilinear $\langle a, b\rangle$ is called non-degenerate on X and Y if the following two conditions are satisfied:

Firstly, for every $a \in X, \langle a, \bar{b}\rangle = 0$, then $\bar{b} = 0$, and secondly, for every $b \in Y, \langle \bar{a}, b\rangle = 0$, then $\bar{a} = 0$[11, 14].

3. Problem Formulation

Consider the semi explicit linear descriptor system

$E\dot{x} = Ax + Bu + f(t)$,

where $E, A \in R^{m\times n}$, with $\text{rank}(E) = n_0 < n$, $x \in R^n$, $u \in R^r$, $B \in R^{m\times r}$, $f \in C(I; R^m)$.

Since the rank $(E) = n_0$, it follows from [9], [15], and [16] that there exists unitary matrices $U \in R^{m\times m}$ and $V \in R^{n\times n}$ such that

$E = U\begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}V^T, \Sigma = \text{diag}(\delta_1, \delta_2, .., \delta_{n_0})$and $\delta_1 \geq \delta_2 \geq .. \geq \delta_{n_0} > 0$,

With the setting $\mathbb{P} = V, \mathbb{Q} = U^{-1}$, where $\mathbb{Q}E\mathbb{P} = \begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}$, one gets

$\mathbb{P}^{-1}x = \begin{pmatrix}x_1\\ x_2\end{pmatrix}, x_1 \in R^{n_0}, x_2 \in R^{n-n_0}$, hence

$$\mathbb{Q}E\mathbb{P}\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \mathbb{Q}A\mathbb{P}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \mathbb{Q}Bu + \mathbb{Q}f(t)$$

Set $\mathbb{Q}A\mathbb{P} = \begin{pmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ \tilde{A}_{21} & \tilde{A}_{22}\end{pmatrix}, \mathbb{Q}B = \begin{pmatrix}\tilde{B}_1\\ \tilde{B}_2\end{pmatrix}, \mathbb{Q}f = \begin{pmatrix}\tilde{f}_1(t)\\ \tilde{f}_2(t)\end{pmatrix}$,

$$\begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ \tilde{A}_{21} & \tilde{A}_{22}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}\tilde{B}_1\\ \tilde{B}_2\end{pmatrix}u + \begin{pmatrix}\tilde{f}_1(t)\\ \tilde{f}_2(t)\end{pmatrix}$$

Hence,, the semi explicit system is transformed into differential algebraic systems:

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u + f_1 \qquad ...(1)$$

$$0 = A_{21}x_1 + A_{22}x_2 + B_2u + f_2 \qquad ...(2)$$

where $A_{11} = \Sigma^{-1}\tilde{A}_{11}, \quad A_{12} = \Sigma^{-1}\tilde{A}_{12}, \quad A_{21} = \tilde{A}_{21}, \quad A_{22} = \tilde{A}_{22},$

$$f_1 = \Sigma^{-1}\tilde{f}_1, \quad f_2 = \tilde{f}_2, \quad B_1 = \Sigma^{-1}\tilde{B}_1, \quad B_2 = \tilde{B}_2.$$

If there exists $A_{22} = 0$ is invertible matrix with $(A_{21}A_{12})$ or A_{22} is not invertible matrix with $(A_{21}A_{12})^{-1}$, then the system (1) and (2) are index two linear DAEs with control u.

From the Jacobean of the algebraic constraint with respect to x_2, one can use the implicit function theorem [5, 17, 18] to solve the following:

$$x_2 = \mathcal{L}(t, x_1, u, \dot{u}) \triangleq -(A_{21}A_{12})^{-1}\left[A_{21}A_{11}x_1 + A_{21}B_1u + A_{21}f_1 + B_2\dot{u} + \dot{f}_2\right] \qquad \ldots(3)$$

Thus, from (1), (2), and (3), one have

$$\dot{x}_1 = A_{11}x_1 + A_{12}\mathcal{L}(t, x, u, \dot{u}) + B_1u + f_1$$
$$0 = A_{21}x_1 + A_{22}x_2 + B_2u + f_2$$

where the class of consistent initial condition at $t = t_0$ is defined according to the given algebraic constraint (2), as follows:

$$\omega^0 = \{(x_1^0, x_2^0) \in R^{rank(E)} \times R^{n-rank(E)} | A_{21}x_1(t_0) + B_2u(t_0) + f_2(t_0) = 0,$$
$$x_2(t_0) = \mathcal{L}\big(t_0, x_1(t_0), u(t_0), \dot{u}(t_0)\big)\} \qquad \ldots(4)$$

Note that, if there is an interest in finding the explicit expression for $\dot{x}_2$ to obtain the state-space $(\dot{x}_1, \dot{x}_2)$, then one has to derive (3) with respect to t, as follows:

$$\dot{x}_2 = \frac{d}{dt}\mathcal{L}(t, x_1, u, \dot{u}) = \frac{\partial\mathcal{L}}{\partial t} + \frac{\partial\mathcal{L}}{\partial x_1}\dot{x}_1 + \frac{\partial\mathcal{L}}{\partial u}\dot{u} + \frac{\partial\mathcal{L}}{\partial \dot{u}}\ddot{u}$$

For $u \in C^2[I, R^r]$, $t \in I = [t_0, t_1]$, $f \in [I, R^{n-n_0}]$

$$\dot{x}_2 = \frac{\partial\mathcal{L}}{\partial t} + \frac{\partial\mathcal{L}}{\partial x_1}[A_{11}x_1 + A_{12}x_2 + B_1u + f_1] + \frac{\partial\mathcal{L}}{\partial u}\dot{u} + \frac{\partial\mathcal{L}}{\partial \dot{u}}\ddot{u}$$
$$= \left(\frac{\partial\mathcal{L}}{\partial x_1}A_{11}\right)x_1 + \left(\frac{\partial\mathcal{L}}{\partial x_1}A_{12}\right)x_2 + \left(\frac{\partial\mathcal{L}}{\partial x_1}B_1\right)u + \frac{\partial\mathcal{L}}{\partial x_1}f_1 + \frac{\partial\mathcal{L}}{\partial u}\dot{u} + \frac{\partial\mathcal{L}}{\partial \dot{u}}\ddot{u} + \frac{\partial\mathcal{L}}{\partial t}$$

Then systems (1) and (2) are equivalent to state- space differential equation defined on manifold:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ \frac{\partial\mathcal{L}}{\partial x_1}A_{11} & \frac{\partial\mathcal{L}}{\partial x_1}A_{12}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix} B_1 \\ \frac{\partial\mathcal{L}}{\partial x_1}B_1\end{pmatrix}u + \begin{pmatrix} 0 \\ \frac{\partial\mathcal{L}}{\partial u}\end{pmatrix}\dot{u}$$
$$+ \begin{pmatrix} 0 \\ \frac{\partial\mathcal{L}}{\partial \dot{u}}\end{pmatrix}\ddot{u} + \begin{pmatrix} f_1 \\ \frac{\partial\mathcal{L}}{\partial x_1}f_1 + \frac{\partial\mathcal{L}}{\partial t}\end{pmatrix} \qquad \ldots(5)$$

where $x_2(t) = \mathcal{L}(t, x_1, u, \dot{u})$, subject to the manifold

$$A_{21}x_1(t) + B_2u(t) + f_2(t) = 0 \qquad \ldots(6)$$

As one can see, the terms $\dot{u}, \ddot{u}$ are not appropriate for an application point of view, and the usage of the implicit function theorem to reduce the number of variables and estimate x_2 by (3) is better than solving problems (5) and (6).

4. Index-2 Time-Varying DAEs and their Variational Formulations

The main theme of this section is to discuss the solvability of index-2 time-varying DAEs using the variational formulation approach.

We are looking for a suitable function, such that its critical points lead to a solution to the proposed problem and vice-versa.

Define

$$\dot{x}_1(t) = A_{11}(t)x_1(t) + A_{12}(t)x_2(t) + B_1(t)u(t) + f_1(t) \qquad \ldots(7)$$
$$0 = A_{21}(t)x_1(t) + B_2(t)u(t) + f_2(t) \qquad \ldots(8)$$

where $A_{21}(t)A_{12}(t)$ is a non - singular matrix $\forall t \in I$, $B_2(t)u(t) + f_2(t) \in Range\big(A_{21}(t)\big)$ $u(t) \in \Delta_u$, where Δ_u is the class of admissible control defined according to the given problem.

If $u(t) \in c'(I, R^r)$, $f_2 \in c(I, R^n)$, $f_2 \in c'(I, R^{n-n_0})$, and $I = [t_0, t_f], t_f > t_0$, one can have the following

$$0 = \dot{A}_{21}x_1 + A_{21}\dot{x}_1 + \dot{B}_2u + B_2\dot{u} + \dot{f}_2$$
$$= \dot{A}_{21}x_1 + A_{21}(A_{11}x_1 + A_{12}x_2 + B_1u + f_1) + \dot{B}_2u + B_2\dot{u} + \dot{f}_2$$

$$= \dot{A}_{21}x_1 + A_{21}A_{11}x_1 + A_{21}A_{12}x_2 + A_{21}B_1u + A_{21}f_1 + \dot{B}_2u + B_2\dot{u} + \dot{f}_2 \quad \ldots(9)$$

Since the problem is index 2, i.e. $(A_{21}A_{12})$ is invertible matrix, then (9) is solvable and gives that

$$x_2(t) = -\tilde{A}(t)x_1(t) - \tilde{B}(t)u(t) - \big((A_{21}(t)A_{12}(t))^{-1}A_{21}(t)\big)f_1(t) - \big((A_{21}(t)A_{12}(t))^{-1}B_2(t)\big)\dot{u}(t) - (A_{21}(t)A_{12}(t))^{-1}\dot{f}_2(t)$$

where $\tilde{A} = (A_{21}(t)A_{12}(t))^{-1}\Big(A_{21}(t)A_{11}(t) + \dot{A}_{21}(t)\Big)$,

$\tilde{B} = (A_{21}(t)A_{12}(t))^{-1}\Big(A_{21}(t)B_1(t) + \dot{B}_2(t)\Big)$.

This leads to $x_2(t) = \mathcal{L}(x_1, u, \dot{u}),\ u(t) \in \Delta_u, f \in c'(I, R^{n-n_0})$,
$I = [t_0, t_f]$, t_0, t_f are given and real numbers with $t_0 < t_f$.

The selection of a consistent initial condition is based on the nature of equations (8), as:
$0 = A_{21}(t_0)x_1(t_0) + B_2(t_0)u(t_0) + f_2(t_0)$
$x_2(t_0) = \mathcal{L}(x_1(t_0), u(t_0), \dot{u}(t_0))$, for a given $x_1(t_0), u(t_0), \dot{u}(t_0)$

Next, we define

$$\begin{aligned}\omega^0 = \{\big(x_1(t_0), x_2(t_0)\big) \in R^{rank(E)} \times R^{n-rank(E)} | x_2(t_0) &= \mathcal{L}\big(x_1(0), u(0), \dot{u}(0)\big) \\ &= -\tilde{A}(t_0)x_1(t_0) - \tilde{B}(t_0)u(t_0) - \big((A_{21}(t_0)A_{12}(t_0))^{-1}A_{21}(t_0)\big)f_1(t_0) \\ &\quad -\big((A_{21}(t_0)A_{12}(t_0))^{-1}B_2(t_0)\big)\dot{u}(t_0) - (A_{21}(t_0)A_{12}(t_0))^{-1}\dot{f}_2(t_0), \\ &\quad \text{for a given } f_1(t_0), f_2(t_0), u(t_0), \dot{u}(t_0)\}.\end{aligned}$$

We redefine the constrained DAEs as

$$L_1x_1(t) = G_1(t), \quad \ldots(10)$$

$$L_2x_1(t) = G_2(t), \quad \ldots(11)$$

$$L_3x_1(t_0) = G_3(t_0) \quad \ldots(12)$$

where

$L_1x_1(t) \triangleq \frac{d}{dt}x_1(t) + \Big(A_{12}(t)\tilde{A}(t) - A_{11}(t)\Big)\ x_1(t)$
$L_2x_1(t) \triangleq -A_{21}(t)x_1(t)$
$L_3x_1(t_0) \triangleq L_2x_1(t_0)$
$G_1(t) \triangleq B_1(t)u(t) + f_1(t) - \big((A_{21}(t)A_{12}(t))^{-1}A_{21}(t)\big)f_1(t) -\big((A_{21}(t)A_{12}(t))^{-1}B_2(t)\big)\dot{u}(t) - (A_{21}(t)A_{12}(t))^{-1}\dot{f}_2(t)$
$G_2(t) \triangleq B_2(t)u(t) + f_2(t)$, $G_3(t_0) = B_3(t_0)u(t_0) + f_3(t_0)$
Set $L = (L_1, L_2, L_3)^T$
$L: D(L) \subset C(I, R^{n_0}) \to \text{Rang}(L)$
$D(L) = \{x_1 \in C'(I, R^{n_0}) | x_1(t_0) \in \omega^0\} \subset C(I, R^{n_0}) \forall u(t) \in c'(I, R^r), f_2 \in c(I, R^n)$, $f_2 \in c'(I, R^{n-n_0})$.

Since the operator $\frac{d}{dt}$ is appeared in $L_1x_1(t)$, the linear operator L is not symmetric with the given usual bilinear form basic concept. Hence, no variational formulation exists unless one can redefine the linear operator or its bilinear form [11, 19].

To create a functional (variational) equivalent to a linear problem $Lu = f$, where L is not symmetric with respect to the chosen bilinear form, by the functional $F\big(x_1, \mathcal{L}, u, \dot{u}, f, \dot{f}\big) = F[x_1]$, we have:

$$F[x_1] = \int_{t_0}^{t_f}\left[\frac{1}{2}(L_1^Tx_1(t) \quad L_2^Tx_1(t) \quad L_3^Tx_1(t_0))\begin{pmatrix} L_1x_1(t) \\ L_2x_1(t) \\ L_3x_1(t_0)\end{pmatrix} - (G_1^T(t) \quad G_2^T(t) \quad G_3^T(t_0))\begin{pmatrix} L_1x_1(t) \\ L_2x_1(t) \\ L_3x_1(t_0)\end{pmatrix}\right]dt$$

$$F[x_1] = \int_{t_0}^{t_f}\Big[\frac{1}{2}L_1^Tx_1(t)L_1x_1(t) + \frac{1}{2}L_2^Tx_1(t)L_2x_1(t) + \frac{1}{2}L_3^Tx_1(t_0)L_3x_1(t_0) - G_1^T(t)L_1x_1(t) - G_2^T(t)L_2x_1(t) - G_3^T(t_0)L_3x_1(t_0)\Big]dt \quad \ldots(13)$$

$$=\frac{1}{2}\langle L_1x_1(t),L_1x_1(t)\rangle+\frac{1}{2}\langle L_2x_1(t),L_2x_1(t)\rangle+\frac{1}{2}\langle L_3x_1(t_0),L_3x_1(t_0)\rangle-\langle G_1(t),L_1x_1(t)\rangle$$
$$-\langle G_2(t),L_2x_1(t)\rangle-\langle G_3(t_0),L_3x_1(t_0)\rangle.$$

where $\langle x,y\rangle=\int_{t_0}^{t_f}x^Ty\,dt\ ,\ x(t),y(t)\in C[t_0,t_f]$.

We define the first variation, due to the linear part of the increment of the functional $F[x_1(t)]$, as:

$$\delta F\big(x_1(t)\big)=F[x_1+\delta x_1]-F[x_1]|_{linear\ part\ in\ \delta x_1}=0$$

$$\delta F=\frac{1}{2}\langle L_1(x_1+\delta x_1),L_1(x_1+\delta x_1)\rangle+\frac{1}{2}\langle L_2(x_1+\delta x_1),L_2(x_1+\delta x_1)\rangle$$
$$+\frac{1}{2}\langle L_3\big(x_1(t_0)+\delta x_1(t_0)\big),L_3\big(x_1(t_0)+\delta x_1(t_0)\big)\rangle-\langle G_1(t),L_1(x_1+\delta x_1)\rangle$$
$$-\langle G_2(t),L_2(x_1+\delta x_1)\rangle-\langle G_3(t_0),L_3\big(x_1(t_0)+\delta x_1(t_0)\big)\rangle$$
$$-\Big[\frac{1}{2}\langle L_1x_1(t),L_1x_1(t)\rangle+\frac{1}{2}\langle L_2x_1(t),L_2x_1(t)\rangle+\frac{1}{2}\langle L_3x_1(t_0),L_3x_1(t_0)\rangle$$
$$-\langle G_1(t),L_1x_1(t)\rangle-\langle G_2(t),L_2x_1(t)\rangle-\langle G_3(t_0),L_3x_1(t_0)\rangle\Big].$$

Such that

$$\langle L_1(x_1+\delta x_1),L_1(x_1+\delta x_1)\rangle$$
$$=\langle L_1x_1,L_1x_1\rangle+\langle L_1x_1,L_1\delta x_1\rangle+\langle L_1\delta x_1,L_1\delta x_1\rangle+\langle L_1\delta x_1,L_1x_1\rangle$$

Since $\langle L_1\delta x_1,L_1\delta x_1\rangle$ is non linear in the term of δx_1, then

$$\langle L_1(x_1+\delta x_1),L_1(x_1+\delta x_1)\rangle=\langle L_1x_1,L_1x_1\rangle+2\langle L_1x_1,L_1\delta x_1\rangle.$$

And by the same way

$$\langle L_2(x_1+\delta x_1),L_2(x_1+\delta x_1)\rangle=\langle L_2x_1,L_2x_1\rangle+2\langle L_2x_1,L_2\delta x_1\rangle$$
$$\langle L_3\big(x_1(t_0)+\delta x_1(t_0)\big),L_3\big(x_1(t_0)+\delta x_1(t_0)\big)\rangle$$
$$=\langle L_3x_1(t_0),L_3x_1(t_0)\rangle+2\langle L_3x_1(t_0),L_3\delta x_1(t_0)\rangle.$$

where $x_1(t_0)$ is an arbitrary selection from the class of consistency initial condition ω^0. Otherwise, one can assume it as fixed numbered and set $\delta x_1(t_0)=0$,

$$\langle G_1(t),L_1(x_1+\delta x_1)\rangle=\langle G_1(t),L_1x_1(t)\rangle+\langle G_1(t),L_1\delta x_1(t)\rangle$$
$$\langle G_2(t),L_2(x_1+\delta x_1)\rangle=\langle G_2(t),L_2x_1(t)\rangle+\langle G_2(t),L_2\delta x_1(t)\rangle$$
$$\langle G_3(t_0),L_3\big(x_1(t_0)+\delta x_1(t_0)\big)\rangle=\langle G_3(t_0),L_3x_1(t_0)\rangle+\langle G_3(t_0),L_3\delta x_1(t_0)\rangle.$$

From the above discussion, let us define $\delta F[x_1(t)]$ as:

$$\delta F[x_1(t)]=\langle L_1x_1,L_1\delta x_1\rangle+\langle L_2x_1,L_2\delta x_1\rangle+\langle L_3x_1(t_0),L_3\delta x_1(t_0)\rangle-\langle G_1(t),L_1\delta x_1(t)\rangle$$
$$-\langle G_2(t),L_2\delta x_1(t)\rangle-\langle G_3(t_0),L_3\delta x_1(t_0)\rangle$$
$$=\langle L_1x_1-G_1(t),L_1\delta x_1\rangle+\langle L_2x_1-G_2(t),L_2\delta x_1\rangle$$
$$+\langle L_3x_1(t_0)-G_3(t_0),L_3\delta x_1(t_0)\rangle$$

$$\delta F[x_1(t)]=\int_{t_0}^{t_f}\Big[\big(L_1x_1-G_1(t)\big)^TL_1\delta x_1+\big(L_2x_1-G_2(t)\big)^TL_2\delta x_1+\big(L_3x_1(t_0)-G_3(t_0)\big)^TL_3\delta x_1(t_0)\Big]dt$$

$$=\int_{t_0}^{t_f}\Big[\big(L_1x_1-G_1(t)\big)^T\quad\big(L_2x_1-G_2(t)\big)^T\quad\big(L_3x_1(t_0)-G_3(t_0)\big)^T\Big]\begin{bmatrix}L_1\delta x_1\\ L_2\delta x_1\\ L_3\delta x_1(t_0)\end{bmatrix}dt$$

It is noticed that, if there is $x_1\in c'(I,R^{n_0})$ satisfying the operator equations (10)-(12), uniquely over the class ω^0, then these equations will be identically satisfied.

Since the aim is to fix $x_1\in c'(I,R^{n_0})$, then this variational problem is well defined.

Also, since from the linearity in x_1,

$$\Rightarrow L_1\delta x_1=\delta L_1x_1,\qquad L_2\delta x_1=L_2\delta x_1,\qquad \delta L_3x_1(t_0)=\delta L_3x_1(t_0)$$

for arbitrary $\delta L_1x_1,\ \delta L_2x_1,\delta L_3x_1(t_0)$,

then

$\delta F[x_1(t)]$
$= \langle [(L_1x_1 - G_1(t)) \quad (L_2x_1 - G_2(t)) \quad (L_3x_1(t_0) - G_3(t_0))], [L_1\delta x_1 \quad L_2\delta x_1 \quad L_3\delta x_1(t_0)] \rangle$
For the arbitrary δx_1, the non-degeneracy property on the range and domain of the bilinear form, and the linearity property, we get
$\delta F[x_1(t)] = 0$
$\Rightarrow [(L_1x_1 - G_1(t)) \quad (L_2x_1 - G_2(t)) \quad (L_3x_1(t_0) - G_3(t_0))] = [0 \quad 0 \quad 0] \; \forall \delta x_1$
$\Rightarrow L_1x_1 = G_1(t), \quad L_2x_1 = G_2(t), \quad L_3x_1(t_0) = G_3(t_0), \quad x_2 = Lx_1$
$\Rightarrow Lx_1 - G(t) = 0 \Leftrightarrow Lx_1 = G(t), L = (L_1, L_2, L_3)^T, G(t) = (G_1, G_2, G_3)$
It should be noticed that

1. If $x_2 = Lx_1 \Rightarrow \delta x_1 = \frac{\partial L}{\partial x_1}\delta x_1$ and this term should be inserted in the variational formulation.
2. If $x_1(t_0) \in \mathcal{M}(x_1(t_0))$ (linear manifold of consistent initial conditions), $\Rightarrow \delta x_1(t_0) = \frac{\partial \mathcal{M}}{\partial x_1}\Big|_{t=t_0} . \delta x_1(t_0),$
3. $x_1(t_0)$ may also be assumed as fixed to produce that $\delta x_1(t_0) = 0$ and this will not affect the previous results.

Then, if $x_1 = G(t)$, with $L = (L_1, L_2, L_3)^T$is the solution of the proposed problem, then $\langle 0, L_1\delta x_1 \rangle = 0 \Leftrightarrow \delta F[x_1(t)] = 0$.
The other direction is clearly understood and the solution $x_1 \in c'(I, R^{n_0})$ is a critical point of variable formulation (10).
From practical point of view, one has to evaluate the functional $F[x_1(t)]$ in order to find its critical points.
Moreover, critical points of a functional are equivalent to solve the necessary Euler equation corresponding to the given problem, which is difficult too. Thus, a direct method of variational problem is adapted to approximate the solution by a finite number of bases functions of separable Banach space $c(I, R)$, as:

$$x_1^j = \sum_{i=0}^{m_j} a_i^j H_i^j(t), \; j = 1,2, \dots, n_0 \;, m_j \text{ arbitrary} \qquad \dots(14)$$

$$x_2^l = \mathcal{L}(x_1^j, u, \dot{u}), \; l = 1,2, \dots, n - n_0 \;, \; j = 1,2, \dots, n_0 \qquad \dots(15)$$

where H_i^j is linearly independent bases function of time t.
By substituting (14) and (15) in (13), we have
$F[x_1] = F(x_1, \mathcal{L}, u, \dot{u}, f, \dot{f})$

$$= F(a_0^1, a_1^1, a_2^1, \dots, a_{m_1}^1, a_0^2, a_1^2, a_2^2, \dots, a_{m_2}^2, \dots, a_0^{n_0}, a_1^{n_0}, a_2^{n_0}, \dots a_{m_{n_0}}^{n_0}) \qquad \dots(16)$$

where $n = n_0 + n - n_0$ is the total number of unknown variables.
The critical point of variational formulation (13) is then equivalent to find the derivative of the functional (16) with respect to $a_i^j, i = 0, .., m_j \,, j = 1, \dots, n_0$.

$$\text{i.e. } \frac{\partial F}{\partial a_i^j} = 0, \; \forall i = 0, .., m_j \,, j = 1, \dots, n_0 \qquad \dots(17)$$

Since the varitional formulation is of quadratic type, the linear system of algebraic equation was obtained from equation (14), with the class of consistency initial condition where the given functions $u(t), \dot{u}(t), f(t), \dot{f}(t)$ are selected from the class of admissible functions.
Once this system (17) is being solved for a_i^j, approximate solutions $x_1(t)$ and $x_2(t)$ are obtained according to equations (14) and (15) and hence the original solution of (7) and (8) is obtained approximately.

5. Illustrations

Example 5.1: (index 2 linear time-varying DAEs with given $u(t)$ over admissible class)
Consider the linear time invariant index-2 semi explicit DAE problem [1],

$$\begin{pmatrix}\dot{x}_{11}\\ \dot{x}_{12}\end{pmatrix}=\begin{pmatrix}-1 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_{11}\\ x_{12}\end{pmatrix}+\begin{pmatrix}0\\ 1+2t\end{pmatrix}x_{21}+\begin{pmatrix}1\\ 0\end{pmatrix}u(t) \quad \ldots(18)$$

$$0=(1 \quad 1)\begin{pmatrix}x_{11}\\ x_{12}\end{pmatrix}-(e^{-t})+u(t),\ t\in[0,1] \quad \ldots(19)$$

Note that the variable x_{21} does not appear in the algebraic constraint explicitly.
Then, by deriving the algebraic constraint with respect to t and $u(t)=-sint$ we have that

$$\frac{d}{dt}(0=x_{11}+x_{12}-e^{-t}-sint)$$
$$\Rightarrow 0=\dot{x}_{11}+\dot{x}_{12}+e^{-t}-cost \quad \ldots(20)$$

We substitute (18) in (20) to get

$$[-x_{11}+x_{12}-sint]+[1+2t]x_{21}+e^{-t}-cost=0$$

$\Rightarrow x_{21}=\frac{x_{11}-x_{12}+sint-e^{-t}+cost}{1+2t}=\mathcal{L}(x_{11},x_{12},t),\ \forall t\in[0,1]$.

The class of consistency initial condition is

$$\omega^0=\left\{(x_{11}(t_0),x_{12}(t_0),x_{21}(t_0))|x_{21}(t_0)=\mathcal{L}(x_{11}(t_0),x_{12}(t_0),t_0)\right.$$
$$\left.=\frac{x_{11}(t_0)-x_{12}(t_0)+sint_0-e^{-t_0}+cost_0}{1+2t_0}\ \forall\, x_{11}(t_0),x_{12}(t_0)\right\}$$

Then, the index-2 semi explicit system will be as

$$\begin{pmatrix}\dot{x}_{11}\\ \dot{x}_{12}\end{pmatrix}=\begin{pmatrix}-1 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_{11}\\ x_{12}\end{pmatrix}+\begin{pmatrix}0\\ 1+2t\end{pmatrix}\mathcal{L}(x_{11},x_{12},t)+\begin{pmatrix}1\\ 0\end{pmatrix}u(t)$$
$$0=(1 \quad 1)\begin{pmatrix}x_{11}\\ x_{12}\end{pmatrix}-(e^{-t})+u(t).$$

And, as we mentioned in algorithm 5 for finding variational formulation,

$$F[x_1]=\frac{1}{2}\langle L_1x_1(t),L_1x_1(t)\rangle+\frac{1}{2}\langle L_2x_1(t),L_2x_1(t)\rangle+\frac{1}{2}\langle L_3x_1(t_0),L_3x_1(t_0)\rangle$$
$$-\langle\tilde{B}(t),L_1x_1(t)\rangle-\langle f(t),L_2x_1(t)\rangle-\langle f(t_0),L_3x_1(t_0)\rangle.$$

where $L_1x_1(t)=\dot{x}_1+\tilde{A}x_1,\quad L_2x_1(t)=-A_{21}x_1,\quad L_3x_1(t_0)=-A_{21}x_1(t_0)$
$x_1=(x_{11},x_{12})^T$, $A_{11}=\begin{pmatrix}-1 & 1\\ 0 & 0\end{pmatrix}$, $A_{12}=\begin{pmatrix}0\\ 1+2t\end{pmatrix}$, $A_{21}=(1 \quad 1)$,
$\tilde{A}=(A_{12}(A_{21}A_{12})^{-1}A_{21}A_{11}-A_{11})$, $f(t)=-e^t+u(t)$,
$\tilde{B}=(1-A_{12}(A_{21}A_{12})^{-1}A_{21})B_1u-A_{12}(A_{21}A_{12})^{-1}(e^{-t}-cost)$
$\mathcal{L}(x_1)=\frac{x_{11}-x_{12}+sint-e^{-t}+cost}{1+2t}$ are defined with the class ω^0 .

The variational formulation with the class of consistent initial condition is defined as:

$$F[x_1]=\frac{1}{2}\int_0^1\Big[\left[\dot{x}_1+\tilde{A}x_1\right]^T\left[\dot{x}_1+\tilde{A}x_1\right]-2\tilde{B}^T\left[\dot{x}_1+\tilde{A}x_1\right]+[-A_{21}x_1]^T[-A_{21}x_1]$$
$$-2[f(t)]^T[-A_{21}x_1]+[-A_{21}x_1(t_0)]^T[-A_{21}x_1(t_0)]$$
$$-2[-(e^{-t_0})+u(t_0)]^T[-A_{21}x_1(t_0)]\Big]dt$$

Now, we set

$$x_{11}(t)=\sum_{i=0}^{m_1}a_i^1H_i^1(t),H_i^1(t)=t^i,i=0,..,m_1;\ \ m_1=5,$$

$$x_{12}(t)=\sum_{i=0}^{m_2}a_i^2H_i^2(t),H_i^2(t)=t^i,i=0,..,m_2;\ \ m_2=5$$

and H_i^1,H_i^2 are linearly independent-bases functions that are vanished on ω^0 .

We estimate $F\left(\overrightarrow{a^1},\overrightarrow{a^2}\right)=F(a_0^1,a_1^1,a_2^1,\dots,a_5^1,a_0^2,a_1^2,a_2^2,\dots,a_5^2)$, and by taking $\frac{\partial F}{\partial \overrightarrow{a^1}}=0$, and $\frac{\partial F}{\partial \overrightarrow{a^2}}=0$, this gives $A\begin{pmatrix}\overrightarrow{a^1}\\ \overrightarrow{a^2}\end{pmatrix}=B$, hence the approximate solution $(x_{11}(t),x_{12}(t),x_{21}(t))$ is obtained.
The numerical results of the unknown coefficients of linear algebraic system were found to be:

$a_0^1=1, a_1^1=-0.99997, a_2^1=0.49967, a_3^1=-0.16510, a_4^1=0.03840, a_5^1=-0.00511$
$, a_0^2=0, a_1^2=0.99998, a_2^2=0.00020, a_3^3=-0.16757, a_4^2=0.00162,\ a_5^2=0.00723.$

And The exact solution that taken from [1] is
$x_{11}=e^{-t},\quad x_{12}=sint,\quad x_{21}(t)=\frac{cost}{1+2t}$ for a given $u(t)=-sint.$

Then one can shows the comparison between the proposed solution and the exact solution in tables 5.1 and 5.2.

Table 5.1-Comparison between differential states in the proposed method and exact solution

t	x_{11} **Propose method**	x_{11} **Exact Sol.**	**Abs. error**	x_{12} **Propose method**	x_{12} **Exact Sol.**	**Abs. error**
0	1	1	0	0	0	0
0.1	0.9048	0.9048	1496×10^{-7}	0.0998	0.0998	1299×10^{-7}
0.2	0.8187	0.8187	5443×10^{-7}	0.1986	0.1987	3842×10^{-7}
0.3	0.7408	0.7408	3639×10^{-7}	0.2955	0.2955	3052×10^{-7}
0.4	0.6703	0.6703	3007×10^{-7}	0.3894	0.3894	1859×10^{-7}
0.5	0.6065	0.6065	6213×10^{-7}	0.4794	0.4794	4818×10^{-7}
0.6	0.5488	0.5488	2583×10^{-7}	0.5646	0.5646	2461×10^{-7}
0.7	0.4965	0.4966	3840×10^{-7}	0.6442	0.6442	2770×10^{-7}
0.8	0.4493	0.4493	5108×10^{-7}	0.7173	0.7174	4324×10^{-7}
0.9	0.4065	0.4066	1605×10^{-7}	0.7833	0.7833	1139×10^{-7}
1	0.3678	0.3679	3043×10^{-10}	0.8414	0.8415	2775×10^{-10}

Table 5.2-Comparison between equality constraint states in the proposed method and exact solution

t	x_{21} **Prposed method**	x_{21} **Exact Sol.**	**Abs. error**
0	1	1	0
0.1	0.8291	0.8292	2329×10^{-7}
0.2	0.7000	0.7000	6633×10^{-7}
0.3	0.5970	0.5971	4182×10^{-7}
0.4	0.5117	0.5117	2703×10^{-7}
0.5	0.4387	0.4388	5516×10^{-7}
0.6	0.3751	0.3752	2293×10^{-7}
0.7	0.3186	0.3187	2754×10^{-7}
0.8	0.2679	0.2680	3628×10^{-7}
0.9	0.2220	0.2220	9804×10^{-7}
1	0.1801	0.1801	1939×10^{-10}

There is another way to test the accuracy of the solution, without knowing the exact solution, by using L_2-norm and substituting the a_i^j values in

$$x_{11}(t) = \sum_{i=0}^{m_1} a_i^1 H_i^1(t)\,, H_i^1(t) = t^i, i = 0, .., m_1;\;\; m_1 = 5\,, \quad \ldots(21)$$
$$x_{12}(t) = \sum_{i=0}^{m_2} a_i^2 H_i^2(t)\,, H_i^2(t) = t^i, i = 0, .., m_2;\;\; m_2 = 5 \quad \ldots(22)$$

So for differential equation, one can check the accuracy as follows

$$\|\dot{x}_1 - A_{11}x_1 - A_{12}x_2 - B_1 u\|_2 = \left[\int_0^1 [|\dot{x}_{11} + x_{11} - x_{12} + sint|^2 + |\dot{x}_{12} - x_{11} + x_{12} - sint + e^{-t} - cost|^2]dt\right]^{1/2}$$
$$= 3 \times 10^{-5} \quad \ldots(23)$$

where $x_1 = (x_{11}, x_{12})^T$, and for equality algebraic constraint

$$\|A_{21}x_1 - e^{-t} + u\|_2 = \left[\int_0^1 |x_{11} + x_{12} - e^{-t} - sint|^2 dt\right]^{1/2} = 2 \times 10^{-7} \quad \ldots(24)$$

And for consistency condition

$$\|A_{21}x_1(t_0) - e^{-t_0} + u(t_0)\|_2 = \left[\int_0^1 |x_{11}(t_0) + x_{12}(t_0) - e^{-t_0} - sint_0|^2 dt\right]^{1/2} = 0 \quad \ldots(25)$$

The L_2 norm errors (23)-(25) explain the overall error of satisfying the equations (10)-(12), $\forall t \in [t_0, t_f] = [0,1]$, for each equation , where the parameterizations (21),(22) are suggested. The approximate and exact solution to the differential-equality states are showed in Figure 5.1.

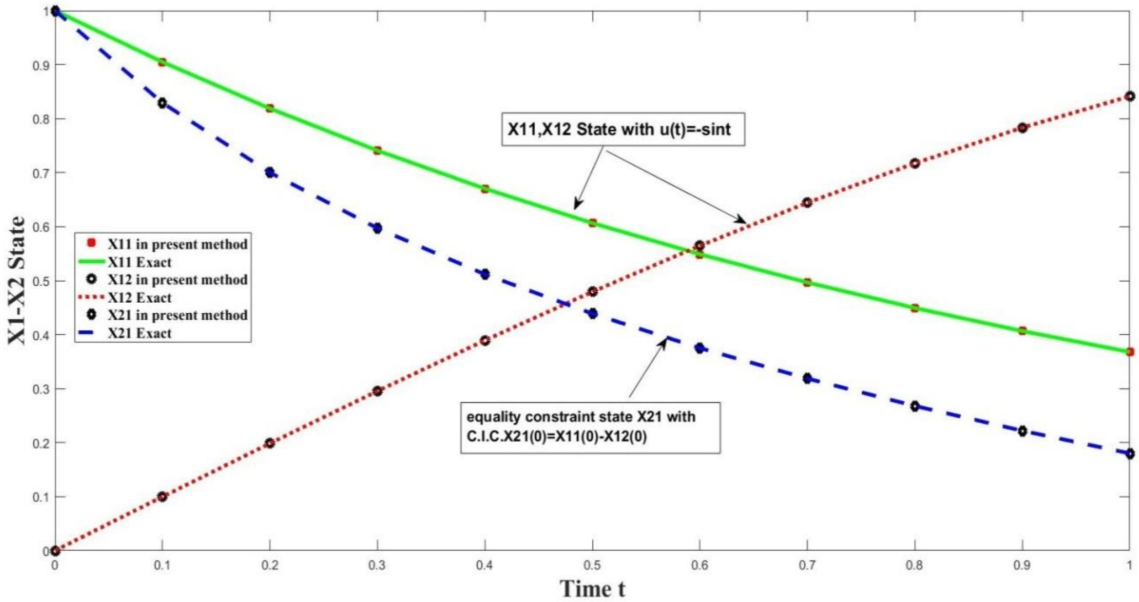

Figure 5.1- Differential-equality states and exact solution of index 2 linear time-varying DAEs with given u(t) =-sint.

Example 5.2: (The algebraic equation appears as a system of equations)

Consider the linear time invariant descriptor system

$$\dot{x}_{11} = x_{11} + 2x_{21}(t) + x_{22}(t) + u(t) + f_{11}(t)$$
$$\dot{x}_{12} = 2x_{11}(t) + x_{12}(t) + 2x_{22}(t) + f_{12}(t)$$
$$0 = x_{11}(t) - u(t) + f_{21}(t)$$
$$0 = x_{12}(t) + f_{22}(t)$$

with $f_{11}(t) = -t^5 + 3t^4 - t^3 - 2t^2 - 1 - u(t),$

$f_{12}(t) = 2t^5 - t^4 + t^3 - t^2 - 4,$

$f_{21}(t) = -1 - t^2 - t^5 + u(t),$

$f_{22}(t) = -2 - t^3 - t^4$, where $\in [0,1]$.

This system is equivalent to the following differential-algebraic system

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1 u(t) + f_1$$

$$0 = A_{21}x_1 + B_2u(t) + f_2.$$

where $x_1 = (x_{11}, x_{12})^T$, $x_2 = (x_{21}, x_{22})^T$, $A_{11} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $A_{12} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, $A_{21} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $f_1 = (f_{11}, f_{12})^T$, $f_2 = (f_{21}, f_{22})^T$, $B_1 = (1 \quad 0)^T$, $B_1 = (-1 \quad 0)^T$, $u(t) \in \Delta_u$, where Δ_u is the class of admissible control.

If we differentiate the equality constraint with respect to t, one can get

$$0 = A_{21}\dot{x}_1 + B_2\dot{u}(t) + \dot{f}_2$$

$$0 = A_{21}(A_{11}x_1 + A_{12}x_2 + B_1u(t) + f_1) + B_2\dot{u}(t) + \dot{f}_2$$

Since $(A_{21}A_{12})$ is nonsingular, then it is possible to rewrite our system as

$$\dot{x}_1 = \left(A_{11} - A_{12}\left((A_{21}A_{12})^{-1}(A_{21}A_{11})\right)\right)x_1 + (B_1 - A_{12}(A_{21}A_{12})^{-1}A_{21}B_1)u(t) + (1 - A_{12}(A_{21}A_{12})^{-1}A_{21})f_1 - A_{12}(A_{21}A_{12})^{-1}B_2\dot{u}(t) - A_{12}(A_{21}A_{12})^{-1}\dot{f}_2$$

$$x_2 = \mathcal{L}(x_1, u, \dot{u}, f_1, \dot{f}_2, t)$$
$$= -(A_{21}A_{12})^{-1}\left((A_{21}A_{11})x_1 + A_{21}B_1u(t) + A_{21}f_1(t) + B_2\dot{u}(t) + \dot{f}_2(t)\right)$$
$$= \begin{pmatrix} \frac{3t^4}{4} - \frac{t^3}{4} + t - \frac{x_{12}}{4} - \frac{1}{2} \\ t^5 + \frac{t^4}{2} + \frac{3t^3}{2} + 2t^2 - x_{11} - \frac{x_{12}}{2} + 2 \end{pmatrix} \text{ defined with the class } \omega^0 .$$

The class of consistency initial condition is

$$\omega^0 = \{(x_{11}(t_0), x_{12}(t_0), x_{21}(t_0), x_{22}(t_0)) | x_{21}(t_0) = \frac{x_{12}(t_0)}{4} - \frac{1}{2}, x_{22}(t_0) = 2 - x_{11}(0) - \frac{x_{12}(0)}{2} \ \forall\, x_{11}(0), x_{12}(0)\}.$$

The variational formulation with the class of consistent initial condition is defined as

$$F[x_1] = \frac{1}{2}\int_0^1 \Big[\left[\dot{x}_1 + \tilde{A}x_1\right]^T\left[\dot{x}_1 + \tilde{A}x_1\right] - 2\tilde{B}^T\left[\dot{x}_1 + \tilde{A}x_1\right] + [-A_{21}x_1]^T[-A_{21}x_1] - 2[B_2u + f_2(t)]^T[-A_{21}x_1] + [-A_{12}x_1^0]^T[A_{12}x_1^0] - 2[B_2u(t_0) + f_2(t_0)]^T[A_{12}x_1^0]\Big]\,dt.$$

where

$$\tilde{A} = (A_{12}(A_{21}A_{12})^{-1}A_{21}A_{11} - A_{11}),$$
$$\tilde{B} = (1 - A_{12}(A_{21}A_{12})^{-1}A_{21})B_1u - (1 - A_{12}(A_{21}A_{12})^{-1}A_{21})f_1 - A_{12}(A_{21}A_{12})^{-1}B_2\dot{u}(t) - A_{12}(A_{21}A_{12})^{-1}\dot{f}_2.$$

Now, we set

$$x_{11}(t) = \sum_{i=0}^{m_1} a_i^1 H_i^1(t),\ H_i^1(t) = t^i, i = 0,..,m_1;\ \ m_1 = 5,$$
$$x_{12}(t) = \sum_{i=0}^{m_2} a_i^2 H_i^2(t),\ H_i^2(t) = t^i, i = 0,..,m_2;\ \ m_2 = 5$$

and H_i^1, H_i^2 are linearly independent-base functions that are vanished on ω^0 , which leads to

$$x_2(t) = \mathcal{L}(x_{11}, x_{12}, t) = \mathcal{L}\left(\sum_{i=0}^{m_1} a_i^1 H_i^1(t), \sum_{i=0}^{m_2} a_i^2 H_i^2(t), t\right)$$

then $F\left(\overrightarrow{a^1}, \overrightarrow{a^2}\right) = F(a_0^1, a_1^1, a_2^1, \dots, a_5^1, a_0^2, a_1^2, a_2^2, \dots, a_5^2)$, and by taking $\frac{\partial F}{\partial \overrightarrow{a^1}} = 0$, and $\frac{\partial F}{\partial \overrightarrow{a^2}} = 0$, this gives $A\begin{pmatrix} \overrightarrow{a^1} \\ \overrightarrow{a^2} \end{pmatrix} = B$, hence the numerical solution for a^1, a^2 using the proposed technique is found as:

$$a_0^1 = 1, a_1^1 = 0, a_2^1 = 1, a_3^1 = 0, a_4^1 = 0, a_5^1 = 1,$$
$$a_0^2 = 2, a_1^2 = 0, a_2^2 = 0, a_3^3 = 1, a_4^2 = 1, \ a_5^2 = 0,$$

where the exact answer is obtained as:

$$x_{11} = 1 + t^2 + t^5, \quad x_{12} = 2 + t^3 + t^4, \quad x_{21} = t + t^4, \quad x_{22} = t^2 + t^3.$$

The numerical results used in the proposed technique and the comparisons with given exact solutions are shown in following tables.

Table 5.3-Comparisons among differential states in the proposed method and exact solutions

t	x_{11} Proposed method	x_{11} exact	Abs. error	x_{12} Proposed method	x_{12} exact	Abs. error
0	1	1	0	2	2	0
0.1	1.0100	1.0100	7×10^{-13}	2.0011	2.0011	0
0.2	1.0403	1.0403	0	2.0096	2.0096	0
0.3	1.0924	1.0924	1×10^{-13}	2.0351	2.0351	0
0.4	1.1702	1.1702	0	2.0896	2.0896	0
0.5	1.2812	1.2813	0	2.1875	2.1875	0
0.6	1.4377	1.4378	0	2.3456	2.3456	0
0.7	1.6580	1.6581	6×10^{-14}	2.5831	2.5831	0
0.8	1.9676	1.9677	0	2.9216	2.9216	0
0.9	2.4004	2.4005	1×10^{-14}	3.3851	3.3851	0
1	3	3	0	4	4	0

Table 5.4-Comparison among equality state in proposed method and exact solutions

t	x_{21} Proposed method	x_{21} exact	Abs. error	x_{22} Proposed method	x_{22} exact	Abs. error
0	0	0	0	0	0	0
0.1	0.1001	0.1001	0	0.011	0.011	0
0.2	0.2016	0.2016	0	0.048	0.048	0
0.3	0.3081	0.3081	0	0.117	0.117	0
0.4	0.4256	0.4256	0	0.224	0.224	0
0.5	0.5625	0.5625	0	0.375	0.375	0
0.6	0.7296	0.7296	0	0.576	0.576	0
0.7	0.9401	0.9401	0	0.833	0.833	0
0.8	1.2096	1.2096	0	1.152	1.152	0
0.9	1.5561	1.5561	0	1.539	1.539	0

1	2	2	0	2	2	0

To test the accuracy of the solution presented by the present method, the L_2-norm is examined.
For equality constraint, we have

$$\|A_{21}x_1 - e^{-t} + u\|_2 = \left[\int_0^1 |x_{11} + x_{12} - e^{-t} - sint|^2 dt\right]^{1/2} = 0$$

This criterion represents a good test to show the extent to which the approximate solution matches the exact solution, and thus indicates the accuracy of the method used.
The differential-equality states are illustrated in Figure 5.2.

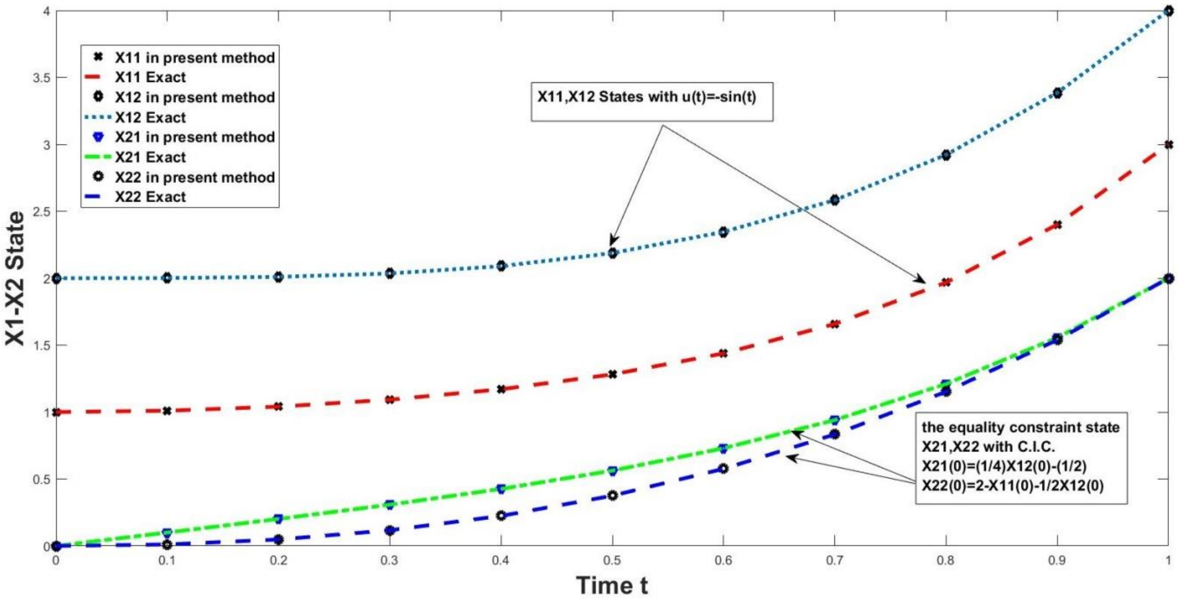

Figure 5.2- Differential-equality states and exact solution with u(t) = -sint. C.I.C. is consistent initial condition.

Example 5.3 (Descriptor index of two control model)
Consider the following linear mechanical system

$$E\dot{x} = Ax + Bu + f(t),$$

with $E = \begin{bmatrix} I & 0 & 0 \\ 0 & \mathcal{M} & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & I & I \\ -\mathcal{K} & -\mathcal{D} & \mathcal{J} \\ \mathcal{H} & G & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ \mathcal{L} \\ 0 \end{bmatrix}, x = \begin{bmatrix} \mathcal{Z} \\ \dot{\mathcal{Z}} \\ \mu \end{bmatrix}$
where the matrices in the table below represent the respective models.

Matrix	Represent in mechanical model
$\mathcal{Z} \in R^n$	the displacement vector
$\mu \in R^q$	the vector of lagrangian multiplier
u	the known input force
$\mathcal{M}$	the inertial matrix
$\mathcal{D}$	the damping matrix
$\mathcal{K}$	the stiffness matrix
$\mathcal{L}$	matrix of force distribution
$G, \mathcal{H}$	the coefficient matrices

All these matrices are known with appropriate dimensions. For more detail about this mechanical model, see [14].
Now, based on these matrices the semi explicit descriptor system can be rewritten as

$$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\end{bmatrix}x=\begin{bmatrix}0&0&1&0&1\\0&0&0&1&1\\-2&0&-1&-1&1\\0&-1&-1&-1&1\\1&1&0&0&0\end{bmatrix}x+\begin{bmatrix}0\\0\\-1\\1\\0\end{bmatrix}u(t)+\begin{bmatrix}1+t\\t\\0\\0\\t^2+t\end{bmatrix}$$

This system needs to be transform firstly into DAEs, as mentioned previously in section 3, with $\operatorname{rank}(E)=4<n$. Then, a singular value decomposition is used to calculate U as an orthonormal eigenvectors matrix of EE^T and V as an orthonormal eigenvectors matrix of E^TE.

Set $\mathbb{P}=V=\begin{bmatrix}0&0&0&1&0\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&1\end{bmatrix}$, $\mathbb{Q}=U^{-1}=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\1&0&0&0&0\\0&0&0&0&1\end{bmatrix}$

Now $\mathbb{Q}E\mathbb{P}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\end{bmatrix}\Rightarrow \Sigma=\mathrm{I}_{4\times4}=\Sigma^{-1},\quad \mathbb{P}^{-1}x=\begin{bmatrix}x_{11}\\x_{12}\\x_{13}\\x_{14}\\x_{21}\end{bmatrix}$

$$\mathbb{Q}A\mathbb{P}=\begin{bmatrix}0&0&1&0&1\\0&-1&-1&-2&1\\-1&-1&-1&0&1\\0&1&0&0&1\\1&0&0&1&0\end{bmatrix},\quad \mathbb{Q}B=\begin{bmatrix}0\\-1\\1\\0\\0\end{bmatrix},\quad \mathbb{Q}f=\begin{bmatrix}t\\0\\0\\t+1\\t^2+t\end{bmatrix}.$$

Under the transformation $(\mathbb{Q},\mathbb{P})$, the mechanical system will be differential algebraic control system as:

$$\dot{x}_1=A_{11}x_1+A_{12}x_2+B_1u+f_1$$
$$0=A_{21}x_1+f_2$$

with $A_{11}=\begin{bmatrix}0&0&1&0\\0&-1&-1&-2\\-1&-1&-1&0\\0&1&0&0\end{bmatrix}, A_{12}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, B_1=\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}, A_{21}=[1\quad 0\quad 0\quad 1],$

$$f_1=\begin{bmatrix}t\\0\\0\\1+t\end{bmatrix}, f_2=t^2+t, x_1=\begin{bmatrix}x_{11}\\x_{12}\\x_{13}\\x_{14}\end{bmatrix}, x_2=x_{21}.$$

which is an index-2 system. By differentiating the equality constraint with respect to t to estimate x_2, and since $(A_{21}A_{12})$ invertible matrix, then

$$x_2=-(A_{21}A_{12})^{-1}\left[A_{21}A_{11}x_1+A_{21}B_1u+A_{21}f_1+B_2\dot{u}+\dot{f}_2\right]$$

$$x_2=-2t-\frac{x_{12}+x_{13}+1}{2}\ \text{ is defined with the class } \omega^0.$$

And the class of consistency initial condition is

$$\omega^0=\{(x_{11}(t_0),x_{12}(t_0),x_{13}(t_0),x_{14}(t_0),x_{21}(t_0))|x_{21}(t_0)$$
$$=\mathcal{L}(x_{11}(t_0),x_{12}(t_0),x_{13}(t_0),x_{14}(t_0),t_0)\ \forall\ x_{12}(t_0),x_{13}(t_0)\}.$$

The variational function with the class of consistent initial condition is defined as

$$F[x_1]=\frac{1}{2}\int_0^1\Big[\left[\dot{x}_1+\tilde{A}x_1\right]^T\left[\dot{x}_1+\tilde{A}x_1\right]-2\tilde{B}^T\left[\dot{x}_1+\tilde{A}x_1\right]+[-A_{21}x_1]^T[-A_{21}x_1]$$
$$-2[B_2u+f_2(t)]^T[-A_{21}x_1]+[-A_{21}x_1(t_0)]^T[-A_{21}x_1(t_0)]$$
$$-2[f_2(t_0)]^T[-A_{21}x_1(t_0)]\Big]\,dt.$$

where

$\tilde{A}=(A_{12}(A_{21}A_{12})^{-1}A_{21}A_{11}-A_{11}),$

$$\tilde{B} = (1 - A_{12}(A_{21}A_{12})^{-1}A_{21})B_1 u - (1 - A_{12}(A_{21}A_{12})^{-1}A_{21})f_1 - A_{12}(A_{21}A_{12})^{-1}B_2\dot{u}(t) - A_{12}(A_{21}A_{12})^{-1}\dot{f}_2$$

Now, we set

$$x_{11}(t) = \sum_{i=0}^{m_1} a_i^1 H_i^1(t)\,, H_i^1(t) = t^i, i = 0,..,m_1;\ \ m_1 = 5\,,$$

$$x_{12}(t) = \sum_{i=0}^{m_2} a_i^2 H_i^2(t)\,, H_i^2(t) = t^i, i = 0,..,m_2;\ \ m_2 = 5$$

$$x_{13}(t) = \sum_{i=0}^{m_3} a_i^3 H_i^3(t)\,, H_i^3(t) = t^i, i = 0,..,m_3;\ \ m_3 = 5$$

$$x_{14}(t) = \sum_{i=0}^{m_4} a_i^4 H_i^4(t)\,, H_i^4(t) = t^i, i = 0,..,m_4;\ \ m_4 = 5$$

and $H_i^1, H_i^2, H_i^3, H_i^4$ are linearly independent-base functions that are vanished on ω^0, which leads to

$$\begin{aligned} x_{21}(t) &= \mathcal{L}(x_{11}, x_{12}, x_{13}, x_{14}, t) \\ &= \mathcal{L}\left(\sum_{i=0}^{m_1} a_i^1 H_i^1(t)\,, \sum_{i=0}^{m_2} a_i^2 H_i^2(t), \sum_{i=0}^{m_3} a_i^3 H_i^3(t)\,, \sum_{i=0}^{m_4} a_i^4 H_i^4(t)\,, t\right) \end{aligned}$$

And $F\left(\overrightarrow{a^1}, \overrightarrow{a^2}, \overrightarrow{a^3}, \overrightarrow{a^4}\right), \overrightarrow{a^1} = \left(a_0^1, a_1^1, a_2^1, \dots, a_{m_1}^1\right), \overrightarrow{a^2} = \left(a_0^2, a_1^2, a_2^2, \dots, a_{m_2}^2\right)$

$$\overrightarrow{a^3} = \left(a_0^3, a_1^3, a_2^3, \dots, a_{m_3}^3\right), \overrightarrow{a^4} = \left(a_0^4, a_1^4, a_2^4, \dots, a_{m_4}^4\right)$$

Since it is a quadratic functional form, for finding the critical point ,this leads to the linear algebraic equation $A\begin{pmatrix}\overrightarrow{a^1}\\ \overrightarrow{a^2}\\ \overrightarrow{a^3}\\ \overrightarrow{a^4}\end{pmatrix} = B$, which is solvable directly for $\begin{pmatrix}\overrightarrow{a^1}\\ \overrightarrow{a^2}\\ \overrightarrow{a^3}\\ \overrightarrow{a^4}\end{pmatrix} = A^{-1}B$.

Since this system is taken from practical applications without exact solution, then we test the accuracy of the solution presented by the present method, using test by L_2-norm:

For differential equation

$$\begin{aligned} &\|\dot{x}_1 - A_{11}x_1 - A_{12}x_2 - B_1 u - f_1\|_2 \\ &= \left[\int_0^1 \left[\left|\dot{x}_{11} - x_{13} + 2t + \tfrac{x_{12}+x_{13}+1}{2} - t\right|^2 + \left|\dot{x}_{12} - x_{12} - x_{13} - 2x_{14} + 2t + \tfrac{x_{12}+x_{13}+1}{2} + e^{-t}\right|^2 + \left|\dot{x}_{13} - x_{11} - x_{12} - x_{13} + 2t + \tfrac{x_{12}+x_{13}+1}{2} - e^{-t}\right|^2 + \left|\dot{x}_{14} - x_{12} + 2t + \tfrac{x_{12}+x_{13}+1}{2} - t - 1\right|^2\right] dt\right]^{1/2} \\ &= 2 \times 10^{-5} \end{aligned}$$

And for equality constraint

$$\|A_{21}x_1 + f_2\|_2 = \left[\int_0^1 |x_{11} + x_{14} + t^2 + t|^2 dt\right]^{1/2} = 4 \times 10^{-4}$$

Which states the accuracy of the present approach even if the exact solution is unknown for the system.

Figure 5.3 shows the differential equality states for open loop control $u(t) \in \Delta_u$, where Δ_u is the class of admissible control.

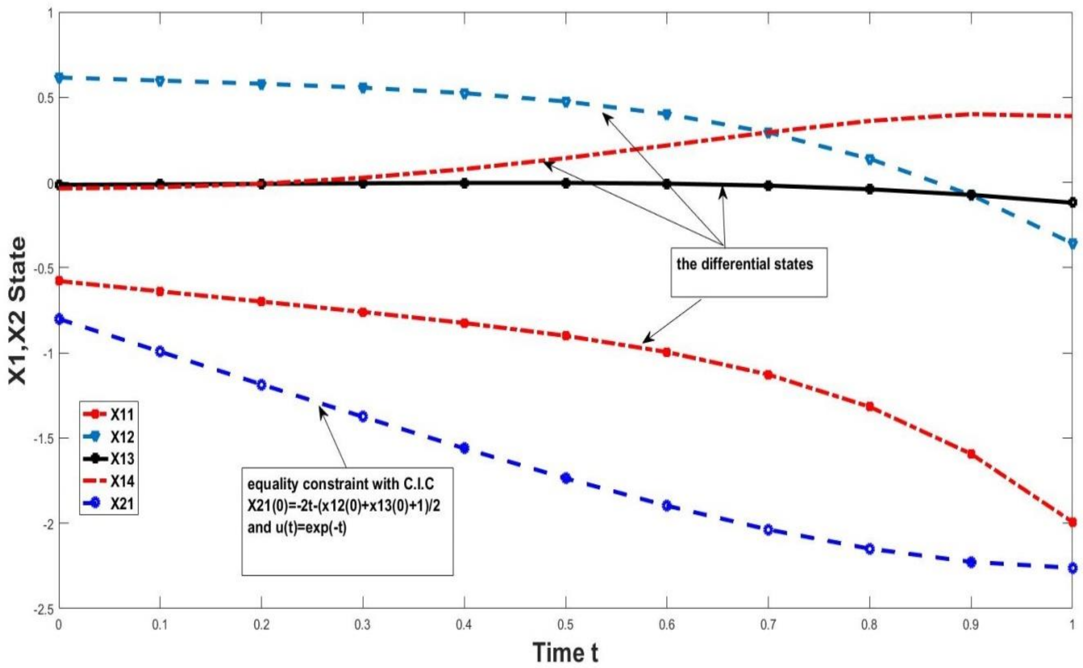

Figure 5.3-Differential-equality states for linear mechanical system of index two control model at $u(t) = e^{-t}$.

6. Results and discussion

The illustrated examples 5.1-5.3 are ranked from simple to more complex. The examples 5.1-5.2 are of semi explicit index-2, time-varying differential algebraic system, with known exact solutions. While the last example 5.3 is a descriptor system which firstly needs transformation, using singular value decomposition, to semi explicit DAEs, and needs to be taken from real life application without knowing its exact form solution.

These examples are taken as a test for the proposed method. By step by step implementation, the approximate solution is parameterized via polynomial base function, which is dense in $c[I, R^{n_0}]$. Even with reasonable small number of these polynomials with unknown coefficient, the obtained solutions are shown to be very accurate and efficient. Figures 5.1-5.3 show the excellent matching between the approximate solution, using the present method, and the given exact solution. The overall error value, using L_2 norm of the linear operators, showed very good results on $t \in [t_0, t_f], t_f > t_0$, for each example. The pointwise error in tables 5.1-5.4 demonstrated the good accuracy.

For example 5.3, due to the absence of the exact form solution, the L_2 norm errors for all the constraints (the differential and the algebraic) are adapted ($\|E_1\|_2 = 2 \times 10^{-5}, \|E_2\|_2 = 4 \times 10^{-4}, E_1 \triangleq L_1 x - G_1(x), E_2 \triangleq L_2 x - G_2(x)$) to test the convergence of the solution.

As an overall evaluation, the method has very good accuracy, being simple and effective as a tool to solve index 2, time-varying control DAEs.

7. Conclusions

As one can see, the present method is suitably applicable for an efficient class of index two DAEs with input $u(t)$ or even with semi-explicit index two, descriptor system. The method is easily implemented and a very good accuracy has been obtained, even for simple types of polynomial bases functions. This approach is reliable and efficient for this class of functions and can be extended to higher index DAEs (index greater than two).

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