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Variational Approximate Solutions of Fractional Delay Differential Equations with Integral Transform

Eman Mohmmed Nemah
Mathematics Department, Ibn Al-Haitham Education College for Pure Science,University of Baghdad, Baghdad, Iraq

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Abstract

The idea of the paper is to consolidate Mahgoub transform and variational iteration method (MTVIM) to solve fractional delay differential equations (FDDEs). The fractional derivative was in Caputo sense. The convergences of approximate solutions to exact solution were quick. The MTVIM is characterized by ease of application in various problems and is capable of simplifying the size of computational operations. Several non-linear (FDDEs) were analytically solved as illustrative examples and the results were compared numerically. The results for accentuating the efficiency, performance, and activity of suggested method were shown by comparisons with Adomian Decomposition Method (ADM), Laplace Adomian Decomposition Method (LADM), Modified Adomian Decomposition Method (MADM) and Homotopy Analysis Method (HAM).

Keywords: Variational Iteration Method; Mahgoub Transform; Delay Differential Equations of Fractional Order; Derivatives of Caputo; Approximate Solution.

الحلول التقريبية المتباينة لمعادلات تفاضلية تباطؤية كسورية مع تحويل تكاملي

إيمان محمد نعمة
قسم الرياضيات , كلية التربية للعلوم الصرفة / ابن الهيثم , جامعة بغداد ، بغداد، العراق

الخلاصة

الفكرة من البحث هي دمج تحويل محقوب مع الطريقة التكرارية للتباين (MTVIM) لحل معادلات تفاضلية تباطؤية كسورية (FDDEs). أن المشتقة الكسورية كانت بصيغة كابيتو. أن تقارب الحلول التقريبية الى الحل المضبوط كانت سريعة. تتميز طريقة MTVIM بسهولة التطبيق في مسائل متنوعة وانها قادرة على تبسيط العمليات الحسابية. تم حل العديد من المسائل الخطية وغير الخطية من نوع (FDDEs) تحليليا مع أمثلة توضيحية ومقارنة النتائج عدديا. ان النتائج قد بينت وأكدت كفاءة واداء وفعالية الطريقة المقترحة بالمقارنة مع الطرق ADM, LADM, HAM, MADM.

1. Introduction

Fractional differential equations (FDEs) have a great effectiveness on modeling processes in fluid mechanics, mathematical biology, physics, population growth, and so on [1-3]. The best way to describe scientific problems has been the usage of differential equations with fractional derivatives. The main importance of this kind of equations is illustrated in pure mathematics and applications, such as water flow in pipe, blood flow, the analysis of

*Email: eman.m.n@ihcoedu.uobaghdad.edu.iq

pollution, and many other applications [4-6]. It is known that the description of standard models mathematically in integer order is not adequate in various cases. The theory of delay differential equations (DDEs) describes a kind of functional differential equations which takes into account the history of a phenomenon [7]. The DDEs can be used to describe dynamical models of numerous phenomena in life [8]. Therefore, the FDDEs have a large number of applications and present remarkably efficient models to formulate scientific problems [9, 10].
It is important to note that finding exact solutions to non-linear FDDEs needs some new methods. The study of exact solution of FDDEs is a complex process. The approximate and numerical methods have been used to solve this class of equations, providing a good tool to solve some types of non-linear problems.
In the last three decades, many researchers focused on approximate solution of the FDDEs. Some approximate methods have been modified, such as Mahgoub Adomian decomposition method [11], solving fractional delay differential equations by new approach [12], iterative decomposition method [13], DGJ method [14], Laplace Adomian decomposition method [15], and homotopy analysis method [16].
The proposed method in this paper consists of Mahgoub transform and the variational iteration method. Mahgoub Transform was previously used with ADM to solve FDDEs [11]. In fact, we formulate the Mahgoub Transform to be associated with the variation iteration method to solve a class of non-linear of FDDEs.
The paper has the following structure ; Section 2 consists of main concepts of Mahgoub transform and fractional calculus. The MTVIM is presented in section 3. In section 4, numerical experiments of fractional order derivative are given to illustrate the efficiency of the considered method. Finally, in section 5, the conclusions are drawn.

2. Mahgoub Transform and Fractional Calculus

This section consists of essential concepts and definitions with features to Mahgoub transform and fractional calculus.

Definition 2.1 [14, 16]: Let $\omega(z)$ be a real valued function, $z > 0$, then $\omega(z)$ belongs to the space $C_{\mathfrak{w}}, \mathfrak{w} \in R$ if there is a real number $p > \mathfrak{w}$ such that $\omega(z) = z^p \omega_1(z)$, where $\omega_1(z) \in C[0, \infty]$, and it belongs to the space $C_{\mathfrak{w}}^r$ if $\omega^{(r)} \in C_{\mathfrak{w}}$, $r \in N \cup \{0\}$.

Definition 2.2 [14, 16]: The fractional integral of Riemann-Liouville of $\varsigma > 0$ of $\omega(z) \in C_{\mathfrak{w}}, \mathfrak{w} > -1$ is:

$$J^{\varsigma}\omega(z) = \frac{1}{\Gamma(\varsigma)} \int_0^z (z-\tau)^{\varsigma-1} \omega(\tau) d\tau, \quad z > 0 \qquad \dots (1)$$

$$J^0 \omega(z) = \omega(z)$$

Definition 2.3 [14, 17]: The fractional derivative of Caputo of $\omega(z) \in C_{\mathfrak{w}}$ is :

$$D_*^{\varsigma}\omega(z) = J^{r-\varsigma} D^r \omega(z) = \frac{1}{\Gamma(r-\varsigma)} \int_0^z (z-\tau)^{r-\varsigma-1} \omega^{(r)}(\tau) d\tau , \quad z > 0 \qquad \dots (2)$$

For $r - 1 < \varsigma \leq r, \ r \in N, \ z > 0, \ \omega \in C_{-1}^r$.

Definition 2.4 [17]: Let A be the set of continuous functions with exponential order and defined by:

$$A = \left\{ \omega(z) : \exists M, k_1, k_2 > 0, |\omega(z)| < M e^{\frac{|z|}{k_i}}, \ z \in (-1)^i \times [0, \infty), \ i = 1,2 \right\} \qquad \dots (3)$$

Let $\omega(z) \in, A$ then Mahgoub transform is defined as:

$$\mu[\omega(z)] = W(v) = v \int_0^{\infty} \omega(z) e^{-uz} \, dz, \ z \geq 0, k_1 \leq v \leq k_2 \qquad \dots (4)$$

In the following table we submit Mahgoub transform for some elementary functions:

Table 1-Mahgoub transform of some functions

$\omega(z)$	$\mu[\omega(z)]$
1	1
z	$\frac{1}{v}$
$z^n, \quad n \geq 1$	$\frac{n!}{v^n}$
e^{az}	$\frac{v}{v-a}$
e^{-az}	$\frac{v}{v+a}$
Sin az	$\frac{av}{v^2+a^2}$
Cos az	$\frac{v^2}{v^2+a^2}$

Theorem 2.5 [17]: Let $W(u)$ be the Mahgoub transform of the functions of Caputo fractional derivative of $\omega(z)$ of order ς, which is given by :

$$\mu\left[D_*^{\varsigma}\omega(z)\right] = v^{\varsigma}W(v) - \sum_{j=0}^{m-1} v^{\varsigma-j}\,\omega^{(j)}(0) \quad \ldots(5)$$

for $m-1 < \varsigma \leq m,\ m \in N$

3. The Mahgoub Variational Iteration Method (MVIM)

To explain the main idea, we study the fractional delay differential equation:

$$D_*^{\varsigma}\omega(z) = \mathfrak{f}(z) + N(\omega(z), \omega(\vartheta(z))), \quad m-1 < \varsigma \leq m,\ z > 0 \quad \ldots(6)$$

$$\omega^{(k)}(0) = w_0^{(k)}, k = 0, 1, \ldots, m-1 \quad \ldots(7)$$

where D_*^{ς} is derivative of order ς in Caputo sense, N is a non-linear bounded operator, $\mathfrak{f}(z)$ is a given continuous function, w is the unknown function, and $\vartheta(z)$ is the delay function.

The main steps involved are given as follows:

1. Take the Mahgoub transform $\mu[.]$of eq.(6),

$$\Rightarrow \mu\left[D_*^{\varsigma}\omega(z)\right] = \mu[\mathfrak{f}(z) + N(\omega(z), \omega(\vartheta(z))]$$

By using Theorem (2.5), the initial conditions and linearity of the Mahgoub transform are then expressed as:

$$v^{\varsigma}W(v) - \sum_{j=0}^{m-1} v^{\varsigma-j}\,\omega^{(j)}(0) = \mu[\mathfrak{f}(z)] + \mu[N(\omega(z), \omega(\vartheta(z))],$$

$$\Rightarrow v^{\varsigma}W(v) - v^{\varsigma}\omega(0) - v^{\propto-1}\omega^{(1)}(0) - v^{\varsigma-2}\omega^{(2)}(0) - \cdots - v^{\varsigma-(m-1)}\omega^{(m-1)}(0) = \mu[\mathfrak{f}(z)] + \mu[N(\omega(z), \omega(\vartheta(z))], \quad \ldots(8)$$

where $W(v) = \mu[\omega(z)] = v\int_0^{\infty}\omega(t)e^{-vt}dt.$

2. By multiplying eq. (8) with Lagrange multipliers and using iteration formula, we have:

$$W_{m+1}(v) = W_m(v) + \lambda(v)\Big[v^{\varsigma}W_m(v) - v^{\varsigma}\omega(0) - v^{\varsigma-1}\omega^{(1)}(0) - v^{\varsigma-2}\omega^{(2)}(0) - \cdots - v^{\propto-(m-1)}\omega^{(m-1)}(0) - \mu[\mathfrak{f}(z)] - \mu[N(\omega_m(z), \omega_m(\vartheta(z))]\Big] \quad \ldots(9)$$

3. Regarding the terms $\mu[N(\widetilde{\omega_m}(z), \widetilde{\omega_m}(\vartheta(z))]$ as restricted varations, we make eq. (9) stationary with respect to W_m:

$$\delta W_{m+1}(v) = \delta W_m(v) + \lambda(v)[v^{\varsigma}\delta W_m(v)] \quad \ldots(10)$$

From (10), we determine the Lagrange multiplier as:

$$\lambda(\upsilon) = -\frac{1}{\upsilon^{\varsigma}} \quad \dots (11)$$

4. The successive approximations are obtained by inverse Maghoub transform $\mu^{-1}[.]$.

$$\omega_{m+1}(z) = \omega_m(z) - \mu^{-1}\left[\frac{1}{\upsilon^{\varsigma}}\left[\upsilon^{\varsigma}W_m(u) - \upsilon^{\varsigma}\omega(0) - \upsilon^{\varsigma-1}\omega^{(1)}(0) - \upsilon^{\varsigma-2}\omega^{(2)}(0) - \cdots - \upsilon^{\varsigma-(m-1)}\omega^{(m-1)}(0) - \mu[\mathfrak{f}(z)] - \mu[N(\omega_m(z), \omega_m(\vartheta(z))]\right]\right] \quad \dots (12)$$

$$\Rightarrow \omega_{m+1}(z) = \omega(0) + \frac{1}{\upsilon}\omega^{(1)}(0) + \frac{1}{\upsilon^2}\omega^{(2)}(0) + \cdots + \frac{1}{\upsilon^{m-1}}\omega^{(m-1)}(0) + \mu^{-1}\left[\frac{1}{\upsilon^{\varsigma}}\mu[\mathfrak{f}(z)]\right] + \mu^{-1}\left[\frac{1}{\upsilon^{\varsigma}}\mu[N(\omega_m(z), \omega_m(\vartheta(z))]\right]$$

$$\left.\begin{aligned} \omega_0(z) &= \omega(0) + \tfrac{1}{\upsilon}\omega^{(1)}(0) + \tfrac{1}{\upsilon^2}\omega^{(2)}(0) + \cdots + \tfrac{1}{\upsilon^{m-1}}\omega^{(m-1)}(0) \\ \Rightarrow\ \omega_{m+1}(z) &= \omega(0) + \tfrac{1}{\upsilon}\omega^{(1)}(0) + \tfrac{1}{\upsilon^2}\omega^{(2)}(0) + \cdots + \tfrac{1}{\upsilon^{m-1}}\omega^{(m-1)}(0) +,\ \ m \geq 0 \\ &\mu^{-1}\left[\tfrac{1}{\upsilon^{\varsigma}}\mu[\mathfrak{f}(z)]\right] + \mu^{-1}\left[\tfrac{1}{\upsilon^{\varsigma}}\mu[N(\omega_m(z), \omega_m(\vartheta(z))]\right] \\ \omega(z) &= \lim_{m\to\infty}\omega_m(z) \end{aligned}\right\} \dots (13)$$

4. Applications and Results

In this part, some applications to the usage of MVIM in solving FDDE are introduced.

Problem 1: Consider the following linear FDDE:

$$D_*^{\varsigma}\omega(z) = \frac{2}{\Gamma(3-\alpha)}z^{2-\varsigma} - \frac{z^2}{9} + \omega\left(\frac{z}{3}\right),\ 0 < \varsigma \leq 1,\ \ 0 \leq z \leq 1. \quad \dots (14)$$

$$w(0) = 0, \quad \dots (15)$$

The exact solution is:

$$w(z) = z^2 \quad \dots (16)$$

Solution

Applying MT and using initial condition yield:

$$\Rightarrow \upsilon^{\varsigma}\mu[\omega(z)] = \mu\left[\frac{2}{\Gamma(3-\varsigma)}z^{2-\varsigma}\right] - \mu\left[\frac{z^2}{9}\right] + \mu\left[\omega\left(\frac{z}{3}\right)\right]$$

$$\Rightarrow \upsilon^{\varsigma}W(\upsilon) = \mu\left[\frac{2}{\Gamma(3-\varsigma)}z^{2-\varsigma}\right] - \mu\left[\frac{z^2}{9}\right] + \mu\left[\omega\left(\frac{z}{3}\right)\right]$$

By using Eq. (9):

$$\Rightarrow W_{n+1}(\upsilon) = W_n(\upsilon) + \lambda(\upsilon)\left(\upsilon^{\varsigma}W_n(\upsilon) - \mu\left[\frac{2}{\Gamma(3-\varsigma)}z^{2-\varsigma}\right] + \mu\left[\frac{z^2}{9}\right] - \mu\left[\omega_m\left(\frac{z}{3}\right)\right]\right),$$

By using Eq. (11):

$$\Rightarrow W_{n+1}(\upsilon) = \frac{1}{\upsilon^{\alpha}}\mu\left[\frac{2}{\Gamma(3-\varsigma)}z^{2-\varsigma}\right] - \frac{1}{\upsilon^{\varsigma}}\mu\left[\frac{z^2}{9}\right] + \frac{1}{\upsilon^{\varsigma}}\mu\left[\omega_m\left(\frac{z}{3}\right)\right]$$

By taking inverse MT"

$$\Rightarrow \omega_{m+1}(z) = \mu^{-1}\left[\frac{1}{\upsilon^{\varsigma}}\mu\left[\frac{2}{\Gamma(3-\varsigma)}z^{2-\varsigma}\right]\right] - \mu^{-1}\left[\frac{1}{\upsilon^{\varsigma}}\mu\left[\frac{z^2}{9}\right]\right] + \mu^{-1}\left[\frac{1}{\upsilon^{\varsigma}}\mu\left[\omega_m\left(\frac{z}{3}\right)\right]\right],$$

By using Eq. (13):

$$\left.\begin{aligned} &\omega_0(z) = \mu^{-1}\left[\frac{1}{\upsilon^{\varsigma}}\mu\left[\frac{2}{\Gamma(3-\varsigma)}z^{2-\varsigma}\right]\right] - \mu^{-1}\left[\frac{1}{\upsilon^{\varsigma}}\mu\left[\frac{z^2}{9}\right]\right] = z^2 - \frac{2}{9\,\Gamma(3+\varsigma)}z^{2+\varsigma} \\ \Rightarrow\ &\omega_{m+1}(z) = z^2 - \frac{2}{9\,\Gamma(3+\varsigma)}z^{2+\varsigma} + \mu^{-1}\left[\frac{1}{\upsilon^{\varsigma}}\mu\left[\omega_m\left(\frac{z}{3}\right)\right]\right] \qquad m \geq 0 \end{aligned}\right\}$$

We will calculate the components $\omega_1, \omega_2, \dots$

$$\omega_1(z) = z^2 - \frac{2}{9\,\Gamma(3+\varsigma)} z^{2+\varsigma} + \mu^{-1}\left[\frac{1}{v^{\varsigma}}\mu\left[\omega_0\left(\frac{z}{3}\right)\right]\right]$$
$$= z^2 - \frac{2}{9\,\Gamma(3+\varsigma)} z^{2+\varsigma} + \mu^{-1}\left[\frac{1}{v^{\varsigma}}\mu\left[\left(\frac{z}{3}\right)^2 - \frac{2}{9\,\Gamma(3+\varsigma)}\left(\frac{z}{3}\right)^{2+\varsigma}\right]\right]$$
$$= z^2 - \frac{2}{9\,\Gamma(3+\varsigma)} z^{2+\varsigma} + \mu^{-1}\left[\frac{1}{v^{\varsigma}}\left(\frac{2}{9v^2} - \frac{2}{3^{4+\varsigma}\,v^{2+\varsigma}}\right)\right]$$
$$= z^2 - \frac{2}{9\,\Gamma(3+\varsigma)} z^{2+\varsigma} + \mu^{-1}\left[\frac{2}{9v^{2+\varsigma}} - \frac{2}{3^{4+\varsigma}\,v^{2+2\varsigma}}\right]$$
$$= z^2 - \frac{2}{9\,\Gamma(3+\varsigma)} z^{2+\varsigma} + \frac{2}{9\,\Gamma(3+\varsigma)} z^{2+\varsigma} - \frac{2}{3^{4+\varsigma}\,\Gamma(3+2\varsigma)} z^{2+2\varsigma}$$

$$\omega_2(z) = z^2 - \frac{2}{9\,\Gamma(3+\varsigma)} z^{2+\varsigma} + \mu^{-1}\left[\frac{1}{v^{\varsigma}}\mu\left[\omega_1\left(\frac{z}{3}\right)\right]\right]$$
$$= z^2 - \frac{2}{9\,\Gamma(3+\varsigma)} z^{2+\varsigma} + \mu^{-1}\left[\frac{1}{v^{\varsigma}}\mu\left[\left(\frac{z}{3}\right)^2 - \frac{2}{9\,\Gamma(3+\varsigma)}\left(\frac{z}{3}\right)^{2+\varsigma}\right]\right]$$
$$+ \mu^{-1}\left[\frac{1}{v^{\varsigma}}\mu\left[\frac{2}{9\,\Gamma(3+\varsigma)}\left(\frac{z}{3}\right)^{2+\varsigma} - \frac{2}{3^{4+\varsigma}\,\Gamma(3+2\varsigma)}\left(\frac{z}{3}\right)^{2+2\varsigma}\right]\right]$$
$$= z^2 - \frac{2}{9\,\Gamma(3+\varsigma)} z^{2+\varsigma} + \mu^{-1}\left[\frac{1}{v^{\varsigma}}\left(\frac{2}{9v^2} - \frac{2}{3^{4+\varsigma}\,v^{2+\varsigma}}\right)\right]$$
$$+ \mu^{-1}\left[\frac{1}{v^{\varsigma}}\left(\frac{2}{3^{4+\varsigma}\,v^{2+\varsigma}} - \frac{2}{3^{6+3\varsigma}\,v^{2+2\varsigma}}\right)\right]$$
$$= z^2 - \frac{2}{9\,\Gamma(3+\varsigma)} z^{2+\varsigma} + \mu^{-1}\left[\frac{2}{9v^{2+\varsigma}} - \frac{2}{3^{4+\varsigma}\,v^{2+2\varsigma}}\right] + \mu^{-1}\left[\frac{2}{3^{4+\varsigma}\,v^{2+2\varsigma}} - \frac{2}{3^{6+3\varsigma}\,v^{2+3\varsigma}}\right]$$
$$= z^2 - \frac{2}{9\,\Gamma(3+\varsigma)} z^{2+\varsigma} + \frac{2}{9\,\Gamma(3+\varsigma)} z^{2+\varsigma} - \frac{2}{3^{4+\varsigma}\,\Gamma(3+2\varsigma)} z^{2+2\varsigma}$$
$$+ \frac{2}{3^{4+\varsigma}\,\Gamma(3+2\varsigma)} z^{2+2\varsigma} - \frac{2}{3^{6+3\varsigma}\,\Gamma(3+3\varsigma)} z^{2+3\varsigma}$$

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$$\omega(z) = \lim_{r\to\infty} \omega_r(z) = z^2 - \frac{2}{9\,\Gamma(3+\varsigma)} z^{2+\varsigma} + \frac{2}{9\,\Gamma(3+\varsigma)} z^{2+\varsigma} - \frac{2}{3^{4+\varsigma}\,\Gamma(3+2\varsigma)} z^{2+2\varsigma}$$
$$+ \frac{2}{3^{4+\varsigma}\,\Gamma(3+2\varsigma)} z^{2+2\varsigma} - \frac{2}{3^{6+3\varsigma}\,\Gamma(3+3\varsigma)} z^{2+3\varsigma} + \cdots$$

The approximate solution in a series form of (14)-(15), when $\varsigma = 1$, is given by

$$\omega(z) = \lim_{r\to\infty} \omega_r(z) = z^2$$

which is the exact solution of (14) -(15).

As shown in table (2), the proposed solutions using MVIM are better from the solutions acquire by ADM, LADM, and the exact solution.

Table 2-Approximate solution of problem 1 for $\varsigma = 1$ by using the proposed method (MVIM) and comparison with ADM, LADM and Exact Solution

z	MVIM	ADM	LADM	Exact

0.1	0.01	0.01	0.01	0.01
0.2	0.04	0.04	0.04	0.04
0.3	0.09	0.09	0.09	0.09
0.4	0.16	0.16	0.16	0.16
0.5	0.25	0.25	0.25	0.25
0.6	0.36	0.36	0.36	0.36
0.7	0.49	0.49	0.49	0.49
0.8	0.64	0.64	0.64	0.64
0.9	0.81	0.81	0.81	0.81

Problem 2: Consider the following nonlinear FDDE:

$$D_*^{\varsigma}\omega(z) = 1 - 2\omega^2\left(\frac{z}{2}\right) \quad 0 < \varsigma \leq 1, \quad 0 \leq z \leq 1. \qquad \ldots (17)$$

$$\omega(0) = 0, \qquad \ldots. (18)$$

The exact solution is:

$$\omega(z) = sinz \qquad \ldots (19)$$

Solution

Applying MT and using initial condition yields:

$$\Rightarrow v^{\varsigma}\mu[\omega(z)] = 1 - \mu\left[2\omega^2\left(\frac{z}{2}\right)\right],$$

By using Eq. (9), we obtain:

$$\Rightarrow W_{n+1}(v) = W_n(v) + \lambda(v)\left(v^{\varsigma}W_n(v) - 1 + \mu\left[2{\omega_n}^2\left(\frac{z}{2}\right)\right]\right),$$

By using Eq. (11):

$$\Rightarrow W_{n+1}(v) = \frac{1}{v^{\varsigma}} - \frac{1}{v^{\varsigma}}\mu\left[2\omega_n^2\left(\frac{z}{2}\right)\right]$$

Taking inverse MT yields:

$$\Rightarrow \omega_{n+1} = \mu^{-1}\left[\frac{1}{v^{\varsigma}}\right] - \mu^{-1}\left[\frac{1}{v^{\varsigma}}\mu\left[2\omega_n^2\left(\frac{z}{2}\right)\right]\right],$$

By using Eq. (13), we obtain:

$$\left.\begin{array}{l} \omega_0(z) = \mu^{-1}\left[\frac{1}{v^{\varsigma}}\right] = \frac{z^{\varsigma}}{\Gamma(\varsigma+1)} \\ \Rightarrow \\ \omega_{n+1}(z) = \frac{z^{\varsigma}}{\Gamma(\varsigma+1)} - \mu^{-1}\left[\frac{1}{v^{\varsigma}}\mu\left[2\omega_n^2\left(\frac{z}{2}\right)\right]\right] \qquad n \geq 0 \end{array}\right\}$$

We will calculate the components $\omega_1, \omega_2, \ldots$

$$\omega_1(z) = \frac{z^{\varsigma}}{\Gamma(\varsigma+1)} - \mu^{-1}\left[\frac{1}{v^{\varsigma}}\mu\left[2\omega_0^2\left(\frac{z}{2}\right)\right]\right]$$

$$= \frac{z^{\varsigma}}{\Gamma(\varsigma+1)} - \mu^{-1}\left[\frac{1}{v^{\varsigma}}\mu\left[2\left(\frac{\left(\frac{z}{2}\right)^{\varsigma}}{\Gamma(\varsigma+1)}\right)^2\right]\right]$$

$$= \frac{z^{\varsigma}}{\Gamma(\varsigma+1)} - \mu^{-1}\left[\frac{1}{v^{\varsigma}}2\frac{\Gamma(2\varsigma+1)}{2^{2\varsigma}\,\Gamma^2(\varsigma+1)v^{2\varsigma}}\right]$$

$$= \frac{z^{\varsigma}}{\Gamma(\varsigma+1)} - \mu^{-1}\left[\frac{2\,\Gamma(2\varsigma+1)}{2^{2\varsigma}\,\Gamma^2(\varsigma+1)v^{3\varsigma}}\right]$$

$$= \frac{z^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{2\,\Gamma(2\varsigma+1)z^{3\varsigma}}{2^{2\varsigma}\,\Gamma^2(\varsigma+1)\Gamma(3\varsigma+1)}$$

$$w_2(z) = \frac{z^{\varsigma}}{\Gamma(\varsigma+1)} - \mu^{-1}\left[\frac{1}{v^{\varsigma}}\mu\left[2w_1^2\left(\frac{z}{2}\right)\right]\right]$$

$$= \frac{z^{\varsigma}}{\Gamma(\varsigma+1)} - \mu^{-1}\left[\frac{1}{v^{\varsigma}}\mu\left[2\left(\frac{\left(\frac{z}{2}\right)^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{2\,\Gamma(2\varsigma+1)\left(\frac{z}{2}\right)^{3\varsigma}}{2^{2\varsigma}\,\Gamma^2(\varsigma+1)\Gamma(3\varsigma+1)}\right)^2\right]\right]$$

$$= \frac{z^{\varsigma}}{\Gamma(\varsigma+1)} - \mu^{-1}\left[\frac{1}{v^{\varsigma}}\mu\left[2\left(\frac{\left(\frac{z}{2}\right)^{\varsigma}}{\Gamma(\varsigma+1)}\right)^2\right]\right] + \mu^{-1}\left[\frac{1}{v^{\varsigma}}\mu\left[\frac{-8\,\Gamma(2\varsigma+1)\left(\frac{z}{2}\right)^{4\varsigma}}{2^{2\varsigma}\,\Gamma^3(\varsigma+1)\Gamma(3\varsigma+1)}\right]\right]$$
$$- \mu^{-1}\left[\frac{1}{v^{\varsigma}}\mu\left[2\left(\frac{-2\,\Gamma(2\varsigma+1)\left(\frac{z}{2}\right)^{3\varsigma}}{2^{2\varsigma}\,\Gamma^2(\varsigma+1)\Gamma(3\varsigma+1)}\right)^2\right]\right]$$

$$= \frac{z^{\varsigma}}{\Gamma(\varsigma+1)} - \mu^{-1}\left[\frac{1}{v^{\varsigma}}2\frac{\Gamma(2\varsigma+1)}{2^{2\varsigma}\,\Gamma^2(\varsigma+1)v^{2\varsigma}}\right] + \mu^{-1}\left[\frac{1}{v^{\varsigma}}\frac{-8\Gamma(4\varsigma+1)}{2^{6\varsigma}\,\Gamma^3(\varsigma+1)\Gamma(3\varsigma+1)v^{4\varsigma}}\right]$$
$$- \mu^{-1}\left[\frac{1}{v^{\varsigma}}\frac{-8\Gamma^2(2\varsigma+1)\Gamma(6\varsigma+1)}{2^{10\varsigma}\,\Gamma^4(\varsigma+1)\Gamma^2(3\varsigma+1)v^{6\varsigma}}\right]$$

$$= \frac{z^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{2\,\Gamma(2\varsigma+1)z^{3\varsigma}}{2^{2\varsigma}\,\Gamma^2(\varsigma+1)\Gamma(3\varsigma+1)} + \frac{8\,\Gamma(2\varsigma+1)\,\Gamma(4\varsigma+1)z^{5\varsigma}}{2^{6\varsigma}\,\Gamma^3(\varsigma+1)\Gamma(3\varsigma+1)\,\Gamma(5\varsigma+1)}$$
$$- \frac{8\Gamma^2(2\varsigma+1)\Gamma(6\varsigma+1)z^{7\varsigma}}{2^{10\varsigma}\,\Gamma^4(\varsigma+1)\Gamma^2(3\varsigma+1)\Gamma(7\varsigma+1)}$$

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$$\omega(z) = \lim_{r\to\infty}\omega_r(z)$$
$$= \frac{z^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{2\,\Gamma(2\varsigma+1)z^{3\varsigma}}{2^{2\varsigma}\,\Gamma^2(\varsigma+1)\Gamma(3\varsigma+1)} + \frac{8\,\Gamma(2\varsigma+1)\,\Gamma(4\varsigma+1)z^{5\varsigma}}{2^{6\varsigma}\,\Gamma^3(\varsigma+1)\Gamma(3\varsigma+1)\,\Gamma(5\varsigma+1)}$$
$$- \frac{8\Gamma^2(2\varsigma+1)\Gamma(6\varsigma+1)z^{7\varsigma}}{2^{10\varsigma}\,\Gamma^4(\varsigma+1)\Gamma^2(3\varsigma+1)\Gamma(7\varsigma+1)} + \cdots$$

The approximate solution in the series form of (17)-(18), when $\varsigma = 1$, is given by

$$\omega(z) = \lim_{r\to\infty}\omega_r(z) = \sin z$$

which is the exact solution of (17) -(18).

As shown in table (3), the proposed method (MVIM) solutions are better from the solutions acquire by HAM, ADM, LADM, MADM and, the exact solution.

Table 3 - Approximate solution of problem 2 for $\varsigma = 1$ by using proposed method (MVIM) and comparison with HAM, ADM, LADM, MADM and Exact Solution

z	MVIM	HAM	ADM	LADM	MADM	Exact
0.1	0.0998334	0.100	0.100	0.100	0.099833	0.0998334
0.2	0.198669	0.199	0.199	0.199	0.198669	0.198669
0.3	0.29552	0.296	0.296	0.296	0.29552	0.29552
0.4	0.389418	0.389	0.389	0.389	0.389419	0.389418
0.5	0.479426	0.479	0.479	0.479	0.479427	0.479426
0.6	0.564645	0.565	0.565	0.565	0.564648	0.564642
0.7	0.644224	0.644	0.644	0.644	0.644234	0.644218

0.8	0.717371	0.717	0.717	0.717	0.717397	0.717356
0.9	0.783361	0.783	0.783	0.783	0.783421	0.783327

Problem 3: Consider the following linear FDDE:

$$D_*^{\varsigma}\omega(z) = \frac{3}{4}\omega(z) + \omega\left(\frac{z}{2}\right) - z^2 + 2,\ 1 < \varsigma \le 2,\ \ 0 \le z \le 1. \quad \ldots (20)$$

$$\omega(0) = 0, \quad \omega^{(1)}(0) = 0 \quad \ldots (21)$$

The exact solution is:

$$\omega(z) = z^2 \quad \ldots (22)$$

Solution

Applying MT and using initial condition yield:

$$\Rightarrow v^{\varsigma}\mu[\omega(z)] = \mu[-z^2 + 2] + \mu\left[\frac{3}{4}\omega(z) + \omega\left(\frac{z}{2}\right)\right],$$

$$\Rightarrow v^{\varsigma}\mu[\omega(z)] = 2 - \frac{2}{v^2} + \mu\left[\frac{3}{4}\omega(z) + \omega\left(\frac{z}{2}\right)\right]$$

$$\Rightarrow v^{\varsigma}W(v) = 2 - \frac{2}{v^2} + \mu\left[\frac{3}{4}\omega(z) + \omega\left(\frac{z}{2}\right)\right]$$

By using Eq. (9):

$$\Rightarrow W_{n+1}(v) = W_n(v) + \lambda(v)\left(v^{\varsigma}W_n(v) - 2 + \frac{2}{v^2} - \mu\left[\frac{3}{4}\omega_n(z) + \omega_n\left(\frac{z}{2}\right)\right]\right),$$

By using Eq. (11):

$$\Rightarrow W_{n+1}(v) = \frac{2}{v^{\varsigma}} - \frac{2}{v^{\varsigma+2}} + \frac{1}{v^{\varsigma}}\ \mu\left[\frac{3}{4}\omega_n(z) + \omega_n\left(\frac{z}{2}\right)\right]$$

Taking the inverse MT yields:

$$\Rightarrow \omega_{m+1}(z) = \mu^{-1}\left[\frac{2}{v^{\varsigma}} - \frac{2}{v^{\varsigma+2}}\right] + \mu^{-1}\left[\frac{1}{v^{\varsigma}}\mu\left[\frac{3}{4}\omega_n(z) + \omega_n\left(\frac{z}{2}\right)\right]\right],$$

By using Eq. (13):

$$\left.\begin{aligned} &\omega_0(z) = \mu^{-1}\left[\frac{2}{v^{\varsigma}} - \frac{2}{v^{\varsigma+2}}\right] = \frac{2z^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{2z^{\varsigma+2}}{\Gamma(\varsigma+3)} \\ \Rightarrow\ &\omega_{n+1}(z) = \frac{2z^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{2z^{\varsigma+2}}{\Gamma(\varsigma+3)} + \mu^{-1}\left[\frac{1}{v^{\varsigma}}\mu\left[\frac{3}{4}\omega_n(z) + \omega_n\left(\frac{z}{2}\right)\right]\right] \qquad n \ge 0 \end{aligned}\right\}$$

We will calculate the components $\omega_1, \omega_2, \ldots$

$$\omega_1(z) = \frac{2z^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{2z^{\varsigma+2}}{\Gamma(\varsigma+3)} + \mu^{-1}\left[\frac{1}{v^{\varsigma}}\mu\left[\frac{3}{4}\omega_0(z) + \omega_0\left(\frac{z}{2}\right)\right]\right]$$

$$= \frac{2z^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{2z^{\varsigma+2}}{\Gamma(\varsigma+3)} + \mu^{-1}\left[\frac{1}{v^{\varsigma}}\mu\left[\frac{3}{4}\left(\frac{2z^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{2z^{\varsigma+2}}{\Gamma(\varsigma+3)}\right) + \frac{2\left(\frac{z}{2}\right)^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{2\left(\frac{z}{2}\right)^{\varsigma+2}}{\Gamma(\varsigma+3)}\right]\right]$$

$$= \frac{2z^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{2z^{\varsigma+2}}{\Gamma(\varsigma+3)} + \mu^{-1}\left[\frac{3}{4v^{\varsigma}}\mu\left[\frac{2z^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{2z^{\varsigma+2}}{\Gamma(\varsigma+3)}\right] + \frac{1}{v^{\varsigma}}\mu\left[\frac{2z^{\varsigma}}{2^{\varsigma}\,\Gamma(\varsigma+1)} - \frac{2z^{\varsigma+2}}{2^{\varsigma+2}\Gamma(\varsigma+3)}\right]\right]$$

$$= \frac{2z^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{2z^{\varsigma+2}}{\Gamma(\varsigma+3)} + \mu^{-1}\left[\frac{3}{4v^{\varsigma}}\left(\frac{2}{v^{\varsigma}} - \frac{2}{v^{\varsigma+2}}\right) + \frac{1}{v^{\varsigma}}\left(\frac{2}{2^{\varsigma}v^{\varsigma}} - \frac{2}{2^{\varsigma+2}v^{\varsigma+2}}\right)\right]$$

$$= \frac{2z^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{2z^{\varsigma+2}}{\Gamma(\varsigma+3)} + \mu^{-1}\left[\frac{3}{4}\left(\frac{2}{v^{2\varsigma}} - \frac{2}{v^{2\varsigma+2}}\right) + \left(\frac{2}{2^{\varsigma}v^{2\varsigma}} - \frac{2}{2^{\varsigma+2}v^{2\varsigma+2}}\right)\right]$$

$$= \frac{2z^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{2z^{\varsigma+2}}{\Gamma(\varsigma+3)} + \frac{3}{2}\left(\frac{z^{2\varsigma}}{\Gamma(2\varsigma+1)} - \frac{z^{2\varsigma+2}}{\Gamma(2\varsigma+3)}\right) + \frac{2}{2^{\varsigma}}\left(\frac{z^{2\varsigma}}{\Gamma(2\varsigma+1)} - \frac{z^{2\varsigma+2}}{4\Gamma(2\varsigma+3)}\right)$$

$$\omega_2(z) = \frac{2z^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{2z^{\varsigma+2}}{\Gamma(\varsigma+3)} - \mu^{-1}\left[\frac{1}{v^{\varsigma}} M\left[\frac{3}{4}\omega_1(z) + \omega_1\left(\frac{z}{2}\right)\right]\right]$$

$$= \frac{2z^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{2z^{\varsigma+2}}{\Gamma(\varsigma+3)} + \mu^{-1}\left[\frac{1}{v^{\varsigma}}\mu\left[\frac{3}{4}\left(\frac{2z^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{2z^{\varsigma+2}}{\Gamma(\varsigma+3)} + \frac{3}{2}\left(\frac{z^{2\varsigma}}{\Gamma(2\varsigma+1)} - \frac{z^{2\varsigma+2}}{\Gamma(2\varsigma+3)}\right)\right.\right.\right.$$
$$\left.\left.\left. + \frac{2}{2^{\varsigma}}\left(\frac{z^{2\varsigma}}{\Gamma(2\varsigma+1)} - \frac{z^{2\varsigma+2}}{4\Gamma(2\varsigma+3)}\right)\right) + \left(\frac{2\left(\frac{z}{2}\right)^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{2\left(\frac{z}{2}\right)^{\varsigma+2}}{\Gamma(\varsigma+3)} + \frac{3}{2}\left(\frac{\left(\frac{z}{2}\right)^{2\varsigma}}{\Gamma(2\varsigma+1)} - \frac{\left(\frac{z}{2}\right)^{2\varsigma+2}}{\Gamma(2\varsigma+3)}\right)\right.\right.$$
$$\left.\left.\left. + \frac{2}{2^{\varsigma}}\left(\frac{\left(\frac{z}{2}\right)^{2\varsigma}}{\Gamma(2\varsigma+1)} - \frac{\left(\frac{z}{2}\right)^{2\varsigma+2}}{4\Gamma(2\varsigma+3)}\right)\right)\right]\right]$$

$$= \frac{2z^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{2z^{\varsigma+2}}{\Gamma(\varsigma+3)} + \frac{3}{2}\left(\frac{z^{2\varsigma}}{\Gamma(2\varsigma+1)} - \frac{z^{2\varsigma+2}}{\Gamma(2\varsigma+3)}\right) + \frac{2}{2^{\varsigma}}\left(\frac{z^{2\varsigma}}{\Gamma(2\varsigma+1)} - \frac{z^{2\varsigma+2}}{4\Gamma(2\varsigma+3)}\right)$$
$$+ \frac{9}{8}\frac{z^{3\varsigma}}{\Gamma(3\varsigma+1)} - \frac{9}{8}\frac{z^{3\varsigma+2}}{\Gamma(3\varsigma+3)} + \frac{3}{2^{\varsigma+1}}\frac{z^{3\varsigma}}{\Gamma(3\varsigma+1)} - \frac{3}{2^{\varsigma}}\frac{z^{3\varsigma+2}}{\Gamma(3\varsigma+3)}$$
$$+ \frac{3}{2^{2\varsigma+1}}\frac{z^{3\varsigma}}{\Gamma(3\varsigma+1)} - \frac{3}{2^{2\varsigma+2}}\frac{z^{3\varsigma+2}}{\Gamma(3\varsigma+3)} + \frac{2}{2^{3\varsigma}}\frac{z^{3\varsigma}}{\Gamma(3\varsigma+1)} - \frac{2}{2^{3\varsigma+4}}\frac{z^{3\varsigma+2}}{\Gamma(3\varsigma+3)}$$

$$\vdots$$

$$\omega(z) = \lim_{m\to\infty}\omega_m(z)$$
$$= \frac{2z^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{2z^{\varsigma+2}}{\Gamma(\varsigma+3)} + \frac{3}{2}\left(\frac{z^{2\varsigma}}{\Gamma(2\varsigma+1)} - \frac{z^{2\varsigma+2}}{\Gamma(2\varsigma+3)}\right)$$
$$+ \frac{2}{2^{\varsigma}}\left(\frac{z^{2\varsigma}}{\Gamma(2\varsigma+1)} - \frac{z^{2\varsigma+2}}{4\Gamma(2\varsigma+3)}\right) + \frac{9}{8}\frac{z^{3\varsigma}}{\Gamma(3\varsigma+1)} - \frac{9}{8}\frac{z^{3\varsigma+2}}{\Gamma(3\varsigma+3)}$$
$$+ \frac{3}{2^{\varsigma+1}}\frac{z^{3\varsigma}}{\Gamma(3\varsigma+1)} - \frac{3}{2^{\varsigma}}\frac{z^{3\varsigma+2}}{\Gamma(3\varsigma+3)} + \frac{3}{2^{2\varsigma+1}}\frac{z^{3\varsigma}}{\Gamma(3\varsigma+1)} - \frac{3}{2^{2\varsigma+2}}\frac{z^{3\varsigma+2}}{\Gamma(3\varsigma+3)}$$
$$+ \frac{2}{2^{3\varsigma}}\frac{z^{3\varsigma}}{\Gamma(3\varsigma+1)} - \frac{2}{2^{3\varsigma+4}}\frac{z^{3\varsigma+2}}{\Gamma(3\varsigma+3)} + \cdots$$

The approximate solution in the series form of (20)-(21), when $\varsigma = 2$, is given by

$$\omega(z) = \lim_{r\to\infty}\omega_r(z) = z^2$$

which is the exact solution of (20) -(21).

As shown in table (4), the proposed method (MVIM) solutions are better from the solutions acquire by HAM, ADM, MADM, and the exact solution.

Table 4- Approximate solution of problem 3 for $\varsigma = 2$ using the proposed method (MVIM) and comparison with HAM, ADM, MADM and the exact solution.

z	MVIM	HAM	ADM	MADM	Exact
0.1	0.01	0.01	0.01	0.009999	0.01
0.2	0.04	0.04	0.04	0.039999	0.04
0.3	0.09	0.09	0.09	0.089999	0.09
0.4	0.16	0.16	0.16	0.159999	0.16
0.5	0.25	0.25	0.25	0.249999	0.25
0.6	0.36	0.36	0.36	0.359999	0.36
0.7	0.49	0.49	0.49	0.489998	0.49
0.8	0.64	0.64	0.64	0.63995	0.64
0.9	0.81	0.81	0.809	0.809987	0.81

5. Conclusions

This paper comprises a new technique that involves the employment of Mahgoub transform with the variational iteration method to solve non-linear FDDEs. The aim of the algorithm is to acquire the not-required unreal suppositions. In example 1, table 2, we achieved the same results obtained in the exact solution. Also, the results were similar to those obtained by the ADM and, LADM. In Example 2, table 3, we reached the result that: the values of t from 0.1 to 0.5 have the same results obtained by the exact solution, while the values for 0.4 and 0.5 are better than those obtained by MADM. The values of t from 0.6 to 0.9 were closer to the values achieved by the exact solution than those by the MADM. Hence, our method showed better results than MADM and all other methods. In example 3, table 4, we found that MVIM gives the exact solution to the problem and the results are similar to those reached by the HAM and ADM, while it was better than MADM.

In conclusion, we demonstrated that the method has an excellent effectiveness in solving FDDEs.

References

[1] Hanan F. Q. and Ghada H. I. "Precise solution of a viscoelastic fluid flow in an annular piple under an impulsive pressure with fractional generalized burgers model". *International Journal of Science and Research*, vol. 6, no. 6, pp. 801-805, 2015.

[2] Garmichael B., Babahosseini H., Mahmoodi S. N., and Agah M. "The fractional viscoelastic response of human breast tissue cells". *IOP Publishing Phys. Biol*., v0l. 12, pp. 1-14, 2015.

[3] Ricardo A., Nuno R.O.B., and Teresa M. T. M. "Modelling some real phenomena by fractional differential equations". *Mathematical Methods in the Applied Sciences*, vol. 13, pp. 1-12, 2015.

[4] Sundos B., Suad N. K., and Ahmed M.A "Influence of inclined MHD on unsteady flow fractional derivative between two inclined coaxial cylinders through a porous medium". *Iraqi Journal of Science*, vol. 61, no. 7, pp.1705-1714, 2020.

[5] Sundos B.H. and Ahmed M.A. "Effects of MHD on the unsteady thin flow of non-new tonian oldroyd-B fluid over an oscillating inclined belt through a porous medium". *IOSR Journal of Mathematics*, vol. 12, no. 4, pp. 28-40, 2016.

[6] Eman M.N. "Homotopy Transforms analysis method for solving fractional Navier-Stokes equations with applications", *Iraqi Journal of Science*, vol. 61, no. 8, pp. 2048-2054, 2020. DOI:10.24996/ijs.2020.61.8.20.

[7] Amit P., Doddabhadrappla G. P. and Pundikala Hussain A.M. and Aqeel F.J. **2020**. "Asymptotic criteria of neutral differential equation with positive and negative coefficients and impulsive integral term". *Iraqi Journal of Science*, vol. 61, no. 9, pp. 3215-2323, 2020. DOI:10.24996/ ijs.2020.61.9.18.

[8] Magdi S.M. **2017**. "Recent progress in stability and stabilization of systems with time- delay". *Mathematical Problems in Engineering*, pp:1-25, 2017.

[9] Bhalekar S., Daftardar-Gejji V., Baleanu D. and Magin R. "Generalized fractional order Bloch equation with extended delay". *International J. of Bifurcation and Chaos*, vol. 22, no. 4, pp. 1-15, 2012.

[10] Sweilam N.H., Nagy A.M., Assiri T.A., and Ali N.Y. "Numerical simulation for variable-order fractional non-linear delay differential equation". *Journal of Fractional Calculus and Applications*, vol. 6, no. 1, pp. 71-82, 2015.

[11] Gethsi R.S. and Katrin S.L. "Numerical solution of fractional delay differential equations using Mahgoub Adomian decomposition method". *The International Journal of analytical and experimental modal analysis*, vol. XI, no. XIII, pp. 2189-2196, 2019.

[12] Varsha D., Yogita S. and Sachin B. "Solving fractional delay differential equations: A new approach". *Fract. Calc. Appl. Anal*., vol. 18, no. 2, pp. 400-418, 2015.

[13] Olutnde S. O., Abass I.T. and Ousola A.D. "An approximation technique for fractional order delay differential equations". Iraqi Journal of Science, vol. 60, no. 7, pp. 1539-1545, 2019. DOI: 10.249961/ijs.2019.60.7.14.

[14] Aman J. and Varsha D. "A new numerical method for solving fractional delay differential equations". Compulational and Applied Mathematics, vol. 166, pp. 1-18, 2019.

[15] Zaid A.M. "Computational method based Laplace Adomian decomposition for solving delay differential equations of fractional order". *Journal of Al-Nahrain University*, vol. 21, no.1, pp. 128-133, 2018.

[16] Osama H.M. and Abbas I.K. "Homotopy analysis method for solving delay differential equations of fractional order". *Mathematical Theory and Modeling*, vol. 4, no. 14, pp. 48-56, 2014.

[17] Eminal K.P.A. and Katrin L.S. "Mahgoub transform method for solving linear fractional differential equations". *International Journal of Mathematics Trends and Technology* (IJMTT), vol. 58, no. 4, pp. 253-257, 2018.