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Some Properties of Algebraically Paranormal Operator

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Abstract
Through this study, the following has been proven, if T is an algebraically paranormal operator acting on separable Hilbert space, then T satisfies the (am) property and $f(T)$ is also satisfies the (am) property for all $f \in H(\sigma(T))$. These results are also achieved for (gam) property.
In addition, we prove that for a polaroid operator with finite ascent then after the property (am) holds for $f(T)$ for all $f \in H(\sigma(T))$.

Keywords: Algebraically paranormal operator, Polaroid operator, Weyl's Theorem

بعض خصائص المؤثرات الجبرية الطبيعة

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الخلاصة
في هذا البحث تمالحصول على مايلي, اذا كان T مؤثر جبري طبيعي يعمل على فضاء هلبرت, فان T تحقق الخاصية (am) و $f(T)$ تحقق خاصية (am) لكل $f \in H(\sigma(T))$ تم تحقيق هذه النتائج أيضًا لخاصية (gam). من ناحية اخرى اثبتنا بالنسبة لمؤثر بولارويد مع صعود محدود ، فإن الخاصية (am) تتحقق لـ $f(T)$ لجميع$f \in H(\sigma(T))$.

1. Introduction

Throughout this study, we suppose that $B(X)$ is the algebra of bounded linear operators acting on an infinite-dimensional separable Hilbert space. Let $T \in B(X)$, $\alpha(T)$, $\beta(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ denote the dimension of the kernel $\ker T$,the codimension of the range $\mathcal{R}(T)$, the spectrum, the point spectrum or eigenvalues spectrum and the approximate point spectrum, respectively. An operator $T \in B(X)$ is an upper semi-Fredholm operator if it has the closed range and $\alpha(T)$ is finite, it is called to be lower semi-Fredholm operator if $\beta(T)$ is finite [1]. In [2], the set of all upper semi-Fredholm (resp. lower semi- Fredholm) operators denotes $SF_+(X)$(resp. $SF_-(X)$). In the sequel, the set of all semi-Fredholm operators is defined by $SF_+^-(X) = SF_+(X) \cup SF_-(X)$, while the set of all Fredholm operators T is defined by $SF(X) = SF_+(X) \cap SF_-(X)$. The index of a Fredholm operator is defined by $ind(T) = \alpha(T) - \beta(T)$. T is said to be Weyl operator if T belongs to $SF(X)$ and $ind(T)$ equals zero and Weyl spectrum of T is defined by $\sigma_W(T) = \{\lambda \in \mathbb{C}: T - \lambda \ is\ not\ Weyl\}$. The upper Weyl spectrum (or Weyl essential approximate point spectrum) $\sigma_{SF_+^-}(T)$ of a bounded linear operator $T$is,

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$\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C}: T - \lambda is\ not\ belongs\ to\ SF_+^-(X)\}$.

$E^0(T)$ is an eigenvalue of T of finite multiplicity that is isolated in spectrum of T (i.e.)
$E^0(T) = \{\lambda \in iso\ \sigma(T): 0 < \alpha(T - \lambda) < \infty\}$.

Recall that, the point y is called an isolated point of the set $\mathcal{S}$ if $y \in \mathcal{S}$ and if there exists a ball $k(y, p)$ which contains no point of $\mathcal{S}$ other than$y$. Where y is a point of the metric space M and p is a positive number
$k(y, p) = \{x \in M: d(x, y) < p\}$,
called a ball with center y and radius p.

An operator $T \in B(X)$ is said to achieve Weyl's theorem if $\sigma(T)\backslash\sigma_w(T) = E^0(T)$[3]. A linear operator T on a vector space X is said to have finite ascent if $\bigcup_{n\in N} \ker T^n = ker\, T^k$ for some positive integer k. Clearly, in such a case there is a smallest positive integer $\wp = \wp(T)$ such that $ker\, T^{\wp} = ker\, T^{\wp+1}$, the positive integer $\wp$ is called the ascent of T. If there is no such integer, then the set $\wp(T) = \infty$. Analogously, T is said to have finite descent if $\bigcap_{n\in N} \ker T^n = T^k(X)$for some k. The smallest integer $q = q(T)$ that satisfies $T^{q+1}(X) = T^{q}(X)$ is called the descent of T. If there is no such integer, then the set $q(T) = \infty$. Noted that if both $\wp(T)$ and $q(T)$ are finite then $\wp(T) = q(T)$.

In the previous paragraph, we mention Weyl's theorem. The following two properties presented in [4] are related to Weyl's theorem, namely the (am) property and (gam) property. The main objective of our study is to answer the following question if T algebraically paranormal operator does it achieve these properties?

Now, suppose that$T \in B(X)$, the class of all upper semi-Browder operators is defined as: [2]
$b_+(X) = \{T \in SF_+: \wp(T) < \infty\}$,
and the class of all lower semi-Browder operators is defined by
$b_-(X) = \{T \in SF_-: q(T) < \infty\}$,
the set of all Browder operators defined by $b(X) = b_+(X) \cap b_-(X)$, and on it, the upper semi-Browder spectrum of T is defined by
$\sigma_{ub}(T) = \{\lambda \in \mathbb{C}: T - \lambda \notin b_+(X)\}$,
the lower semi-Browder spectrum of T is defined by
$\sigma_{ub}(T) = \{\lambda \in \mathbb{C}: T - \lambda \notin b_-(X)\}$,
whilst, the Browder spectrum of T is defined by, (see [8])
$\sigma_b(T) = \{\lambda \in \mathbb{C}: T - \lambda \notin b(X)\}$.

In [4], a bounded linear operator T is said to have property (am) if $\sigma_a(T)\backslash\sigma_b(T) = E_a^0(T)$, where $E_a^0(T) = \{\lambda \in iso\ \sigma_a(T): 0 < \alpha(T - \lambda) < \infty\}$.

Recall that from [3], a continuous linear operator $T \in B(X)$ has a single valued extension property at a point $\lambda_0 \in \mathbb{C}$ (Shortly $\mathcal{SVEP}$), if for every open disc $\mathcal{U}$ centered at λ_0 , then only analytic function $f: \mathcal{U} \longrightarrow X$ which satisfies $(T - \lambda)f(\lambda) = 0$ is the function $f \equiv 0$. T is said to have the $\mathcal{SVEP}$ if T has $\mathcal{SVEP}$ at every point $\lambda \in \mathbb{C}$. Evidently, T has $\mathcal{SVEP}$ at every isolated point of the spectrum, consequently, note that the single valued extension property plays an important role in Fredholm and spectral theory, if T has $\mathcal{SVEP}$, then $\sigma_a(T) = \sigma(T)$.

2. Property (am) For Algebraically Paranormal Operators

In 1909 H. Weyl [5] proved the spectrum of all compact perturbations of self-adjoint operators and showed that the self-adjoint operator satisfies Weyl's theorem. This result was later expanded to hyponormal operators, P- hyponormal operators. Lately, the authors [6] showed that if T algebraically paranormal acting on separable Hilbert space, then T satisfies Weyl's theorem. In this paper, we prove that property (am) and property (gam) satisfied for algebraically paranormal operators. T is called paranormal if $\|Tx\|^2 \le \|T^2x\|\|x\|$ for all $x \in X$, and it is called algebraically paranormal if there exists a non-constant polynomial $q(z)$ such that $q(T)$ is paranormal, (see [7], [8]). Generally, we have

$$\text{hyponormal} \to \text{P} - \text{hyponormal} \to \text{paranormal} \to \text{algebraically paranormal}$$

Aiena in [9] proved that every paranormal operator on a separable Banach space has $\mathcal{SVEP}$. Paranormal operators on Hilbert space have $\mathcal{SVEP}$.

Suppose that, $T \in B(X)$ and Δ is clopen subset of $\sigma(T)$, $\Delta \subseteq \sigma(T)$ that is relatively open and closed. Thus, $\sigma(T) = \Delta \cup (\sigma(T)\backslash\Delta)$, by proposition (4.11) in [10]

$$E(\Delta) = E(\Delta\,; T) = \frac{1}{2\pi i}\int_{\Gamma} (z - T)^{-1} dz$$

where Γ is a positively oriented Jordan system such that Δ is a subset of $\text{ins.}\,\Gamma$ and $\sigma(T)\backslash\Delta$ is a subset $\text{out.}\,\Gamma$ is idempotent. Furthermore, $E(\Delta)S = SE(\Delta)$ wherever $TS = ST$ and if $X_\Delta = E(\Delta)X$, $\sigma(T|X_\Delta) = \Delta$. Call $E(\Delta)$ the Riesz idempotent corresponding to Δ. If $\Delta = t$ singleton set $\{\lambda\}$, let $E(\lambda) = E(\{\lambda\})$ and $X_\lambda = X_{\{\lambda\}}$.

Now, we need to recall some notations and preliminary results.

An operator $T \in B(X)$ is called polaroid if every isolated point of $\sigma(T)$ is pole of the resolvent of T, equivalent $0 < p(T-\lambda) = q(T-\lambda) < \infty$, and it is called a-polaroid if every isolated point of $\sigma_a(T)$ is pole of the resolvent of T. The operator T is said to be hereditarily Polaroid if every part of T is polaroid. Aiena in [9, corollary 2.6] proved that every algebraically paranormal operator is hereditarily Polaroid.

Lemma 2.1

Suppose that $T \in B(X)$ is algebraically paranormal and let $\lambda \in \sigma_a(T)$ be an isolated point of $\sigma_a(T)$. Then, $X_T\{\lambda\} = \left\{x \in X: \|(T-\lambda)^n x\|^{\frac{1}{n}} \to 0\right\} = E_\lambda X$, where E_λ denotes the Riesz idempotent for λ.

Proof

Assume T is algebraically paranormal this leads to that T has $\mathcal{SVEP}$ [11, corollary 2.10]. From [13, corollary (2.4)], $X_T(\{0\}) = \left\{x \in X: \|(T)^n x\|^{\frac{1}{n}} \to 0\right\}$. And form [12, p. 424], $X_T\{\lambda\} = \left\{x \in X: \|(T-\lambda)^n x\|^{\frac{1}{n}} \to 0\right\} = E_\lambda X$.

Lemma 2.2. [6]

Let $T \in B(X)$ be a paranormal, $\lambda \in \mathbb{C}$, and suppose that $\sigma(T) = \{\lambda\}$. Then, $T = \lambda$.

Theorem 2.3

Let $T \in B(X)$ be algebraically paranormal. Then, T satisfies (am) property.

Proof

Let $\lambda \in \sigma_a(T)\backslash\sigma_b(T)$. Since T is algebraically paranormal then T has $\mathcal{EP}$. Thus, $\sigma_a(T) = \sigma(T)$, hence $\lambda \in \sigma(T)\backslash\sigma_b(T)$. Hence $\lambda \in \sigma(T)$ and $\lambda \notin \sigma_b(T)$, that is the ascent and the descent of T are finite. Then, from theorem (3.4) in [3], we have $T - \lambda \in SF_+^-(X)$. But, $\sigma(T)\backslash\sigma_b(T) = \pi^0(T)$ and since $\pi^0(T) \subseteq \pi(T)$ then λ is a pole of the resolvent of T. Then, by [3, corollary 3.21], λ is an isolated point of $\sigma(T)$ and $\alpha(T-\lambda) < \infty$, then $\lambda \in E^0(T)$.But, $E^0(T) \subseteq E_a^0(T)$. Hence, we have $\lambda \in E_a^0(T)$.

Opposite direction, let $\lambda \in E_a^0(T) = \{\lambda \in iso\,\sigma_a(T): 0 < \alpha(T-\lambda) < \infty\}$, and let $E_\lambda X$ be the Riesz idempotent for λ. We can represent T as a direct sum

$$T = \begin{pmatrix} T|E_\lambda X & 0 \\ 0 & T|(I-E_\lambda)X \end{pmatrix},$$

where $\sigma(T|E_\lambda X) = \{\lambda\}$ and $\sigma(T|(I-E_\lambda)X) = \sigma(T)\backslash\{\lambda\}$. Because T is algebraically paranormal, then $p(T)$ is paranormal for some non-constant polynomial p. Since $\sigma(T|E_\lambda X) = \{\lambda\}$, so we should have $\sigma\big(p(T|E_\lambda X)\big) = p\big(\sigma(T|E_\lambda X)\big) = \{p(\lambda)\}$. Thus, $p(T|E_\lambda X) - p(\lambda)$ is quasi-nilpotent, since $p(T|E_\lambda X)$ is paranormal and then form Lemma 2.2 that $p(T|E_\lambda X) - p(\lambda) = 0$. Put $q(z) \coloneqq p(z) - p(\lambda)$, hence $q(T|E_\lambda X) = 0$, and then $T|E_\lambda X$ is algebraically paranormal. Since $T|E_\lambda X - \lambda$ is quasi-nilpotent and algebraically paranormal, and from [6, Lemma 2.2] $T|E_\lambda X - \lambda$ is nilpotent. Then

$$\dim E_\lambda X \leq \dim N((T|E_\lambda X - \lambda)^m)$$

$$\leq \dim N((T-\lambda)^m)$$

$\leq m \dim N(T-\lambda) < \infty$.

Thus, E_λ is a finite rank, from properties XI 6.9 in [10], we have $T-\lambda \in SF$ and $ind\ (T-\lambda) = 0$, that is $\alpha(T) = \beta(T)$. Since T is algebraically paranormal then T is polaroid [9, theorem 1.3] and so $0 < \mathcal{p}(T-\lambda) = \mathcal{q}(T-\lambda) < \infty$. Then, $\lambda \notin \sigma_b(T)$. From corollary (6.13) in [10], we have $\lambda \in \sigma_a(T)$.

Recall that $H(\sigma(\mathcal{S}))$ is the set of analytic functions on the neighborhood $\mho$ of the spectrum of $\mathcal{S}$, where f is non-constant on every component of its domain.

Theorem 2.4

Let $T \in B(X)$ be a polaroid and $T-\lambda$ has a finite ascent for all $\lambda \in \mathbb{C}$. Then $f(T)$ has (am) property, where $f(z)$ is an analytic function on some open neighborhood of $\sigma(T)$.

Proof

Since $T-\lambda$ has a finite ascent for all $\lambda \in \mathbb{C}$, then T has $\mathcal{SVEP}$, [13, proposition 1.8]. According to theorem 2.40 of [3], we have $f(T)$ has $\mathcal{SVEP}$, and hence $\sigma\big(f(T)\big) = \sigma_a\big(f(T)\big)$. Since T polaroid, then $f(T)$ polaroid [9, theorem 2.4]. Finally, it has followed from theorem 3.1 in [4] , the $f(T)$ satisfied property (am).

Theorem 2.5

Assume that T is an algebraically paranormal. Then $f(T)$ satisfies (am) property, where $f(z)$ is an analytic function on some open neighborhood of $\sigma(T)$.

Proof

Since T is algebraically paranormal, then T is a hereditarily polaroid [9, corollary 2.6], and consequently, T is a polaroid. According to [11, corollary 2.10], T has $\mathcal{SVEP}$. Therefore, $\sigma(T) = \sigma_a(T)$. Then from theorem 3.1 in [4], T satisfies (am) property. It follows from [3, theorem 2.40] and [9, theorem 2.3], $f(T)$ is a polaroid and has $\mathcal{SVEP}$. Thus, $\sigma(f(T)) = \sigma_a(f(T))$, then $f(T)$ satisfies (am) property.

The next theorem proves that the property (gam) holds for algebraically paranormal operators, we recall the definition of property (gam).

If T has a finite ascent and descent, then T is called Drazin invertible, the Drazin spectrum is given by,[14]

$\sigma_D(T) = \{\lambda \in \mathbb{C}: T-\lambda \textit{ is not a Drazin invertible}\}$.

T is called left Drazin invertible (in symbol $LD(X)$),if $LD(X) = \{T \in B(X): \mathcal{p}(T) < \infty \textit{ and } \mathcal{R}\big(T^{\mathcal{p}(T)+1}\big) \textit{ is closed}\}$, and left Drazin invertible spectrum is defined by

$\sigma_{LD}(T) = \{\lambda \in \mathbb{C}: T-\lambda \notin LD(X)\}$, [6].

In [4], T is said to satisfy property (gam) if $\sigma_a(T)\backslash\sigma_D(T) = E_a(T)$, where $E_a(T) = \{\lambda \in iso\ \sigma_a(T): 0 < \alpha(T-\lambda)\}$ is the set of all eigenvalues of T which are isolated in approximate point spectrum.

Theorem 2.6

Let $T \in B(X)$ be algebraically paranormal. Then T satisfies (gam) property.

Proof

Let $\lambda \in \sigma_a(T)\backslash\sigma_D(T)$, that is T has a finite ascent and finite descent. Since T is algebraically paranormal, then T has $\mathcal{SVEP}$, it follows from theorem 3.81 of [3], λ is an isolated point of the spectrum of T. Let E_λ the Riesz idempotent for λ. Hence, we can represent T as a direct sum

$$T = T|E_\lambda X \oplus T|(I-E_\lambda)X$$

and $\sigma(T|E_\lambda X) = \{\lambda\}$, $\sigma(T|(I-E_\lambda)X) = \sigma(T)\{\lambda\}$. Hence, from theorem 2.3 $T|E_\lambda X$ is algebraically paranormal, it follows that lemma 2.2 in [6], $T|E_\lambda X$ is a nilpotent (i.e.$(T|E_\lambda X-\lambda)^m = 0$). Then, $E_\lambda X \subset N(T|E_\lambda X-\lambda)^m \subset N(T-\lambda)^m$. Thus, $N(T-\lambda)^m \neq \{0\}$ and $N(T-\lambda)^m \neq \{0\}$, then $\lambda \in E$, and since $E \subset E_a$, we have $\lambda \in E_a$.

Conversely, since T is algebraically paranormal, this implies that every isolated point of $\sigma(T)$ is a pole of the resolvent of T, and by Remark 3.7 in [3], we have $\lambda \in \sigma(T)$. Since T has $\mathcal{SVEP}$, then $\lambda \in \sigma_a(T)$. T has finite ascent and finite descent because λ pole of the resolvent of T, consequently, $\lambda \notin \sigma_D(T)$.

Theorem 2.7

Let $T \in B(X)$ be polaroid, and $T - \lambda$ has a finite ascent for all $\lambda \in \mathbb{C}$. Then $f(T)$ has (gam) property, where $f(z)$ is an analytic function on some open neighborhood of $\sigma(T)$.

Proof

Since $-\lambda < \infty, \lambda \in \mathbb{C}$, then from proposition 1.8 in [13], T has $\mathcal{SVEP}$. From theorem 2.40 of [3], we obtain $f(T)$ has $\mathcal{SVEP}$, and then $\sigma\big(f(T)\big) = \sigma_a\big(f(T)\big)$. Since T is polaroid, then $f(T)$ is polaroid [9, theorem 2.4]. It follows from theorem 3.2 in [4] that roperty (gam) holds for $f(T)$.

Theorem 2.8

Suppose that T is algebraically paranormal. Then, $f(T)$ satisfies (gam) property, where $f(z)$ is an analytic function on some open neighborhood of $\sigma(T)$.

Proof

Since T is algebraically paranormal, then T is a hereditarily polaroid [9, corollary 2.6], and T is a polaroid. T has $\mathcal{SVEP}$, [11, corollary 2.10] and so $\sigma(T) = \sigma_a(T)$. According to theorem 3.2 in [4], T satisfies (am) property. Hence, $f(T)$ is polaroid and has $\mathcal{SVEP}$, [3, theorem 2.40] and [9, theorem 2.3]. Then, $\sigma(f(T)) = \sigma_a(f(T))$, then $f(T)$ satisfies (gam) property.

Conclusions

In this work, the properties (am) and (gam) have been investigated and discussed for an algebraically paranormal operator acting on separable Hilbert space. We also show that $f(T)$ has also satisfied these properties for all $f \in H(\sigma(T))$. Furthermore, the property (am) holds for $f(T)$ for all $f \in H(\sigma(T))$ is proven for some types of operators, namely a polaroid operator with the finite ascent. Some other results have been given.

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