Idempotent Divisor Graph of Commutative Ring

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Abstract

This work aims to introduce and to study a new kind of divisor graph which is called idempotent divisor graph, and it is denoted by \( I(R) \). Two non-zero distinct vertices \( v_1 \) and \( v_2 \) are adjacent if and only if \( v_1, v_2 = e \), for some non-unit idempotent element \( e^2 = e \in R \). We establish some fundamental properties of \( I(R) \), as well as its connection with \( J(R) \). We also study planarity of this graph.

Keywords: Idempotent Elements, Zero Divisor Graph, Idempotent Divisor Graph, Planar Graph.

1. Introduction

Let \( R \) be a finite commutative ring with unity \( 1 \neq 0 \). We denote \( Z(R) \), \( I(R) \), and \( U(R) \) the set of zero divisors, the set of idempotent elements and the set of unit elements respectively. In [1], Beck introduced the idea that connects between ring theory and graph theory when studied the coloring of commutative ring. Later in [2], Anderson and Livingston modified this idea when studied the zero divisor graph \( I(R) \) that have vertices \( Z(R) \) and edges if and only if \( v_1, v_2 \in Z(R) \). Many authors studied this notion see for examples [3], [4], [5] and [6]. Recently, there are other concepts of zero divisor graph, see for examples [7], [8], [9], and [10].

In graph theory “(v)” denotes by the eccentricity of a vertex \( v \) of a connected graph \( G \) which is the number \( max_{u \in V(G)} d(u, v) \). That means \( e(v) \) is the distance between \( v \) and a vertex furthest from \( v \). The radius of \( G \), which is denoted by \( radG \), is \( max_{u \in V(G)} d(u, v) \), while the diameter of \( G \) is the maximum eccentricity and it is denoted by \( diamG \). Consequently, \( diamG \) is the greatest distance between any two vertices of \( G \). Also, a graph \( G \) has radius 1 if and only if \( G \) contains a vertex \( u \) adjacent to all other vertices of \( G \). A vertex \( v \) is a central

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vertex if \( e(v) = \text{rad} G \) and the center \( \text{Cent}(G) \) is the sub-graph of \( G \) that induced by its central vertices. The girth of a graph \( G \) is the length of a shortest cycle contained in \( G \), it is denoted by \( g(G) \). The neighborhood of \( x \) in a graph \( G \) denotes by \( N_{G}(x) \), is the set of all \( y \in V(G) \) such that \( y \) is adjacent to \( x \) in \( G \). In our graph in this case, \( N_{G}(x) = \{ y \in V(G) \setminus \{ x \} | xy = 0 \} \). \( K_{n}K_{m} \) symbolized complete graph and complete bipartite graph respectively. \( K_{1,m} \) we call star graph. A clique number of \( G \) symbolized \( \omega(G) \) is greats complete subgraph of \( G \). If a connected graph does not contain cycle, we call tree. Let \( H \) and \( G \) two graphs, \( G \cup H \) is a graph with \( V(G \cup H) = V(G) \cup V(H) \) and \( E(G \cup H) = E(G) \cup E(H) \), and for \( n \in \mathbb{Z}^{+} \), \( nH = \bigcup_{i=1}^{n} H \). The graph \( G + H \) is a graph with \( V(G + H) = V(G) \cup V(H) \) and \( E(G + H) = E(G) \cup E(H) \cup \{ \{ u, v \} : u \in V(G), v \in V(H) \} \). A path graph of order \( n \) is denoted by \( P_{n} \) is a graph with \( V(P_{n}) = \{ v_{i} : i = 1, 2, ..., n \} \) and \( E(P_{n}) = \{ \{ v_{j}, v_{j} + 1 \} : j = 1, 2, ..., n - 1 \} \). A clique number of \( G \) symbolized by \( \omega(G) \) is greats complete subgraph of \( G \). If \( n \) finite non local ring, then the cardinality of \( R \) symbolized by \( |R| \) equal \( p^{t} \), where \( p \) prime number and \( t \in \mathbb{Z}^{+} \). In ring theory, a ring \( R \) is said to be local if has exactly one maximal ideal. Also, if \( R \) finite local ring, then the cardinality of \( R \) symbolized by \( |R| \) equal \( p^{t} \), where \( p \) prime number and \( t \in \mathbb{Z}^{+} \), as well as the cardinality of maximal ideal \( M = p^{r} \), where \( 0 < r < t \). A ring \( R \) is called Boolean, if every element is an idempotent. We denote \( F_{q} \) is a field order \( q \). In section two we defined a new graph on the ring and prove some basic properties of about this graph and we give all possible graphs less than or equal 6 vertices. In section three, we give all graphs to be planer.

2. Examples and Basic Properties

In this section, we introduce a new class of divisor graph manly idempotent divisor graph, we give some of about this graph, and we also provide some examples.

**Definition 2.1:** The undirected graph is called idempotent divisor graph, and which is symbolized by \( \Pi(R) \) which a simple graph with vertices set in \( R - \{ 0 \} \), and two non-zero distinct vertices \( v_{1} \) and \( v_{2} \) are adjacent if and only if \( v_{1}v_{2} = e \), for some non-unit idempotent element \( e \in R \) \( (i.e. e e^{2} = e \neq 1) \).

**Example 1:** Let \( R = Z_{6} \), since the idempotent elements \( I(R) = \{ 0, 1, 3, 4 \} \), then \( \Pi(R) \) is:

![Figure 2.1](image)

**Remarks:**

1- If 0 idempotent element in \( R \), then \( \Pi(R) \subseteq \Pi(R) \).
2- If \( R \) has only idempotent elements 0 and 1, then \( \Pi(R) = \Pi(R) \). Consequently, when \( R \) local, then \( \Pi(R) = \Pi(R) \).
3- If \( R \) finite non local ring, then \( R \cong R_{1} \times R_{2} \ldots \times R_{n} \). Since \((1,0,\ldots,0)^{2} = (1,0,\ldots,0)\), then \( R \) has idempotent element distinct \( \{ 0, 1 \} \).
If $R$ non-local ring, then there are at greater than or equal two non-trivial idempotent elements in $R$. if $e^2 = e \neq 0$ or 1, then $1 - e$ also idempotent and $e \neq 1 - e$ (because if $e = 1 - e$, then $e + e = 1$ and $e + e = (e + e) + (e + e) = (e + e)^2 = 1$ implies that $1 = 0$ which is a contradiction. Therefore, $e \neq 1 - e$). Hence if $u \in U(R)$, then $u$ adjacent with $u^{-1}e$, for every $e \in \mathcal{I}(R) - \{0, 1\}$, so that $\mathcal{V}(\mathcal{J}(R)) = R^* = R - \{0\}$.

**Example 2:** We shall give all possible idempotent divisor graphs, with $\mathcal{J}(R) \leq 6$.

If $|\mathcal{J}(R)| = 1$, then $R$ is local and $|\mathcal{Z}(R)| = 2$, so by [12] $R \cong Z_4$ or $F_2[Y](Y^2)$.

If $|\mathcal{J}(R)| = 2$, then $R$ is local and $|\mathcal{Z}(R)| = 3$, so by [12] $R \cong Z_9$ or $F_3[Y]/(Y^2)$.

If $|\mathcal{J}(R)| = 3$, and $R$ is local, then $|\mathcal{Z}(R)| = 4$, so that by [12].

$R \cong Z_8, F_2[Y]/(Y^2), Z_4[Y]/(2Y, Y^2 - 2), F_4[Y]/(Y^2), Z_4[Y]/(2, Y^2), Z_4[Y]/(Y^2 + Y + 1)$ or $F_2[y_1, y_2]/(y_1, y_2)^2$. If $R$ non-local, then $|R| = 4$, therefore $R \cong F_2 \times F_2$.

If $|\mathcal{J}(R)| = 4$, then $R$ is local and $|\mathcal{Z}(R)| = 5$, which implies $R \cong Z_{25}$ or $F_5[Y]/(Y^2)$.

If $|\mathcal{J}(R)| = 5$, then $R$ is non-local and $|R| = 6$. Hence $R \cong F_2 \times F_3$.

If $|\mathcal{J}(R)| = 6$, then $R$ is local with $|\mathcal{Z}(R)| = 7$. So $R \cong Z_{49}$ or $F_7[Y]/(Y^2)$.

**Table 2.1 - Rings with $|\mathcal{J}(R)| \leq 6$**

<table>
<thead>
<tr>
<th>Vertices</th>
<th>Ring(s) type</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$Z_4$ or $F_2[Y]/(Y^2)$</td>
<td>$K_1$</td>
</tr>
<tr>
<td>2</td>
<td>$Z_9$ or $F_3[Y]/(Y^2)$</td>
<td>$K_2$</td>
</tr>
<tr>
<td>3</td>
<td>$Z_8, F_2[Y]/(Y^3), F_4[Y]/(Y^2)$ or $F_4[Y]/(2Y, Y^2 - 2), Z_4[Y]/(2Y, Y^2), F_2[y_1, y_2]/(y_1, y_2)^2$</td>
<td>$K_3$</td>
</tr>
<tr>
<td>4</td>
<td>$Z_{25}$ or $F_5[Y]/(Y^2)$</td>
<td>$K_4$</td>
</tr>
<tr>
<td>5</td>
<td>$F_2 \times F_3$</td>
<td>$K_6$</td>
</tr>
<tr>
<td>6</td>
<td>$Z_{49}$ or $F_7[Y]/(Y^2)$</td>
<td>$K_6$</td>
</tr>
</tbody>
</table>

Now, we give some basic properties of idempotent divisor graph.

**Theorem 2.2:** For any ring $R$, $\mathcal{J}(R)$ is connected graph. Moreover, $diam(\mathcal{J}(R)) \leq 3$.

**Proof:** Since if $R$ local ring, then $\mathcal{I}(R) = \mathcal{J}(R)$, so by [2, Theorem 2.3] $R$ connected. Now we investigate the case when $R$ is non-local. Let $a, b \in \mathcal{J}(R)$. Since $R$ finite ring, then $R^* = Z(R)^* \cup U(R)$. So there are three cases:

**Case 1:** If $a, b \in Z(R)^*$. Since $0 \neq 1$ is an idempotent element in $R$, then by [2, Theorem 2.3] there exist a path between $a, b \in \mathcal{I}(R)$ and $d(\mathcal{I}(R), a, b) \leq 3$. So there is a path between $a$ and $b$ in $\mathcal{J}(R)$ and $d(\mathcal{J}(R), a, b) \leq 3$.

**Case 2:** If $a, b \in U(R)$, then there are $x, y \in U(R)$ such that $ax = by = 1$. Also for any idempotent element $e^2 = e \notin \{0, 1\}$.

$a(xe) = e$ and $b(y(1 - e)) = 1 - e$. Since $e(1 - e) = 0$, then $a \rightarrow xe \rightarrow (1 - e) \rightarrow \rightarrow b$ is a path and $d(\mathcal{J}(R), a, b) \leq 3$. 

\[ Figure 2.2 - \mathcal{J}(Z_8) \quad Figure 2.3 - \mathcal{J}(F_2 \times F_3) \]
Case 3: if \( a \in U(R) \) and \( b \in Z(R)^* \). First, if there exists \( e^2 = e \not\in \{0,1\} \) such that \( be = 0 \), then \( a - a^{-1}(1-e) - e - b \) is a path. So \( d_{I(R)}(a,b) \leq 3 \). If for any \( e^2 = e \not\in \{0,1\} \), \( be \neq 0 \). Since \( b \in Z(R)^* \), then there is \( c \neq c^2 \) so that \( bc = 0 \).

If \( ce = 0 \), then \( a - a^{-1}e - c - b \). So \( d_{I(R)}(a,b) \leq 3 \). If \( ce \neq 0 \), then \( a - a^{-1}(1-e) - ce - b \). Therefore for any cases \( d_{I(R)}(a,b) \leq 3 \).

**Theorem 2.3:** For any ring \( R \), the \( g(I(R)) = 3 \) except the cases \( R \cong Z_9, F_3[Y]/(Y^2), Z_9, F_2[Y]/(Y^2) \) or \( Z_4 \), then \( g(I(R)) = \infty \).

**Proof:** Clearly If \( R \cong Z_9, F_3[Y]/(Y^2), Z_9, F_2[Y]/(Y^2) \) or \( Z_4 \), then \( g(I(R)) = \infty \). Suppose \( R \) is non-isomorphic to \( Z_9, F_3[Y]/(Y^2), Z_9, F_2[Y]/(Y^2) \) or \( Z_4 \), then there are two cases:

**Case1:** If \( R \) is local ring, then \( I(R) = I(R) \). So there is \( z \in Z(R)^* \) adjacent with any elements in \( Z(R)^* \). Since \( R \) is non-isomorphic to \( Z_9, F_3[Y]/(Y^2), Z_9, F_2[Y]/(Y^2) \) or \( Z_4 \), then either \( I(R) \) is star graph or has circle of length 3. If \( R \) is star graph which is a contradiction by [2, Theorem 2.5]. So \( I(R) = I(R) \) has circle of length 3. Hence the \( g(I(R)) = 3 \).

**Case2:** If \( R \) is non-local ring, then there exists \( e^2 = e \not\in \{0,1\} \) and \( 1 - e - (1-e) - 1 \) is a circle of length 3. So \( g(I(R)) = 3 \).

**Corollary 2.4:** Let \( I(R) \) is an idempotent divisor graph of ring \( R \), then \( I(R) \) is tree if and only if \( R \cong Z_9, F_3[Y]/(Y^2), Z_9, F_2[Y]/(Y^2) \) or \( Z_4 \).

**Corollary 2.5:** For any non-local ring \( R \), \( \alpha(I(R)) \geq 3 \).

**Proposition 2.6:** If \( R \cong F_2 \times F_2 \times \ldots \times F_2 \) (n-times), then \( I(R) = K_{2^n-1} \).

**Proof:** Since every element in \( R \) is an idempotent, then every non-zero two elements are adjacent in \( I(R) \). Hence \( I(R) \) is complete and \( V(I(R)) = |R| \), so \( I(R) = K_{2^n-1} \).

**Proposition 2.7:** \( I(R) \) is a complete graph if and only if \( R \) is a Boolean ring or local with \( Z(R)^2 = 0 \).

**Proof:** Suppose that \( I(R) \) is a complete, if \( R \) is local, then \( I(R) \) is a complete. Then \( R \) is a Boolean ring.

The converse is obvious.

**Proposition 2.8:** For every non-local ring \( R \), then \( deg_{I(R)}(u) = |I(R)| - 2 \), for every \( u \in U(R) \).

**Proof:** Let \( u \in U(R) \), then for every \( e \in I(R) - \{0,1\} \) we have \( u - u^{-1}e \). Since \( u^{-1}e \neq u \), then \( u^{-1}e \in N_{I(R)}(u) \) and \( deg_{I(R)}(u) = |I(R)| - 2 \).

**Theorem 2.9:** For any non-local ring \( R \), if \( diam(I(R)) \leq 2 \), then \( Cent(I(R)) \subseteq I(R) \)

**Proof:** Since \( diam(I(R)) \leq 2 \), then \( rad(I(R)) = 0 \) or 1. If \( rad(I(R)) = 0 \), then \( diam(I(R)) = 0 \), which is a contradiction since \( R \) is non-local.

If \( rad(I(R)) = 1 \), then either \( I(R) \) complete, so by Proposition 2.7 \( R \) is a Boolean ring and every element idempotent, therefore every element in \( I(R) \) central, we are done. If \( R \) not complete graph, then for any \( a \in Cent(I(R)) \), adjacent with every elements in \( R^* \) and \( a - 1 \), therefore \( a = a \) is an idempotent element in \( R - \{0,1\} \). So \( Cent(I(R)) \subseteq I(R) \).

**Theorem 2.10:** For any non-local ring \( R \), a graph \( I(R) \) has no end vertex.

**Proof:** For any \( a \in R^* \), there are three cases:

- **Case1:** If \( a \in U(R) \), since \( a \not\in \{a^{-1}e, a^{-1}(1-e)\} \), for every idempotent element \( e = e^2 \not\in \{0,1\} \) and \( a^{-1}e \neq a^{-1}(1-e) \), then \( \{a^{-1}e, a^{-1}(1-e)\} \subseteq N_{I(R)}(a) \). So \( deg_{I(R)}(a) \geq 2 \).

- **Case2:** If \( a \in I(R) - \{0,1\} \), then \( \{1 - a, 1\} \subseteq N_{I(R)}(a) \). So \( deg_{I(R)}(a) \geq 2 \).

- **Case3:** If \( a \in Z(R)^* - I(R) \). Since \( R \) finite, then either \( a = a^m \) or \( a^n = 0 \) for some \( n, m \in Z^+ \).
If \( a = a^m \), then there is \( k \in \mathbb{Z}^+ \) such that \( a^k \) idempotent element in \( R \) and since \( a \in Z(R)^* \), then there are \( b \in Z(R)^* - \{a\} \) so that \( ab = 0 \). Therefore \( \{b, a^{k-1}\} \subseteq N_{\mathcal{I}(R)}(a) \). So \( deg_{\mathcal{I}(R)}(a) \geq 2 \).

If \( a^n = 0 \) and \( n = 2 \). But \( ab = 0 \) for some \( b \in Z(R)^* - \{a\} \). Therefore \( \{b, a - b\} \subseteq N_{\mathcal{I}(R)}(a) \). So \( deg_{\mathcal{I}(R)}(a) \geq 2 \).

If \( n \geq 3 \), then \( a, a^{n-1} = 0 \). Which implies that \( a^{n-1}R = \{0, a^{n-1}\} \). Now for any idempotent element \( e \notin \{0, 1\} \). Either \( a^{n-1}e = 0 \) or \( a^{n-1} \) for all cases, there are idempotent element \( f \notin \{0, 1\} \) such that \( a^{n-1}f = 0 \). If \( a^{n-2}f = 0 \), then \( \{a^{n-1}, a^{n-2}f\} \subseteq N_{\mathcal{I}(R)}(a) \). So \( deg_{\mathcal{I}(R)}(a) \geq 2 \). If \( a^{n-2}f = 0 \), then \( \{a^{n-2}, a^{n-3}f\} \subseteq N_{\mathcal{I}(R)}(a) \). If we repeat this process, we can get \( af = 0 \). This means that there is at least two elements adjacent to \( a \).

3. Planarity and Cliques of Idempotent Divisor Graph

In this part, we investigate the planarity, and the clique number of the idempotent divisor graph.

**Proposition 3.1**: Suppose that \( R \cong K \times K^* \), where \( K \) and \( K^* \) are fields, then \( \omega(\mathcal{I}(R)) = 3 \).

**Proof**: Since \( R \cong K \times K^* \), then the only idempotent elements in \( R \) are \( \{(0, 0), (1, 0), (0, 1), (1, 1)\} \). For any \( (a, b) \in R \). If \( a \) and \( b \neq 0 \), then \( (a, b) \) adjacent with only elements \( (a^{-1}, 0), (0, b^{-1}) \). So \( (a, b) \notin K_4 \). Also if \( a = 0 \) and \( b \neq 0 \), then \( (a, b) \) adjacent with only elements \( (x, b^{-1}) \), for every \( x \in K \). But \( (x, b^{-1}) \) adjacent with only elements \( (x^{-1}, 0) \) or \( (0, b^{-1}) \) and non-adjacent with \( (0, b^{-1}) \). So \( (a, b) \notin K_4 \). Similarly if \( a \neq 0 \) and \( b = 0 \), then we have \( (a, b) \notin K_4 \) and hence \( \omega(\mathcal{I}(R)) = 3 \).

**Theorem 3.2**: If \( R \cong R_1 \times R_2 \), where \( R_1 \) and \( R_2 \) are local rings but not fields, then \( \omega(\mathcal{I}(R)) = 3 \) if \( R \cong Z_4 \times Z_4, Z_4 \times F_2[Y] / (Y^2) \) or \( F_2[Y] / (Y^2) \times F_2[Y] / (Y^2) \). Otherwise \( \omega(\mathcal{I}(R)) \geq 4 \).

**Proof**: If \( R \cong Z_4 \times Z_4, Z_4 \times F_2[Y] / (Y^2) \) or \( F_2[Y] / (Y^2) \times F_2[Y] / (Y^2) \), then \( \omega(\mathcal{I}(R)) = 3 \) see Fig 3.1. Suppose \( R \) is non-isomorphic \( Z_4 \times Z_4, Z_4 \times F_2[Y] / (Y^2) \) or \( F_2[Y] / (Y^2) \times F_2[Y] / (Y^2) \). Since \( R_1 \) and \( R_2 \) are local but not fields, then there exists \( (z_1, z_2) \in R \) with \( z_1 \in Z(R_1)^* \) and \( z_2 \in Z(R_2)^* \), thus there are \( a_1 \in Z(R_1)^* - \{z_1\} \) and \( a_2 \in Z(R_2)^* \) such that \( z_1a_1 = z_2a_2 = 0 \). Therefore the set \( \{(z_1, z_2), (a_1, 0), (0, a_2), (a_1, z_2)\} \) induced a sub-graph \( K_4 \). So \( \omega(\mathcal{I}(R)) \geq 4 \).

Recall that “a graph \( G \) is said to be planar if it can be drawn in the plane in such a way that pairs of edges intersect only at vertices, if at all. If \( G \) has no such representation, \( G \) is called
non-planar. It we know that a graph $G$ is planar if and only if contained no sub-graph $K_5$ or $K_{3,3}$ [11].

**Proposition 3.3:** For any local ring $R$, a graph $\mathcal{J}(R)$ is planar if and only if $R$ is isomorphic to one of the following table:

| Table 3.1- local rings with $|\mathcal{J}(R)|$ is planar |
|---------------------------------------------------------|
| **Ring(s) type**                                      | **Graph** |
| $Z_4$ or $F_2[Y]/(Y^2)$                               | $K_1$     |
| $Z_6$ or $F_3[Y]/(Y^2)$                               | $K_2$     |
| $F_2(Y_1,Y_2)/(Y_1^2,Y_2)$, $Z_4[Y]/(2Y,Y^2)$,      | $K_3$     |
| or $F_4[Y]/(Y^2)$                                     |           |
| $Z_4[Y]/(2Y,Y^2 - 2), Z_6$ or $F_2[Y]/(Y^3)$          | $K_{1,2}$ |
| $Z_{25}$ or $F_5[Y]/(Y^2)$                            | $K_4$     |
| $Z_{27}, F_3[Y]/(Y^3)$ or $Z_5[Y]/(3Y, Y^2 \pm 3)$   | $K_{2,6}$ |
| $Z_6, F_2[Y]/(Y^2), Z_4[Y]/(Y^2), Z_4[Y]/(2Y,Y^3 - 2), Z_4[Y]/(2Y,Y^2 - 2), Z_4[Y]/(2Y,Y^3 - 2)$ | $K_1 + (4K_1 \cup K_2)$ |
| $Z_4[Y]/(Y^3 - 2), Z_6[Y]/(2Y,Y^4 - 4), F_2(Y_1,Y_2)/(Y_1^2 - Y_2^2, Y_1,Y_2)$ or | $K_1 + (K_2 \cup C_4)$ |
| $Z_4[Y_1,Y_2]/(Y_1^2 - 2Y_1,Y_2^2 - 2Y_2)$           |           |
| $Z_4[Y_1,Y_2]/(Y_1^2 - 2Y_1,Y_2^2 - 2Y_2,Y_1,Y_2)$   |           |
| $Z_4[Y_1,Y_2]/(Y_1^2 - Y_2^2,Y_1,Y_2)$               |           |

**Proof:** Since $R$ local, then $\mathcal{L}(R) = \mathcal{L}(R)$. Therefore the prove follows by Propositions 2,3 and 4 in [13].

**Theorem 3.4:** If $R \cong F_{q_1} \times F_{q_2}$, then $\mathcal{J}(R)$ is a planar if and only if $F_{q_1} = F_2$ or $F_3$ for $i = 1,2$.

**Proof:** Without loss generality, let $F_{q_1} = F_2$ or $F_3$. First, if $F_{q_1} = F_2$, then $R \cong F_2 \times F_{q_2}$, since $\phi(\mathcal{J}(R)) = 3$, by Proposition 3.1. Therefore, $\mathcal{J}(R)$ does not contain a sub-graph $K_5$.

Now we shall to prove $\mathcal{J}(R)$ does not contain $K_{3,3}$ sub-graph. If not, then there exist disjoint two subsets $V_1 = \{(a_1,b_1), (a_2,b_2), (a_3,b_3)\}$ and $V_2 = \{(x_1,y_1),(x_2,y_2),(x_3,y_3)\}$ such that every element in $V_1$ adjacent with every element in $V_2$, and $a_1,a_2,a_3,x_1,x_2$ and $x_3 \in F_2$, and $b_1,b_2,b_3,y_1,y_2$ and $y_3 \in F_{q_2}$. Since $R$ have exactly idempotent elements $(0,0),(1,0),(0,1)$ and $(1,1)$, then $(a_i,b_i) \in V_1 \subset (0,0),(1,0),(0,1))$. So $a_i \equiv y_j \equiv 0$ or 1, if $a_i \equiv 0$ or 1 for all $i = 1,2,3$, then $y_j \equiv 0$ or $y_i \equiv 1$ for all $j = 1,2,3$. But $x_i \in F_2$, then we have $V_2 = \{(0,0,b_i), (1,0,b_i),(0,1)\}$. Therefore $V_1 = \{(0,b_i), (1,b_i), (0,1)\}$. But $(1,b_i)(1,b_i) = (1,1)$ a contradiction. Also, if $b_i \equiv 0$ or 1 for all $i = 1,2,3$ we get a contradiction. Therefore, $\mathcal{J}(R)$ does not contain $K_{3,3}$ sub-graph and $\mathcal{J}(R)$ is a planar. Similarly, we can show that if $F_{q_1} \cong F_3$, then $\mathcal{J}(R)$ is a planar. Finally, if $F_{q_1} \neq F_2$ or $F_3$ for $i = 1,2$. Then there exist $a_1,a_2 \in F_{q_1} - \{0,1\}$ and $b_1,b_2 \in F_{q_2} - \{0,1\}$. Whence $V_1 = \{(a_1,0),(a_2,0),(1,0)\}$ and $V_2 = \{(0,b_1),(0,b_2),(0,1)\}$ are disjoint sub-sets induced $K_{3,3}$ sub-graph in $\mathcal{J}(R)$. Therefore $R$ not planar.

**Theorem 3.5:** For any ring $R$, a graph $\mathcal{J}(R)$ is planar if and only if $R$ isomorphic one of the following rings in table 3.1 or $R$ isomorphic one of the following rings:

$F_2 \times F_{q_2}, F_3 \times F_{q_2}, F_2 \times Z_4, F_2 \times F_2[Y]/(Y^2), F_2 \times Z_9$ or $F_2 \times F_3[Y]/(Y^2)$

**Proof:** If $R \cong R_1 \times R_2 \times \ldots \times R_n$, where $R_i$ local ring for all $i = 1,2,\ldots,n$ and $n \geq 3$. The set $\{(1,0,\ldots,0),(0,1,0,\ldots,0),(1,1,0,\ldots,0),(0,0,\ldots,1),(1,1,\ldots,1)\} \subseteq \mathcal{V}(\mathcal{J}(R))$ so induced a sub-graph $K_5$, therefore $\mathcal{J}(R)$ is not planar. If $n = 2$, then $R \cong R_1 \times R_2$, where $R_1,R_2$ are local rings, there are three cases:
Case 1: If $R_1$ and $R_2$ are fields, then by Theorem 3.4 $\Lambda(R)$ is planar if and only if $R \cong F_2 \times F_{q_2}$ or $F_3 \times F_{q_2}$, where $F_{q_2}$ is a field order $q_2$.

Case 2: If $R_1$ and $R_2$ are not fields, then $|R_1|, |R_2| \geq 4$. Obviously $\Lambda(R)$ not planar.

Case 3: If $R_1$ is a field and $R_2$ not field. Let $R_2 = F_2$ or $F_3$ and $|Z(R_2)| = 2$, then $|R_2| = 4$, which implies that $R_2 \cong Z_4$ or $F_2[Y]/(Y^2)$, so $\Lambda(R)$ is planar see Fig. 3.2. If $Z(R_2) \geq 3$, then there exists $a, b \in Z(R_2)$, so that $ab = 0$. Therefore the vertices $(1, 0), (1, a), (1, b), (0, a), (0, b)$ are adjacent, whence $\Lambda(R)$ induces a sub-graph $K_5$, therefore $\Lambda(R)$ not planar. If $|R_1| \geq 4$, then it is easy to show that a graph $\Lambda(R)$ is not planar. Finally, if $n = 1$, then R is local and a complete proved it’s follow by proposition 3.3. and table 3.1.

References