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Convergence of Iterative Algorithms in Cat(0) Spaces

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Abstract
In this article, results have been shown via using a general quasi contraction multi-valued mapping in Cat(0) space. These results are used to prove the convergence of two iteration algorithms to a fixed point and the equivalence of convergence. We also demonstrate an appropriate conditions to ensure that one is faster than others.

Keywords: Cat(0) spaces, fixed points, iterative sequences.

تقارب خوارزميات تكرارية في فضاءات *Cat(0)*

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الخلاصة
في هذا البحث ، تم اثبات النتائج عن طريق استخدام تطبيق متعدد القيم شبه انكماش معمم في فضاء cat(0). بينت هذه النتائج تقارب خوارزميتين للتكرار الى نقطة ثابتة وتكافؤ تقاربهما. كذلك تم شرح الشروط المناسبة للتأكد من ان احدها أسرع من الاخرى.

1. Introduction and Preliminaries

Axiomatically, in a cat (k) triangles are slimmer than corresponding triangles in a usual space of fixed curvature k. In a cat (0), the curvature is limited from above by k. A notable special case is k = 0. Complete spaces are known as Hadamard spaces. An example of these spaces is R^m with usual distance [1].

Let $(\sum, \omega)$ be a metric space and $u, v \in \sum$ with $\omega(u, v) = x$.

Definition 1.1: [2] A geodesic path from u to v (geodesic path joining u to v) is an isometry $c: [0, x] \rightarrow c([0, x]) \subset \sum$ such as $c(0) = u$ and $c(x) = v$.

The image of every geodesic path between u and v is named geodesic segment, which is denoted by $[u, v]$. Each point y in the segment is appeared by $\vartheta u \oplus (1-\vartheta)v$, where $\vartheta \in [0, 1]$ that is $[u, v] = \{\vartheta u \oplus (1-\vartheta)v : \vartheta \in [0, 1]\}$.

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Definition 1.2: [3] The space $(\sum, \omega)$ is called (i) a geodesic if each two elements of G are connected through a geodesic.(ii) a uniquely geodesic if there exists one geodesic jouning u and v $\forall u, v \in \sum$.
Definition1.3: [3] A subset H of $\sum$ is named convex if $\forall\, u, v \in \sum$, the geodesic segment $[u, v] \subset$ H
A geodesic triangle $\Delta(u_1, u_2, u_3)$ is a geodesic metric space $(\sum, \omega)$ consists of three points u_1, u_2, u_3 in $\sum$ (the vertices Δ) and a geodesic segment between every pair of vertices (the edges of Δ). A comparison triangle for geodesic triangle $\Delta(\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3)$ in $(\sum, \omega)$ is a $\bar{\Delta}(\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3) \coloneqq \Delta(\overline{\mathrm{u}_1}, \overline{\mathrm{u}_2}, \overline{\mathrm{u}_3})$ in R^2 such as $\omega_{\mathrm{R}^2}(\overline{\mathrm{u}_\mathrm{k}}, \overline{\mathrm{u}_\mathrm{i}}) = \omega(\mathrm{u}_\mathrm{k}, \mathrm{u}_\mathrm{i})$, for k, i.
Definition 1.4: [1] A geodesic space is named CAT(0) space if whole geodesic triangles accomplish the following comparison axiom.
Definition 1.5: [1]Let Δ be a geodesic triangle in $\sum$, $\bar{\Delta} \subset R^2$ be a correspondent triangle for Δ. Then Δ accomplishes the CAT(0) inequality if $\forall\, u, v \in \Delta$, $\forall\, \bar{u}, \bar{v} \in \bar{\Delta}$, $\omega(u, v) \leq \omega_{R^2}(\bar{u}, \bar{v})$.
Next lemma gives the definition of CN inequality that is found in [4].
Lemma 1.6: If u, v_1, v_2 are points in CAT(0) and $v_0 = \frac{1}{2}(v_1 \oplus v_2)$ then CAT(0) inequality leads to

$$\omega(u, v_0)^2 \leq \frac{1}{2}\omega(u, v_1)^2 + \frac{1}{2}\omega(u, v_2)^2 - \frac{1}{4}\omega(v_1, v_2)^2$$

In verity, a geodesic space is a CAT(0) space if and only if it accomplishes CN inequality.
The aim of this work is to prove some approximating results for below iterative schemes for multivalued mappings: Let $\sum$ be a Cat (0), $\emptyset \neq \mathcal{A} \subseteq \sum$ and $F: \mathcal{A} \to 2^{\Sigma}$ be a multi-valued mapping. For $x_0 \in \mathcal{A}$ if the sequence $\langle x_n \rangle \subset \mathcal{A}$with $\langle \partial_n \rangle, \langle \rho_n \rangle$ are sequences in (0, 1)

$$\begin{cases} x_0 \in \mathcal{A} \\ x_{n+1} = (1 - \partial_n)\mu_n \oplus \partial_n\, \xi_n \text{ for } n \geq 0 \\ y_n = (1 - \rho_n)x_n \oplus \rho_n \mu_n \end{cases} \quad \text{.... 1}$$

where $\mu_n \in Fx_n$, $\xi_n \in Fy_n$ [5]

$$\begin{cases} x_0 \in \mathcal{A} \\ x_{n+1} = (1 - \partial_n)y_n \oplus \partial_n\, \xi_n \text{ for } n \geq 0 \\ y_n = (1 - \rho_n)x_n \oplus \rho_n \mu_n \end{cases} \quad \text{... 2}$$

where $\mu_n \in Fx_n$, $\xi_n \in Fy_n$ [6].

In nonlinear analysis, one of the most important theorems is Banach's contraction principle see [7] which substantially, shows that any contraction mapping on a complete metric space $\sum$ that is
$F: \sum \to \sum,\ \omega(Fx, Fy) \leq a\omega(x, y)$,for all $x, y \in \sum, 0 \leq a < 1$ ... 3
It has a unique fixed point. In fact, any contraction on $\sum$ is continuous. A usual question is that there exists a contraction condition that does not imply the continuity of F throughout space $\sum$ or not ?. This issue was positively answered in 1968 by Kannan [8], who extended Banach's theorem to mappings that does not require to be continuous by using the next case instead of (3) there exists $b \in [0, 0.5)$ such that
$\omega(Fx, Fy) \leq b[\omega(x, Fx) + \omega(y, Fy)]$,for all $x, y \in \sum$ ... 4
After the Kannan's theorem established, many studies were devoted to obtain other fixed point results for various types of contractive conditions that do not require the continuity of F. In particular, the duality of Kannan's theorem was studied by Chatterjea's [9].
$c \in [0, 0.5)$ exists, $\omega(Fx, Fy) \leq c[\omega(x, Fy) + \omega(y, Fx)]$, $\forall x, y \in \sum$ ... 5
In [10], Rhoades showed that the conditions (3), (4) and (5) are independent ,while in [7], Zamfirescu's presented the generalization of the aforementioned conditions, which is called the z-operator ,that means an operator F is called the z-operator if it satisfies at least one

condition of (3), (4), or (5). Berinde [11]completed Kannan's theorem and Zamfirescu's theorem with error estimates of Picard iterations and the convergence rate. The z-operator leads to the following conclusions for all $x, y \in \sum$:
(i) $d(Fx, Fy) \leq \delta d(x, y) + 2\delta d(x, Fx)$ using condition (4) and
(ii) $d(Fx, Fy) \leq \delta d(x, y) + 2\delta d(x, Fy)$ using condition (5)
where $\delta = \max\{a, \frac{b}{1-b}, \frac{c}{1-c}\}$ and $\delta \in [0.1)$. Any mapping that satisfies condition (i) or (ii) is called a quasi-contraction mapping [12].
The following contractive condition has been mentioned in the [13], for single valued mappings in metric space case, we present the contractive condition for multivalued mappings in Cat (0) spaces. Let Ω (A,B) be Hausdorff between A, B $\in 2^{\Sigma}$, where 2^{Σ} is a collection of all nonempty subsets of $\sum$ and Ω(A,B) = max$\{sup_{a\in A}\, \omega$(a,B), $sup_{b\in B}\omega$(b,A)$\}$,
where ω(a,B) = $inf_{b\in B}\omega$(a,b).
Definition 1.7: A mapping $F: \sum \to 2^{\Sigma}$ is called general quasi contraction if there exist $q \in (0,1)$ and $\emptyset$ is continuous strictly increasing function $\emptyset: [0, \infty) \to [0, \infty)$ with $\emptyset(0) = 0$ such that

$$\Omega(Fx, Fy) \leq q\omega(x, y) + \emptyset(\omega(x, Fx)), \ \forall x, y \in \sum \qquad \dots 6$$

Remark 1.8: If F is satisfies condition (6) and $Fix_F \neq \emptyset$ then Fix_F is singleton. Suppose that $z, z^* \in Fix_F$ is two fixed point of F, we get
$\omega(z, z^*) \leq q\omega(z, z^*) + \emptyset(\omega(z, \zeta_n)) = q\omega(z, z^*)$, where $\zeta_n \in Fz$
that is $(1 - q)\omega(z, z^*) = 0$, e.i., $z = z^*$
We present the definition of an approximate mapping for multi-valued mappings:
Definition 1.9: We say that F^* is an approximate mapping of F where $F, F^*: \sum \to 2^{\Sigma}$ if, for some $\varepsilon > 0$ we have $\Omega(Fx, F^*x) \leq \varepsilon$ for all $x \in \sum$
The following lemmas are needed:
Lemma 1.10 [14]: let $\{a_n\}_{n=0}^{\infty}$be a non-negative real sequence and $\exists\, n_0 \in \mathbb{N} \ni$ for all $n \geq n_0$
$a_{(n+1)} \leq (1 - \alpha_n)a_n + \alpha_n\beta_n$, where $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\alpha_n \in (0,1)$ for all $n \in \mathbb{N}$
and $\beta_n \geq 0$, $\forall n \in \mathbb{N}$. then $0 \leq \lim_{n\to\infty} sup a_n \leq \lim_{n\to\infty} sup\beta_n$.
Lemma1.11: [15] Let $\{a_n\}_{n=0}^{\infty}$ be a non-negative real sequence and there exists $n_0 \in \mathbb{N}$ *such that* for all $n \geq n_0$. $\sigma_n = 0(\lambda_n)$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. This is satisfying the following inequality:

$$a_n \leq (1 - \lambda_n)a_n + \sigma_n, \qquad \text{then} \lim_{n\to\infty} a_n = 0.$$

Lemma 1.12: [16] [17] Let $\{\alpha_n\}$ be a sequence non-negative such that for all $n \in \mathbb{N}$, $\alpha_n \in (0,1]$. if $\sum_{n=1}^{\infty} \alpha_n = \infty$ then $\prod_{n=1}^{\infty}(1 - \alpha_n) = 0$.
Definition 1.13: [17] Let $\{a_n\}$ and $\{b_n\}$ be two real sequences which are convergent to the limits a and b, respectively if they satisfy the following
$\lim_{n\to\infty} \left|\frac{a_n - a}{b_n - b}\right| = 0$, then $\{a_n\}$ converges faster than$\{b_n\}$.
This work includes many new results about the approximate fixed point in the field of geodesic spaces which are always varied and renewable. It is appropriate to refer to other related results such as in [18-20].

2. Main Results

Let $(\sum, \omega)$ be CAT (0), $CB(\mathcal{A}) = \{C \subset \mathcal{A}: \emptyset \neq C$ is closed and bounded$\}$, and $F: \mathcal{A} \to CB(\mathcal{A})$ satisfies condition (6) with$Fix_F \neq \emptyset$,then we have the following results.
Theorem 2.1: The sequence $\langle x_n \rangle$ in (1) with$\sum_{n=1}^{\infty} \alpha_n = \infty$, converges to a unique fixed point of F.
Proof: The uniqueness comes from Remark (1.5). Use (1), (2) and used Lemma (1.12), we get

$$\omega(x_{(n+1)}, z) \leq (1 - \partial_n)\omega(\mu_n, z) + \partial_n\omega(\xi_n, z)$$

$\leq (1-\partial_n) q\omega(x_n, z) + (1-\partial_n)\emptyset(\omega(z, \zeta_n))$
$+\partial_n q\omega(y_n, z) + \partial_n \emptyset(\omega(z, \zeta_n))$ … 7
and, $\omega(y_n, z) \leq (1-\rho_n)\omega(x_n, z) + \rho_n \omega(\mu_n, z)$
$\leq (1-\rho_n)\omega(x_n, z) + \rho_n q\omega(x_n, z) + \rho_n \emptyset(\omega(z, \zeta_n))$ … 8
substitution of (8) in (7)
$\omega(x_{(n+1)}, z) \leq (1-\partial_n)\omega(\mu_n, z) + \partial_n \omega(\xi_n, z)$
$\leq (1-\partial_n) q\omega(x_n, z) + (1-\partial_n)\emptyset(\omega(z, \zeta_n))$
$+\partial_n q(1-\rho_n)\omega(x_n, z) + \partial_n \rho_n q^2 \omega(x_n, z) + \partial_n q\rho_n \emptyset(\omega(z, \zeta_n)) + \partial_n \emptyset(\omega(z, \zeta_n))$
$\leq [(1-\partial_n)q + \partial_n q(1-\rho_n) + \partial_n q^2 \rho_n]\omega(x_n, z)$
$+[(1-\partial_n) + \partial_n q\rho_n]\emptyset(\omega(z, \zeta_n)) + \partial_n \emptyset(\omega(z, \zeta_n))$
$\leq [1-(1-q)\partial_n \rho_n] q\omega(x_n, z)$
$\leq \prod_{k=1}^{n}[1-(1-q)\partial_k] q\omega(x_0, z)$
$\leq [\prod_{k=1}^{n}(1-q\partial_k)] q^n \omega(x_0, z)]$
Then proof is complete and $x_n \to z, as\ n \to \infty$.
Theorem 2.2: Let $z \in Fix_F$. Then $\langle x_n \rangle$ in (1) converges to z if and only if $\langle u_n \rangle$ in 2 converges to z, where $0 < \alpha_n < A < 1$, for all $n \in \mathbb{N}$.
Proof: Suppose that $\langle x_n \rangle$ defined by (1) converges to z. To prove $\langle u_n \rangle$, which is defined by (2) converges to z. We have the following estimates:
$\omega(x_{(n+1)}, u_{(n+1)}) \leq (1-\partial_n)\omega(\mu_n, v_n) + \partial_n \omega(\xi_n, \gamma_n)$, where$\omega_n \in Fv_n$
$\leq (1-\partial_n)\omega(\mu_n, v_n) + \partial_n q\omega(y_n, v_n) + \partial_n \emptyset(\omega(y_n, \xi_n))$ … 9
$\omega(\mu_n, v_n) \leq (1-\rho_n)\omega(\mu_n, v_n) + \rho_n \omega(\mu_n, \theta_n)$, where $\theta_n \in Fu_n$
$\leq (1-\rho_n)[\omega(\mu_n, x_n) + \omega(x_n, u_n)] + \rho_n q\omega(x_n, u_n) + \rho_n \emptyset(\omega(x_n, \mu_n))$ … 10
$\omega(y_n, v_n) \leq (1-\rho_n)\omega(x_n, u_n) + \rho_n \omega(\mu_n, \theta_n)$
$\leq (1-\rho_n)\omega(x_n, u_n) + \rho_n q\omega(x_n, u_n) + \rho_n \emptyset(\omega(x_n, \mu_n))$ . .. 11
By combining (9), (10), and (11) we obtain
$\omega(x_{(n+1)}, u_{(n+1)}) \leq [(1-\rho_n(1-q))(1-\partial_n(1-q))]\omega(x_n, u_n)$
$+(1-\partial_n)(1-\rho_n)\omega(x_n, \mu_n)$
$+[\rho_n(1-\partial_n(1-q))]\emptyset(\omega(x_n, \mu_n))$
$+\partial_n \emptyset(\omega(y_n, \xi_n))$ … 12
since ∂_n, ρ_n, $q \in (0,1)$ for all n $\in \mathbb{N}$,
$1-\partial_n(1-q) < 1-\partial_n, 1-\rho_n(1-q) < 1$ … 13
using (13) and the assumption $\partial_n \geq A > 0$, for all n$\in \mathbb{N}$ in (12) then
$\omega(x_{(n+1)}, u_{(n+1)}) \leq [1-A(1-q)]\omega(x_n, u_n) + (1-\partial_n)(1-\rho_n)\omega(x_n, \mu_n)$
$+[1-A(1-q)]\emptyset(\omega(x_n, \mu_n)) + \partial_n \emptyset(\omega(y_n, \xi_n))$ … 14
Define $a_n \coloneqq \omega(x_n, u_n), \lambda_n \coloneqq A(1-q) \in (0,1)$, and
$\sigma_n \coloneqq [1-A(1-q)]\emptyset(\omega(x_n, \mu_n)) + \partial_n \emptyset(\omega(y_n, \xi_n)) + \partial_n \emptyset(\omega(y_n, \xi_n))$
$+ (1-\partial_n)(1-\rho_n)\omega(x_n, \mu_n)$.
Since $\lim_{n\to\infty} \omega(x_n, z) = 0$ and $\zeta_n = z \in Fix_F$, it follow from (6) that
$0 \leq \omega(x_n, \mu_n)$
$\leq \omega(x_n, z) + \omega(\zeta_n, \mu_n)$
$\leq \omega(x_n, z) + q\omega(z, x_n) + \emptyset(\omega(z, \zeta_n))$
$= (1+q)\omega(x_n, z) \to 0\ as\ n \to \infty$.
Then $\lim_{n\to\infty} \omega(x_n, \mu_n) = 0$, Now using the same argument to get
$0 \leq \omega(y_n, \xi_n) \leq \omega(y_n, z) + \omega(\zeta_n, \xi_n)$
$\leq \omega(y_n, z) + q\omega(z, y_n) + \emptyset(\omega(z, \zeta_n))$
$= (1+q)\omega(y_n, z)$
$\leq (1+q)(1-\rho_n)\omega(x_n, z) + (1+q)\rho_n \omega(\mu_n, \zeta_n)$

$$= (1+q)\omega(x_n, z) \to 0 \ as \ n \to \infty,$$

that is, $\lim_{n\to\infty} \omega(y_n, \xi_n) = 0$, namely $\sigma_n = o(\lambda_n)$. Hence, by Lemma (1.11) and (14) lead to $\lim_{n\to\infty} \omega(x_n, u_n) = 0$. Since $x_n \to z \ as \ n \to \infty$. Therefore

$$\omega(u_n, z) \le \omega(u_n, x_n) + \omega(x_n, z)$$

and this implies that $\lim_{n\to\infty} u_n = z$.

Now, if $\langle u_n \rangle$ converges to z, we will prove that $\langle x_n \rangle$ converges to z.

Using (6), (1) and (2) we have

$$\omega(x_{(n+1)}, u_{(n+1)}) \le (1-\partial_n)\omega(v_n, \mu_n) + \partial_n \omega(\gamma_n, \xi_n)$$
$$\le (1-\partial_n)\omega(v_n, \mu_n) + \partial_n q\omega(v_n, y_n) + \ \partial_n \emptyset\big(\omega(v_n, \gamma_n)\big) \quad \dots 15$$

$$\omega(v_n, \mu_n) \le (1-\rho_n)\omega(u_n, \mu_n) + \rho_n \omega(\theta_n, \mu_n)$$
$$\le (1-\rho_n)\omega(u_n, \theta_n) + (1-\rho_n)\omega(\theta_n, \mu_n)$$
$$+\rho_n q\omega(u_n, x_n) + \ \rho_n \emptyset\big(\omega(u_n, \theta_n)\big)$$
$$\le (1-\rho_n)\omega(u_n, \theta_n) + q\omega(u_n, x_n) + \ \emptyset\big(\omega(u_n, \theta_n)\big) \quad \dots 16$$

$$\omega(v_n, y_n) \le (1-\rho_n)\omega(u_n, x_n) + \rho_n \omega(\theta_n, \mu_n)$$
$$\le (1-\rho_n)\omega(u_n, x_n) + \rho_n q\omega(u_n, x_n) + \ \rho_n \emptyset\big(\omega(u_n, \theta_n)\big)$$
$$\le [1-\rho_n(1-q)]\omega(u_n, x_n) + \ \rho_n \emptyset\big(\omega(u_n, \theta_n)\big) \quad \dots 17$$

by combining (15), (16), and (17) we obtain the following

$$\omega(x_{(n+1)}, u_{(n+1)}) \le \{(1-\partial_n)q + \partial_n q[1-\rho_n(1-q)]\}\omega(u_n, x_n)$$
$$+(1-\partial_n + \partial_n \rho_n q)\omega(u_n, \theta_n)$$
$$+ [1-\partial_n(1-\rho_n q)]\emptyset(\omega(u_n, \theta_n))$$
$$+ \partial_n \emptyset(\omega(v_n, \gamma_n)) \quad \dots 18$$

since q, $\partial_n, \rho_n \in (0,1)$ for all n$\in \mathbb{N}$,

$$(1-\partial_n)q < 1-\partial_n, 1-\rho_n(1-q) < 1 \quad \dots 19$$

using (19) and the assumption $\partial_n \ge A > 0$, for all n$\in \mathbb{N}$ in (18) , it follows that

$$\omega(x_{(n+1)}, u_{(n+1)}) \le [1-A(1-q)]\omega(u_n, x_n)$$
$$+[1-A(1-q)]\emptyset\big(\omega(u_n, \theta_n)\big)$$
$$+[1-\partial_n(1-\rho_n)]\omega(u_n, \theta_n) + \partial_n \emptyset(\omega(v_n, \gamma_n)). \quad \dots 20$$

Define

$a_n \coloneqq \|u_n - x_n\|$,

$\lambda_n \coloneqq A(1-q) \in (0,1)$,

$\sigma_n \coloneqq [1-A(1-q)]\emptyset\big(\omega(u_n, \theta_n)\big) + [1-\partial_n(1-\rho_n)]\omega(u_n, \theta_n) + \partial_n \emptyset(\omega(v_n, \gamma_n))$

Since $\lim_{n\to\infty}\|u_n - z\| = 0$ and $\zeta_n = z \in Fix_F$, it follow from (20) that

$$0 \le \ \omega(u_n, \theta_n)$$
$$\le \omega(u_n, z) + \omega(\zeta_n, \theta_n)$$
$$\le \omega(u_n, z) + q\omega(z, u_n) \oplus \emptyset(\omega(z, \zeta_n))$$
$$= (1+q)\omega(u_n, z) \to 0 \ as \ n \to \infty.$$

Then $\lim_{n\to\infty} \omega(u_n, \theta_n) = 0$. Similarly, we can get

$$0 \le \ \omega(v_n, \gamma_n) \le \ \omega(v_n, z) + \ \omega(\zeta_n, \gamma_n)$$
$$\le \omega(v_n, z) + q\omega(z, v_n) \oplus \emptyset(\omega(z, \zeta_n))$$
$$= (1+q)\omega(v_n, z)$$
$$\le (1+q)(1-\rho_n)\omega(u_n, z) + (1+q)\rho_n \omega(\theta_n, \zeta_n)$$
$$= (1+q)\omega(u_n, z) \to 0 \ as \ n \to \infty,$$

This gives, $\lim_{n\to\infty} \omega(v_n, \gamma_n) = 0$, namely $\sigma_n = o(\lambda_n)$. Hence an application Lemma (1.11) to (14) getting $\lim_{n\to\infty} \omega(u_n, x_n) = 0$. since$u_n \to z \ as \ n \to \infty$ by assumption, driving

$$\omega(x_n, z) \le \omega(x_n, u_n) + \omega(u_n, z)$$

which implies that $\lim_{n\to\infty} x_n = z$.

Theorem 2.3: If the sequences $\langle x_n\rangle$ and $\langle u_n\rangle$, that defined in (1) and (2), respectively converge to z. Then $\langle x_n\rangle$ converges to z faster than $\langle u_n\rangle$.

Proof: By Theorem (2.1) we have

$$\omega\big(x_{(n+1)}, z\big) \le \prod_{k=1}^{n}[1-(1-q)\partial_k]q\omega(x_0,z)$$

by using the same technique of proof of Theorem (2.1) with $\langle u_n\rangle$ then we have

$$\begin{aligned}\omega\big(u_{(n+1)}, z\big) &\le (1-\partial_n)\omega(v_n,z) + \partial_n\omega(\gamma_n,z)\text{ , where } \gamma_n \in Fv_n\\ &\le (1-\partial_n)[1-\rho_n(1-q)]\omega(u_n,z)\\ &\quad +\partial_n q[1-\rho_n(1-q)]\omega(u_n,z)\\ &\le [1-\rho_n(1-q)][\,1-\partial_n(1-q)]\,\omega(u_n,z)\end{aligned}$$

since ∂_n, ρ_n, $q \in (0,1)$for all n$\in \mathbb{N}$,

$1-\rho_n(1-q) < 1-\partial_n(1-q)$, then

$$\begin{aligned}\omega\big(u_{(n+1)}, z\big) &\le [\,1-\partial_n(1-q)]\,\omega(u_n,z)\\ &\le \prod_{k=1}^{n}[\,1-\partial_k(1-q)]\,\omega(u_0,z)\end{aligned}$$

$$\lim_{n\to\infty}\frac{\omega\big(x_{(n+1)},z\big)}{\omega\big(u_{(n+1)},z\big)} = \lim_{n\to\infty}\frac{\prod_{k=1}^{n}[1-(1-q)\partial_k]q\omega(x_0,z)}{\prod_{k=1}^{n}[\,1-\partial_k(1-q)]\,\omega(u_0,z)}$$

$$\lim_{n\to\infty} q^n\prod_{k=1}^{n}\frac{[1-(1-q)\partial_k]\omega(x_0,z)}{[\,1-\partial_k(1-q)]\,\omega(u_0,z)}$$

since $0<q<1$, then, $\lim_{n\to\infty} q^n = 0$.

Finally, $\lim_{n\to\infty}\frac{\omega(x_{(n+1)},z)}{\omega(u_{(n+1)},z)} = 0$.

Therefore from definition (1.6), we conclude that the convergence of $\langle x_n\rangle$ is faster than $\langle u_n\rangle$.

Theorem 2.4: Let F^* be an approximate mapping of a general quasi contraction mapping $F\colon\mathcal{A}\to CB(\mathcal{A})$ with $Fix_F \ne \emptyset, Fix_{F^*}\ne\emptyset$. let $\langle x_n\rangle$ and $\langle u_n\rangle$ be as in (10) with $\langle\partial_n\rangle,\langle\rho_n\rangle \in [0,1)$ satisfying (1)$\frac{1}{2}\le\partial_n, \forall n\in\mathbb{N}$ (2) $\sum_{n=0}^{\infty}\partial_n = \infty$,then $\omega(z,w)\le\frac{3\varepsilon}{1-q}$, where $\varepsilon>0, q\in(0,1)z\in \mathrm{Fix_F}$and $\mathrm{w}\in\mathrm{Fix_{F^*}}, \langle \mathrm{u_n}\rangle$, .

Proof: Define of F^* by

$u_0 \in M, u_{(n+1)} = (1-\partial_n){\mu_n}^* + \partial_n{\xi_n}^*,$ ... (21)

$v_n = (1-\rho_n)u_n + \rho_n{\mu_n}^*$

where${\mu_n}^*\in Fu_n, {\xi_n}^*\in Fv_n$

using (6), (2) and (21), we obtain the following where $\tau_n \in Fu_n, \vartheta_n\in Fv_n$

$$\begin{aligned}&\omega\big(x_{(n+1)},u_{(n+1)}\big) \le (1-\partial_n)\omega(\mu_n,{\mu_n}^*) + \omega(\xi_n,{\xi_n}^*)\\ &= (1-\partial_n)[\omega(\mu_n,\,\tau_n)+\omega(\,\tau_n,{\mu_n}^*)] + \partial_n[\omega(\xi_n,\vartheta_n)+\omega(\vartheta_n,{\xi_n}^*)]\\ &\le (1-\partial_n)\big\{q\omega(x_n,u_n)\oplus\emptyset\big(\omega(x_n,\mu_n)\big)+\varepsilon\big\}\\ &\quad +\partial_n\{q\omega(y_n,v_n)\oplus\emptyset(\omega(y_n,\xi_n))+\varepsilon\}\end{aligned} \quad \dots 22$$

$$\begin{aligned}&\omega(y_n,v_n)\le(1-\rho_n)\omega(x_n,u_n)+\rho_n\omega(\mu_n,{\mu_n}^*)\\ &= (1-\rho_n)\omega(x_n,u_n)+\rho_n[\omega(\mu_n,\,\tau_n)+\omega(\,\tau_n,{\mu_n}^*)]\\ &\le(1-\rho_n)\omega(x_n,u_n)+\rho_n\big\{q\omega(x_n,u_n)+\ \emptyset\big(\omega(x_n,\mu_n)\big)+\varepsilon\big\}\\ &= [1-\rho_n(1-q)]\,\omega(x_n,u_n)+\rho_n\emptyset\big(\omega(x_n,\mu_n)\big)+\rho_n\varepsilon\end{aligned}\quad \dots 23$$

combining (22) and (23), we get

$$\begin{aligned}\omega\big(x_{(n+1)},u_{(n+1)}\big) &\le \{(1-\partial_n)q+\partial_n q[1-\rho_n(1-q)]\}\omega(x_n,u_n)\\ &\quad +\{1-\partial_n+\partial_n q\rho_n\}\emptyset\big(\omega(x_n,\mu_n)\big)+\partial_n\emptyset(\omega(y_n,\xi_n))\\ &\quad +\partial_n q\rho_n\varepsilon+(1-\partial_n)\varepsilon+\partial_n\varepsilon\end{aligned}\quad \dots 24$$

For $\{\partial_n\}_{n=0}^{\infty}, \{\rho_n\}_{n=0}^{\infty} \subset [0,1)$ and $q \in [0,1)$

$(1-\partial_n)q < 1-\partial_n, \quad 1-\rho_n(1-q) < 1, \quad \partial_n q \rho_n < \partial_n$ … 25

it follows from (1) that $(1-\partial_n) < \partial_n, \quad for\ all\ n \in \mathbb{N}$

Therefore, combine (25) and (24) to (23), and we get

$$\omega(x_{(n+1)}, u_{(n+1)}) \leq [1-\partial_n(1-q)]\omega(x_n, u_n) + 2\partial_n \emptyset(\omega(x_n, \mu_n)) + \partial_n \emptyset(\omega(y_n, \xi_n)) + \partial_n \varepsilon + \partial_n \varepsilon + \partial_n \varepsilon$$

this is equivalent to

$$\omega(x_{(n+1)}, u_{(n+1)}) \leq [1-\partial_n(1-q)]\omega(x_n, u_n) + \partial_n(1-q)\frac{\{2\emptyset(\omega(x_n, \mu_n) + \emptyset(\omega(y_n, \xi_n)) + 3\varepsilon\}}{1-q}.$$

Now define $a_n = \omega(x_n, u_n)$

$$\lambda_n = \alpha_n(1-q) \in (0,1)$$

$\sigma_n = \frac{\{2\emptyset(\omega(x_n, u_n) \oplus \emptyset(\omega(y_n, \xi_n)) + 3\varepsilon\}}{1-q}$ … 26

From Theorem (2.1), we have $\lim_{n\to\infty} \omega(x_n, z) = 0$, because of $Fz = z \in Fix_F$, and F satisfies condition (6), We can use the same argument that applied to the proof of Theorem (2.2).

Then $\lim_{n\to\infty} \omega(x_n, \mu_n) = \lim_{n\to\infty} \omega(y_n, \xi_n) = 0$. Since $\emptyset$ is continuous function , then

$$\lim_{n\to\infty} \emptyset(\omega(x_n, \mu_n)) = \lim_{n\to\infty} \emptyset(\omega(y_n, \xi_n)) = 0.$$

By applying lemma (1.11) and (26), we get $\omega(z, w) \leq \frac{3\varepsilon}{1-q}$.

Open problem: We suggest a similar study with the application of the results contained in [21] or [22] .

References

[1] W. Kirk and N. Shahzad. " *Fixed point theory in distance spaces",Springer*, 2014.

[2] H. Fukhar-Ud-Din, A. Domlo, and A. Khan.“Strong convergence of an implicit algorithm in CAT (0) spaces”, *Fixed Point Theory Appl.*, vol. 2011, pp. 1–11, 2011.

[3] A. Abkar and M. ESLAMIAN. “Geodesic metric spaces and generalized nonexpansive multivalued mappings”. 2013.

[4] F. Bruhat and J. Tits. “Groupes réductifs sur un corps local”, *Publ. Mathématiques l'Institut des Hautes Études Sci.*, vol. 41, no. 1, pp. 5–251, 1972.

[5] R. P. Agarwal, D. O Regan, and D. R. Sahu. “Iterative construction of fixed points of nearly asymptotically nonexpansive mappings”. *J. Nonlinear Convex Anal.*, vol. 8, no. 1, p. 61, 2007.

[6] V. Karakaya, N. E. H. Bouzara, K. Dogan, and Y. Atalan. “On different results for a new two-step iteration method under weak-contraction mapping in Banach spaces”, *arXiv Prepr. arXiv1507.00200*, pp. 10, 2015.

[7] T. Zamfirescu. “Fix point theorems in metric spaces” , *Arch. der Math.*, vol. 23, no. 1, pp. 292–298, 1972.

[8] R. Kannan. “Some results on fixed points”, *Bull. Cal. Math. Soc.*, vol. 60, pp. 71–76, 1968.

[9] S. K. Chatterjea. “Fixed-point theorems”, *Dokl. na Bolg. Akad. na Nauk.*, vol. 25, no. 6, pp. 727-230, 1972.

[10] B. E. Rhoades. “A comparison of various definitions of contractive mappings”, *Trans. Am. Math. Soc.*, vol. 226, pp. 257–290, 1977.

[11] V. Berinde. “Error estimates for approximating fixed points of quasi contractions,”, *Gen. Math.*, vol. 13, no. 2, pp. 23–34, 2005.

[12] V. Berinde.“On the convergence of the Ishikawa iteration in the class of quasi contractive operators,”, *Acta Math. Univ. Comenianae. New Ser.*, vol. 73, no. 1, pp. 119–126, 2004.

[13] C. O. Imoru and M. O. Olatinwo.“On the stability of Picard and Mann iteration processes”, *Carpathian J. Math.*, pp. 155–160, 2003.

[14] Ş. M. Şoltuz and T. Grosan. “Data dependence for Ishikawa iteration when dealing with contractive-like operators", *Fixed Point Theory Appl.*, vol. 2008, pp. 1–7, 2008.

[15] S. S. Abd and S. A. Khadum. "The Equivalence Convergence of Iterative Sequences for Multi-Valued Mappings" , vol. 22, no. 6, 2011.

[16] S. H. Khan. "A Picard-Mann hybrid iterative process" , *Fixed Point Theory Appl.*, vol. 2013, no. 1, pp. 1–10, 2013.

[17] S. M. Soltuz. "Data dependence for Ishikawa iteration", *Lect. Matemáticas*, vol. 25, no. 2, pp. 149–155, 2004.

[18] P. Chuadchawna, A. Farajzadeh and A. Kaewcharoen. "Convergence theorems for total asymptotically nonexpansive single-valued and quasi nonexpansive multi-valued mappings in hyperbolic spaces", accepted in *Journal of Applied Analysis*, doi:10.1515/jaa-2020-2038.

[19] P. Chuadchawna, A. Farajzadeh and A. Kaewcharoen. " Fixed-point approximations of generalized nonexpansive mappings via generalized M-iteration process in hyperbolic spaces", *Inter. J. of Math. and Math. Scie*, Vol. 2020, Article ID 6435043, 2020.

[20] P. Chuadchawna, A. Farajzadeh and A. Kaewcharoen. "On convergence theorems for two generalized nonexpansive multivalued mappings in hyperbolic spaces", *Thai J. of Math.*, vol. 17, no.2, 445-461,2019.

[21] S. S. Al-Bundi. "Iterated function system in $\emptyset-$metric space", accepted in *Bol. Soc. Paran. Math.* doi:10.5269/bspm.52556.

[22] S. H. Malih, S. S. Abed, "Approximating random fixed points under a new iterative sequence," *J. Interdiscip. Math.*, vol. 22, no. 8, pp. 1407–1414, 2019.