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Idempotent Divisor Graph of Commutative Ring

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Abstract
This work aims to introduce and to study a new kind of divisor graph which is called idempotent divisor graph, and it is denoted by $Л(R)$. Two non-zero distinct vertices v_1 and v_2 are adjacent if and only if $v_1.v_2 = e$, for some non-unit idempotent element $e^2 = e \in R$. We establish some fundamental properties of $Л(R)$, as well as it's connection with $\Gamma(R)$. We also study planarity of this graph.

Keywords: Idempotent Elements, Zero Divisor Graph, Idempotent Divisor Graph, Planar Graph.

بيانات قواسم العناصر المتحايدة للحلقات الابدالية

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الخلاصة
في هذا البحث تم تقديم تعريف جديد لبيان قواسم الصفر هو بيان قواسم العناصر المتحايدة $Л(R)$. ان كل راسين مختلفين وغير صفريين متجاوران اذا وفقط اذا $v_1.v_2 = e$, حيث e عنصر متحايد لا يساوي 1. كذلك وجدنا بعض الخواص الأساسية لهذا البيان وعلاقته مع بيان قواسم الصفر $\Gamma(R)$.

1. Introduction

Let R be a finite commutative ring with unity $1 \neq 0$. We denote $Z(R), I(R)$, and $U(R)$ the set of zero divisors, the set of idempotent elements and the set of unit elements respectively.
In [1], Beck introduced the idea that connects between ring theory and graph theory when studied the coloring of commutative ring. Later in [2], Anderson and Livingston modified this idea when studied the zero divisor graph $\Gamma(R)$ that have vertices $Z(R)^* = Z(R) - \{0\}$ and for $v_1, v_2 \in Z(R)^*$, v_1v_2 edges if and only if $v_1.v_2 = 0$. Many authors studied this notion see for examples [3], [4], [5] and [6]. Recently, there are other concepts of zero divisor graph, see for examples [7], [8], [9],and [10].
In graph theory " (v) denotes by the eccentricity of a vertex v of a connected graph G which is the number $max_{u \in V(G)}\, d(u,v)$. That means $e(v)$ is the distance between v and a vertex furthest from v. The radius of G ,which is denoted by $radG$, is $max_{u \in V(G)}\, d(u,v)$, while the diameter of G is the maximum eccentricity and it is denoted by $diamG$. Consequently, $diamG$ is the greatest distance between any two vertices of G. Also, a graph G has radius 1 if and only if G contains a vertex u adjacent to all other vertices of G. A vertex v is a central

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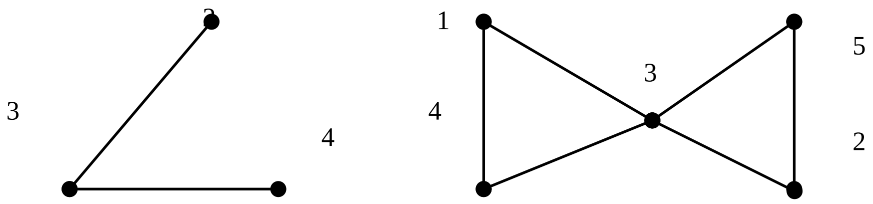
vertex if $e(v) = radG$ and the center $Cent(G)$ is the sub-graph of G that induced by its central vertices. The girth of a graph G is the length of a shortest cycle contained in G, it is denoted by $g(G)$. The neighborhood of x in a graph G denotes by $N_{G(x)}$, is the set of all $y \in V(G)$ such that y is adjacent to x in G. In our graph in this case, $N_{G(x)} = \{y \in V(G) \backslash \{x\} \mid xy = 0\}$. K_n $K_{n,m}$ symbolized complete graph and complete bipartite graph respectively. $K_{1,m}$ we call star graph. A clique number of G symbolized $\omega(G)$ is greats complete sub-graph of G. If a connected graph does not contain cycle, we call tree. Let H and G two graphs, $G \cup H$ is a graph with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$, and for $n \in Z^*$, $nH = \bigcup_{i=1}^{n} H$. the graph $G + H$ is a graph with $V(G \cup H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G), v \in V(H)\}$. A path graph of order n is denoted by P_n is a graph with $V(P_n) = \{v_i : i = 1,2, \dots, n\}$ and $E(P_n) = \{\{v_j, v_j + 1\} : j = 1,2, \dots, n-1\}$, so that C_n is a graph $P_n + \{v_1, v_n\}$ and it called a cycle graph of order n for $n \in Z^+$. For more details see for example" [11].
In ring theory, a ring R is said to be local if has exactly one maximal ideal. Also, if R finite local ring, then the cardinality of R symbolized $|R|$ equal p^t, where p prime number and $t \in Z^+$, as well as the cardinality of maximal ideal $M = p^r$, where $0 < r < t$. A ring R is called Boolean, if every element is an idempotent. We denote F_q is a field order q. In section two we defined a new graph on the ring and prove some basic properties of about this graph and we give all possible graphs less than or equal 6 vertices. In section three, we give all graphs to be planer.

2. Examples and Basic Properties

In this section, we introduce a new class of divisor graph manly idempotent divisor graph, we give some of about this graph, and we also provide some examples.

Definition 2.1: The undirected graph is called idempotent divisor graph, and which is symbolized by $Л(R)$ which a simple graph with vertices set in $R^* = R - \{0\}$, and two non-zero distinct vertices v_1 and v_2 are adjacent if and only if $v_1 v_2 = e$, for some non-unit idempotent element $e \in R$ $(i.e\ e^2 = e \neq 1)$.**Example 1:** Let $R = Z_6$, since the idempotent elements $I(R) = \{0,1,3,4\}$, then $Л(R)$ is:



$\Gamma(Z_6)$ $Л(Z_6)$

Figure 2.1

Remarks:

1- If 0 idempotent element in R, then $\Gamma(R) \subseteq Л(R)$.

2- If R has only idempotent elements 0 and 1, then $\Gamma(R) = Л(R)$. Consequently, when R local , then $\Gamma(R) = Л(R)$.

3- If R finite non local ring, then $R \cong R_1 \, x \, R_2 \dots x \, R_n$. Since $(1,0, \dots, 0)^2 = (1,0, \dots, 0)$, then R has idempotent element distinct $\{0, 1\}$.

4- If R non- local ring, then there are at greater than or equal two non-trivial idempotent elements in R. if $e^2 = e \neq 0$ or 1, then $1 - e$ also idempotent and $e \neq 1 - e$ (because if $e = 1 - e$, then $e + e = 1$ and $e + e = (e + e) + (e + e) = (e + e)^2 = 1$ implies that $1 = 0$ which is a contradiction. Therefore, $e \neq 1 - e$). Hence if $u \in U(R)$, then u adjacent with $u^{-1}e$, for every $e \in I(R) - \{0,1\}$, so that $V(\text{Л}(R)) = R^* = R - \{0\}$.

Example 2: We shall give all possible idempotent divisor graphs, with $\text{Л}(R) \leq 6$.
If $|\text{Л}(R)| = 1$, then R is local and $|Z(R)| = 2$, so by [12] $R \cong Z_4$ or $F_2[Y]/(Y^2)$.
If $|\text{Л}(R)| = 2$, then R is local and $|Z(R)| = 3$, so by [12] $R \cong Z_9$ or $F_3[Y] / (Y^2)$.
If $|\text{Л}(R)| = 3$, and R is local, then $|Z(R)| = 4$, so that by [12].
$R \cong Z_8, F_2[Y] / (Y^3),\ Z_4[Y] / (2Y, Y^2 - 2), F_4[Y]/(Y^2), Z_4[Y] / (2, Y)^2, Z_4[Y] / (Y^2 + Y + 1)$ or $F_2[Y_1, Y_2 2] / (Y_1, Y_2)^2$. If R non-local, then $|R| = 4$, therefor $R \cong F_2 x\, F_2$.
If $|\text{Л}(R)| = 4$, then R is local and $|Z(R)| = 5$, which implies $R \cong Z_{25}$ or $F_5[Y] / (Y^2)$.
If $|\text{Л}(R)| = 5$, then R is non –local and $|R| = 6$. Hence $R \cong F_2\ x\ F_3$.
If $|\text{Л}(R)| = 6$, then R is local with $|Z(R)| = 7$. So $R \cong Z_{49}$ or $F_7[Y] / (Y^2)$.

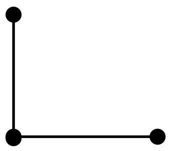


Figure 2.2- $\text{Л}(Z_8)$

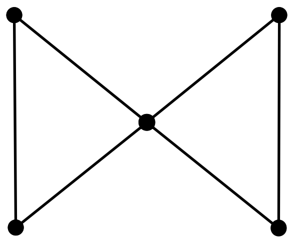


Figure 2.3 -Л $(F_2\ x\ F_3)$

Table 2.1- Rings with $|\text{Л}(R)| \leq 6$

Vertices	Ring(s) type	Graph
1	Z_4 or $F_2[Y]/(Y^2)$	K_1
2	Z_9 or $F_3[Y]/(Y^2)$	K_2
3	$Z_8,\ F_2[Y] / (Y^3),\ F_4[Y] / (Y^2)$ or $Z_4[Y]/(2Y, Y^2 - 2)$	$Fig. 2.2$
	$F_4[Y] / (Y^2\ 2),\ Z_4[Y/(2,Y)^2,\ F_2[Y_1, Y_2]/ (Y_1, Y_2)^2$ or $F_2\ xF_2$	K_3
4	Z_{25} or $F_5[Y] / (Y^2)$	K_4
5	$F_2\ x\ F_3$	$Fig. 2.3$
6	Z_{49} or $F_7[Y] / (Y^2)$	K_6

Now, we give some basic properties of idempotent divisor graph.

Theorem2.2: For any ring R, $\text{Л}(R)$ is connected graph. Moreover, $diam(\text{Л}(R)) \leq 3$.

Proof: Since if R local ring, then $\Gamma(R) = \text{Л}(R)$, so by [2, Theorem 2.3] R connected. Now we investigate the case when R is non-local. Let $a, b \in \text{Л}(R)$. Since R finite ring, then $R^* = Z(R)^* \cup U(R)$. So there are three cases:

Case1: If $a, b \in Z(R)^*$. Since $0 \neq 1$ is an idempotent element in R, then by [2, Theorem2.3] there exist a path between $a, b \in \Gamma(R)$ and $d_{\Gamma(R)}(a, b) \leq 3$. So there is a path between a and b in $\text{Л}(R)$ and $d_{\text{Л}(R)}\ (a, b) \leq 3$.

Case2: If $a, b \in U(R)$, then there are $x, y \in U(R)$ such that $ax = by = 1$. Also for any idempotent element $e^2 = e \notin \{0, 1\}$.
$a(xe) = e$ and $b(y(1 - e)) = 1 - e$. Since $e(1 - e) = 0$, then $a — xe — y(1 - e) — b$ is a path and $d_{\text{Л}(R)}(a, b) \leq 3$.

Case 3: if $a \in U(R)$ and $b \in Z(R)^*$. First, if there exists $e^2 = e \notin \{0,1\}$ such that $be = 0$, then $a — a^{-1}(1-e) — e — b$ is a path. So $d_{Л(R)}(a,b) \leq 3$. If for any $e^2 = e \notin \{0,1\}$, $be \neq 0$. Since $b \in Z(R)^*$, then there is $c \neq c^2$ so that $bc = 0$.
If $ce = 0$, then $a — a^{-1}e — c — b$. So $d_{Л(R)}(a,b) \leq 3$.
If $ce \neq 0$, then $a — a^{-1}(1-e) — ce — b$. Therefore for any cases $d_{Л(R)}(a,b) \leq 3$.

Theorem 2.3: For any ring R, the $g(Л(R)) = 3$ except the cases $R \cong Z_9$, $F_3[Y]/(Y^2)$, Z_9 , $F_2[Y]/(Y^2)$ or Z_4, then $g\ (Л(R)) = \infty$.

Proof : Clearly If $R \cong Z_9$, $F_3[Y]/(Y^2)$, Z_9 , $F_2[Y]/(Y^2)$ or Z_4, then $g\ (Л(R)) = \infty$. Suppose R is non-isomorphic to Z_9, $F_3[Y]/(Y^2)$, Z_9 , $F_2[Y]/(Y^2)$ or Z_4, then there are two cases:

Case1: If R is local ring, then $Л(R) = \Gamma(R)$. So there is $z \in Z(R)^*$ adjacent with any elements in $Z(R)^*$. Since R is non isomorphic to Z_9, $F_3[Y]/(Y^2)$, Z_9 , $F_2[Y]/(Y^2)$ or Z_4, then either $\Gamma(R)$ is star graph or has circle of length 3. If R is star graph which is a contradiction by [2, Theorem 2.5]. So $Л(R) = \Gamma(R)$ has circle of length 3. Hence the $g\ (Л(R)) = 3$.

Case2: If R is non- local ring, then there exists $e^2 = e \notin \{0,1\}$ and $1 — e — (1-e) — 1$ is a circle of length 3. So $g\ (Л(R)) = 3$.

Corollary 2.4: Let $Л(R)$ is an idempotent divisor graph of ring R, then $Л(R)$ is tree if and only if $R \cong Z_9$, $F_3[Y]/(Y^2)$, Z_9 , $F_2[Y]/(Y^2)$ or Z_4.

Corollary 2.5: For any non-local ring R, $\omega(Л(R)) \geq 3$.

Proposition 2.6: If $R \cong F_2\ x\ F_2\ x\ ...\ x\ F_2$ (n-times), then $Л(R) = K_{2^n-1}$.

Proof: Since every element in R is an idempotent, then every non zero two elements are adjacent in $Л(R)$. Hence $Л(R)$ is complete and $V\big(Л(R)\big) = |R^*|$, so $Л(R) = K_{2^n-1}$.

Proposition 2.7: $Л(R)$ is a complete graph if and only if R is a Boolean ring or local with $Z(R)^2 = 0$.

Proof: Suppose that $Л(R)$ is a complete, if R local, then by [9, Theorem 2.5] $Z(R)^2 = 0$. If R is a non-local ring, and for any $a \neq 1$ since $a.1 = a$ and $Л(R)$ is a complete, then a is an idempotent element in R. Therefore, R Boolean ring.
The converse is obvious.

Proposition 2.8: For every non - local ring R, then $deg_{Л(R)}(u) = |I(R)| - 2$, for every $u \in U(R)$.

Proof: Let $u \in U(R)$, then for every $e \in I(R) - \{0,1\}$ we have $u — u^{-1}e$. Since $u^{-1}e \neq u$, then $u^{-1}e \in N_{Л(R)}(u)$ and $deg_{Л(R)}(u) = |I(R)| - 2$.

Theorem 2.9: For any non - local ring R, if $diam(Л(R)) \leq 2$, then $Cent(Л(R)) \subseteq I(R)$

Proof: Since $diam(Л(R)) \leq 2$, then $rad(Л(R)) = 0$ or 1.
If $rad(Л(R)) = 0$, then $diam(Л(R)) = 0$, which is a contradiction since R is non- local.
If $rad(Л(R)) = 1$, then either $Л(R)$ complete, so by Proposition 2.7 R is a Boolean ring and every element idempotent, therefore every element in $Л(R)$ is central, we are done . If $Л(R)$ not complete graph, then for any $a \in Cent(Л(R))$, adjacent with every elements in R* and $a — 1$, therefore $a.1 = a$ is an idempotent element in $R - \{0,1\}$. So $Cent(Л(R)) \subseteq I(R)$.

Theorem 2.10: For any non - local ring R, a graph $Л(R)$ has no end vertex.

Proof: For any $a \in R^*$, there are three cases:

Case1: If $a \in U(R)$, since $a \notin \{a^{-1}e,\ a^{-1}(1-e)\}$, for every idempotent element $e = e^2 \notin \{0,1\}$ and $a^{-1}e \neq a^{-1}(1-e)$, then $\{a^{-1}e, a^{-1}(1-e)\} \subseteq N_{Л(R)}(a)$. So $deg_{Л(R)}(a) \geq 2$.

Case2: If $a \in I(R) - \{0,1\}$, then $\{1-a, 1\} \subseteq N_{Л(R)}(a)$. So $deg_{Л(R)}(a) \geq 2$.

Case3: If $a \in Z(R)^* - I(R)$. Since R finite, then either $a = a^m$ or $a^n = 0$ for some $n, m \in Z^+$.

If $a = a^m$, then there is $k \in Z^+$ such that a^k idempotent element in R and since $a \in Z(R)^*$, then there are $b \in Z(R)^* - \{a\}$ so that $ab = 0$. Therefore $\{b, a^{k-1}\} \subseteq N_{Л(R)}(a)$. So $deg_{Л(R)(a)} \geq 2$.
If $a^n = 0$ and $n = 2$. But $ab = 0$ for some $b \in Z(R)^* - \{a\}$.
Therefore $\{b, a - b\}\} \subseteq N_{Л(R)}(a)$. So $deg_{Л(R)(a)} \geq 2$.
If $n \geq 3$, then $a. a^{n-1} = 0$. Which implies that $a^{n-1}R = \{0, a^{n-1}\}$. Now for any idempotent element $e \not\in \{0,1\}$. Either $a^{n-1}e = 0$ or a^{n-1} for all cases, there are idempotent element $f \not\in \{0,1\}$ such that $a^{n-1}f = 0$. If $a^{n-2}f \neq 0$,then $\{a^{n-1}, a^{n-2}f\} \subseteq N_{Л(R)}(a)$. So $deg_{Л(R)}(a) \geq 2$. If $a^{n-2}f = 0$, then $\{a^{n-2}, a^{n-3}f\} \subseteq N_{Л(R)}(a)$. If we repeat this process, we can get $af = 0$. This means that there is at least two elements adjacent to a.

3. Planarity and Cliques of Idempotent Divisor Graph

In this part, we investigate the planarity, and the clique number of the idempotent divisor graph.

Proposition 3.1: Suppose that $R \cong K x K`$, where $K$ and $K`$ are fields, then $\omega(Л(R)) = 3$.

Proof: Since $R \cong K x K`$, then the only idempotent elements in R are $\{(0,0), (1,0), (0,1), (1,1)\}$. For any $(a,b) \in R$. If a and $b \neq 0$, then (a,b) adjacent with only elements $(a^{-1}, 0)$, $(0, b^{-1})$. So $(a,b) \not\in K_4$. Also if $a = 0$ and $b \neq 0$, then (a,b) adjacent with only elements (x, b^{-1}), for every $x \in K$. But (x, b^{-1}) adjacent with only elements $(x^{-1}, 0)$ or $(0, b^{-1})$ and non-adjacent with $(0, b^{-1})$. So (a,b)) $\not\in K4$. Similarly if $a \neq 0$ and $b = 0$, then we have (a,b)) $\not\in K_4$ and hence $\omega(Л(R)) = 3$.

Theorem 3.2: If $R \cong R_1 x R_2$, where R_1 and R_2 are local rings but not fields, then $\omega(Л(R)) = 3$ if $R \cong Z_4 x Z_4$, $Z_4 x F_2[Y] / (Y^2)$ or $F_2[Y] / (Y^2) x F_2[Y]/(Y^2)$. Otherwise $\omega(Л(R)) \geq 4$.

Proof: If $R \cong Z_4 x Z_4$, $Z_4 x F_2[Y] / (Y^2)$ or $F_2[Y] / (Y^2) x F_2[Y]/(Y^2)$, then $\omega(Л(R)) = 3$ see Fig 3.1. Suppose R is non-isomorphic $Z_4 x Z_4$, $Z_4 x F_2[Y]/(Y^2)$ or $F_2[Y] / (Y^2) x F_2[Y]/ (Y^2)$. Since R_1 and R_2 are local but not fields, then there exists $(z_1, z_1) \in R$ with $z_1 \in Z(R_1)^*$ and $z_2 \in Z(R_2)^*$, thus there are $a_1 \in Z(R_1)^* - \{z_1\}$ and $a_2 \in Z(R_2)^*$ such that $z_1 a_1 = z_2 a_2 = 0$. Therefore the set $\{(z_1, z_2), (a_1, 0), (0, a_2), (a_1, z_2)\}$ induced a sub-graph K_4. So $\omega(Л(R)) \geq 4$.

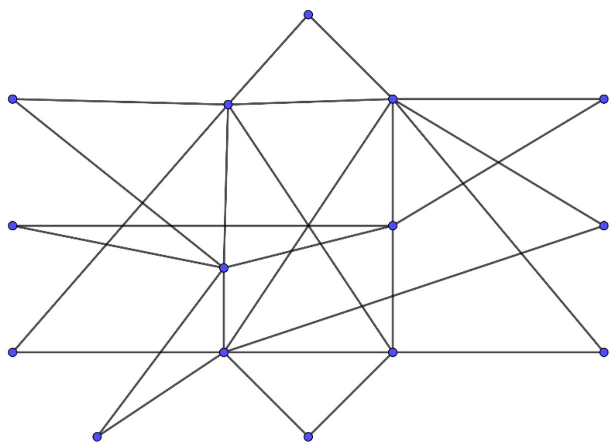


Figure 3.1- Л $(A_1 \; x \; A_2)$, where A_1 and $A_2 \cong Z_4$ or

Recall that "a graph G is said to be planar if it can be drawn in the plane in such a way that pairs of edges intersect only at vertices, if at all. If G has no such representation, G is called non-planar. It we know that a graph G is planar if and only if contained no sub-graph K_5 or $K_{3,3}$ " [11].

Proposition 3.3: For any local ring R, a graph $Л(R)$ is planar if and only if R is isomorphic to one of the following table:

Table 3.1- local **rings with $|\Gamma(R)|$ is planar**

Ring(s) type	Graph
Z_4 or $F_2[Y]/(Y^2)$	K_1
Z_9 Z or $F_3[Y]/(Y^2)$	K_2
$F_2[Y_1,Y_2]/(Y_1,Y_2)^2$, $Z_4[Y]$ / $(2Y,Y^2)$, or $F_4[Y]/(Y^2)$	K_3
$Z_4[Y]/(2Y,Y^2-2)$, Z_8 or $F_2[Y]/(Y^3)$	$K_{1,2}$
Z_{25} or $F_5[Y]/(Y^2)$	K_4
$Z_{27}, F_3[Y]/(Y^3)$ or $Z_9[Y]/(3Y,\ Y^2 \pm 3)$	$K_{2,6}$
Z_{16}, $F_2[Y]/(Y^4)$, $Z_4[Y]/[Y^2]$, $Z_4[Y]/[2Y,Y^3-2]$,$Z_4[Y]/$ $[2Y,Y^2-2Y-2]$ or $Z_4[Y]/[2Y,Y^3-2]$	$K_1+(4K_1 \cup K_2)$
$Z_4[Y]/(Y^2$-2), $Z_8[Y]/(2Y,Y^2-4)$, $F_2[Y_1,Y_2]/(Y_1^2-Y_2^2,Y_1Y_2)$ or $Z_4[Y_1,Y_2]/(Y_1^2-2,Y_1Y_2,Y_2^2-2,2Y_1)$	$K_1+(K_2 \cup C_4)$
$Z_4[Y_1,Y_2]/(Y_1^2,Y_1Y_2-2,Y_2^2)$, $Z_4[Y]/(Y^2)$ or $Z_4[Y]/(Y^2+\text{Y}+1)$	$K_1+(2K_1 \cup C_4)$

Proof: Since R local, then $Л(R) = \Gamma(R)$. Therefore the prove follows by Propositions 2,3 and 4 in [13].

Theorem 3.4: If $R \cong F_{q_1}\, x\, F_{q_2}$, then $Л(R)$ is a planar if and only if $F_{q_i} = F_2$ or F_3 for $i = 1,2$.

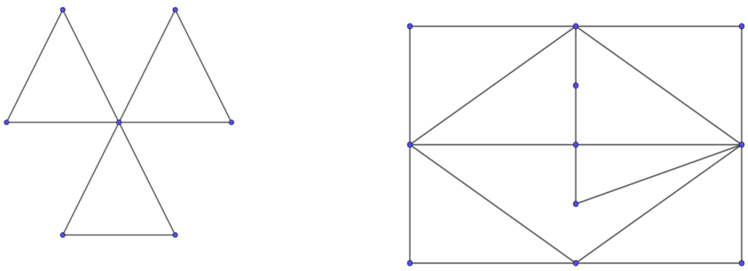
Proof: Without loss generality, let $F_{q_1} = F_2$ or F_3. First, if $F_{q_1} = F_2$, then $R \cong F_2\, x\, F_{q_2}$, since $\omega(Л(R)) = 3$, by Proposition 3.1. Therefore, $Л(R)$ does not contain a sub-graph K_5. Now we shall to prove $Л(R)$ does not contain $K_{3,3}$ sub-graph. If not, then there exist disjoint two subsets $V_1 = \{(a_1,b_1),(a_2,b_2),(a_3,b_3)\}$ and $V_2 = \{(x_1,y_1),(x_2,y_2),(x_3,y_3)\}$ such that every element in V_1 adjacent with every element in V_2, and a_1,a_2,a_3,x_1,x_2 *and* $x_3 \in F_2$, and b_1,b_2,b_3,y_1,y_2 *and* $y_3 \in F_{q_2}$. Since R have exactly idempotent elements $(0,0),(1,0),(0,1)$ and (1,1), then $(a_i,b_i)(x_j,y_j) \in \{(0,0),(1,0),(0,1)\}$. So $b_iy_j = 0$ or 1, if $b_i \neq 0$ or 1 for all $i = 1,2,3$, then $y_j = 0$ or b_i^{-1} for all $j = 1,2\,3$. But $x_i \in F_2$, then we have $V_2 = \{(0,b_i^{-1}),(1,b_i^{-1}),(1,0)\}$. Therefore $V_1 = \{(0,b_i),(1,b_i),(0,1)\}$. But $(1,b_i)(1,b_i^{-1}) = (1,1)$ a contradiction. Also, if $b_i = 0$ or 1 for all $i = 1,2,3$ we get a contradiction. Therefore, $Л(R)$ does not contain $K_{3,3}$ sub- graph and $Л(R)$ is a planar. Similarly, we can show that if $F_{q_1} \cong F_3$, then $Л(R)$ is a planar. Finally, if $F_{q_i} \neq F_2$ or F_3 for $i = 1,2$. Then there exist $a_1,a_2 \in F_{q_1} - \{0,1\}$ and $b_1, b_2 \in F_{q_2} - \{0,1\}$. Whence $V_1 = \{(a_1,0),(a_2,0),(1,0)\}$ and $V_2 = \{(0,b_1),(0,b_2),(0,1)\}$ are disjoint sub-sets induced $K_{3,3}$ sub-graph in $Л(R)$. Therefor R not planar.

Theorem3.5: For any ring R, a graph $Л(R)$ is planar if and only if R isomorphic one of the following rings in table 3.1 or R isomorphic one of the following rings:

$F_2\, x\, F_{q_2}, F_3\, x\, F_{q_2}$, $F_2\, x\, Z_4, F_2\, x\, F_2[Y]\,/\,(Y^2)$, $F_2\, x\, Z_9$ or $F_2\, x\, F_3[Y]\,/\,(Y^2)$

Proof: If $R \cong R_1\, x\, R_2\, x\, \ldots\, x\, R_n$, where R_i local ring for all $i = 1,2,\ldots n$ and $n \geq 3$. The set $\{(1,0,\ldots,0),(0,1,0,\ldots,0),(1,1,0,\ldots,0),(0,0,\ldots,1),(1,1,\ldots,1)\} \subseteq V(Л(R))$ so induced a sub-graph K_5, therefore $Л(R)$ is not planer. If $n = 2$, then $R \cong R_1\, x\, R_2$, where R_1, R_2 are local rings, there are three cases:

Case1: If R_1 and R_2 are fields, then by Theorem3.4 $Л(R)$ is planar if and only if $R \cong F_2\ x\ Fq_2\ or\ F_3\ x\ Fq_2$, where Fq_2 is a field order q_2.
Case2: If R_1 and R_2 are not fields, then $|R_1|, |R_2| \geq 4$. Obviously $Л(R)$ not planar.
Case3: If R_1 is a field and R_2 not field. Let $R_1 = F_2$ or F_3 and $|Z(R_2)| = 2$, then $|R_2| = 4$, which implies that $R_2 \cong Z_4$ or $F_2[Y] / (Y^2)$, so $Л(R)$ is planar see Fig. 3.2 . |If $Z(R_2)| \geq 3$, then there exists $a, b \in Z(R_2)$, so that $ab = 0$. Therefore the vertices $(1,0), (1, a), (1, b), (0, a), (0, b)$ are adjacent, whence $Л(R)$ induced a sub-graph K_5, therefore $Л(R)$ not planar. If $|R_1| \geq F_4$, then it is easy to show that a graph $Л(R)$ is not planar. Finally, if $n = 1$, then R is local and a complete proved it's follow by proposition 3.3. and table 3.1



$Л(F_2 x\ Z_4)$ or $Л(F_2 x\ F_2[Y]/(Y^2)$ $Л(F_3 x\ Z_4)$ or $Л(F_3 x\ F_2[Y]/(Y^2)$

Figure 3.2

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