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COMMON FIXED POINT OF JUNGCK PICARD ITRATIVE FOR TWO WEAKLY COMPATIBLE SELF-MAPPINGS

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Abstract
We develop the previously published results of Arab by using the function $\rho: \mathbb{R}_+^6 \to [0, \infty)$ under certain conditions and using G-α-general admissible and triangular α-general admissible to prove coincidence fixed point and common fixed point theorems for two weakly compatible self –mappings in complete b-metric spaces.

Keywords: Common Fixed Point, b-Metric Space, Coincidence Points, Weakly Compatible Self-Mapping, Upper Semi-Continuous Mapping.

النقط الصامدة المشتركة لتكرار من النمط جانك–بكارد لاثنين من التطبيقات الذاتية المتوافقة بشكل ضعيف

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الخلاصة
نحن قمنا بتطوير نتيجة عرب باستخدام الدالة $\rho: \mathbb{R}_+^6 \to [0, \infty)$ تحت شروط معينة واستخدام G−α− general admissible و triangular α−general admissible لاثبات تطابق النقطة الصامدة ونظريات النقطة الصامدة المشتركة لاثنين من التطبيقات الذاتية المتوافقة بشكل ضعيف على فضاءات مترية تامة من نمط b.

1. Introduction

The fixed point theory is a very useful tool to solve many kinds of equations; for more detail see [1, 2].

The fixed point theory is an interesting mixture of analyses (pure and applied), topology, and geometry, which is one of the most powerful and fruitful tools of modern mathematics and may be considered a core subject of nonlinear functional analysis.

In 1976, Jungck [3] proved a common fixed point theorem for commuting maps, generalizing the Banach's fixed point theorem. In 1982, Sessa [4] defined weak commutativity and proved common fixed point theorem for weakly commuting maps. Further, in 1986, the compatibility concept was introduced by Jungck [5]. Many authors studied various fixed point theorems for compatible mappings satisfying contractive type conditions and assuming continuity for at least one of the mappings [6]. It has been known from the paper of Kannan [7] that there exist discontinuous maps which have fixed points. In 1998, Jungck and Rhoades [8] presented the concept of weakly compatible self-mappings;

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recall that the pair of $\text{self} - \text{mappings}$ (F, G) on a nonempty set X is said to be weakly compatible if $Fx = Gx$ for some x in X, then

$$FGx = GFx$$

Note that compatible maps are weakly compatible but the converse needs not to be true.

2. Preliminaries

The aim of this section is to present some notions and propositions used in the paper.

Proposition 2.1 [9]. Let F and G be weakly compatible self maps of a set X. If F and G have a unique fixed point of coincidence, say $w = Fx = Gx$, then w is the unique common fixed point of F and G.

Definition 2.2 [10, 11]. Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}_+ = [0, \infty)$ is a b −metric on X, if for all $x, y, z \in X$, the following conditions hold:

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$, ($b$ −triangular inequality).

Then, the triple (X, d, s) is called a b −metric space.

When $s = 1$, the b-metric space is a metric space while if $s > 1$, the b-metric space is not a metric space.

Example 2.3 [12]. Let $X = \{x_1, x_2, x_3, x_4\}$ and $m \geq 1$. Define $d : X \times X \to \mathbb{R}_+$ as

$$d(x_i, x_j) = \begin{cases} 0 & \text{if } i = j\ ;\ i, j = 1, 2, 3, 4. \\ m & \text{if } i = 1, j = 2 \text{ or vice versa} \\ 1 & \quad\quad o.w \end{cases}$$

Thus for all $n, i, j = 1, 2, 3, 4$,

$$d(x_i, x_j) \leq \frac{m}{2}\left[d(x_i, x_n) + d(x_n, x_j)\right].$$

Note that when $m > 2$, we get an example of a b −metric space which is not a metric space.

Definition 2.4 [13]. Let (X, d, s) be a b-metric space and $\{x_n\}$ is a sequence in X . Then:

1) $\{x_n\}$ is A convergent sequence if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$, written as $\lim_{n\to\infty} x_n = x$.

2) $\{x_n\}$ is called Cauchy sequence if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

3) A b-metric space (X, d, s) is said to be a complete b-metric space if every Cauchy sequence in X is convergent.

Definition 2.5 [9]. Let (X, d, s) be a b −metric space. Then a subset $Y \subset X$ is called b −closed (for simplicity we call it closed), if and only if for each sequence $\{x_n\}$ in Y which converges to an element x, then x is in Y.

Lemma 2.6 [15]. Let (X, d, s) be a b −metric space and $\{x_n\}$ be a sequence in X such that:

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$$

If $\{x_n\}$ is not a Cauchy sequence, then there are two subsequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers with $m(k) \leq n(k)$, such that the following four sequences:

$$\{d(x_{m(k)}, x_{n(k)})\}, \{d(x_{m(k)}, x_{n(k)+1})\},$$
$$\{d(x_{m(k)+1}, x_{n(k)})\}, \{d(x_{m(k)+1}, x_{n(k)+1})\}$$

Tend to $\epsilon > 0$:

$$\epsilon \leq \limsup_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) \leq s\epsilon$$
$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d\left(x_{m(k)}, x_{n(k)+1}\right) \leq s^2\epsilon$$
$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)}\right) \leq s^2\epsilon$$
$$\frac{\epsilon}{s^2} \leq \limsup_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right) \leq s^3\epsilon .$$

From the proof of the theorem in [3, page 262], the Jungck Picard iterative scheme appears.

Definition 2.7: [3] Let A be a non-empty subset of X and let $\mathrm{T}, \mathrm{G} : \mathrm{A} \to \mathrm{A}$ be two self- mappings such that $T(A) \subset G(A)$, then for $x_0 \in A$:

$$Gx_{n+1} = Tx_n\ ;\ n \in \mathbb{N}. \quad (1)$$

The objective of this paper is to develop the result of Arab [14] by using the function $\rho: \mathbb{R}_+^6 \to [0, \infty)$ under certain conditions and using G-α-general admissible and triangular α-general admissible to prove coincidence fixed point and common fixed point theorems for two weakly compatible self – mappings in complete b-metric spaces.

3. Main Results

We introduce the following definitions.

Definition 3.1: Let X be a nonempty set, $x, y \in X$, $\alpha: X \times X \to \mathbb{R}$ be a function, and $T : X \to X$ be a mapping. Then T is said to be α −general admissible if
$\alpha(x, y) \geq m$, then $\alpha(Tx, Ty) \geq m$ for all $m \in \mathbb{N}$.

Definition 3.2: Let X be a nonempty set, $\alpha: X \times X \to \mathbb{R}$, and $T, G : X \to X$. The mapping T is $G - \alpha$ −general admissible if, for all $x, y \in X$ such that $\alpha(Gx, Gy) \geq m$, we have $\alpha(Tx, Ty) \geq m$ for all $m \in \mathbb{N}$. If G is the identity mapping, then T is called α −ganeral admissible.

Definition 3.3: An α −ganeral admissible map T is said to be triangular α −general admissible if $\alpha(x, z) \geq m$ and $a(z, y) \geq m$ then $\alpha(x, y) \geq m$ for all $m \in \mathbb{N}$,
and $x, y, z \in X$.

Definition 3.4: Let (X, d, s) be a b −metric space, $\alpha : X \times X \to \mathbb{R}$, and $G, T : X \to X$. T is α −genreal regular with respect to G, if for every sequence $\{x_n\} \subseteq X$ such that $\alpha(Gx_n, Gx_{n+1}) \geq m$ for all $n \in \mathbb{N}_0, m \in \mathbb{N}$ and $Gx_n \to Gx \in GX$ as $n \to \infty$, then there exists a subsequence $\{Gx_{n(k)}\}$ of $\{Gx_n\}$ such that for all $k \in \mathbb{N}_0$, $\alpha(Gx_{n(k)}, Gx) \geq m$. If G is the identity mapping, then T is called α −general regular.

Definition 3.5: Let $\rho: \mathbb{R}_+^6 \to [0, \infty)$ be a mapping with the following conditions:

1. ρ is upper semi-continuous and non-decreasing in each coordinate
2. $max \left\{ \begin{matrix} \rho(0,0,t,t,0,t), \rho(t,0,0,t,t,0), \rho(t,t,t,t,0,t), \\ p(t,0,0,t,t,t), p(0,0,t,t,0,0) \end{matrix} \right\} < t$ for all $t > 0$.

The following example is to clarify the previous definition,

Example 3.6: $\rho(t_1, t_2, t_3, t_4, t_5, t_6) = at_1 + b \max\{t_2, t_3, t_4, t_5, t_6\}$, where $a > 0$, $b \geq 0$ and $a + b < 1$. Note that:

1. ρ is upper semi continuous mapping and nondecreasing in each coordinate.
2. $\rho(0,0,t,t,0,t) = bt < t, \rho(t,0,0,t,t,0) = (a + b)t < t$.

Lemma 3.7: Let $\alpha : X \times X \to \mathbb{R}$ be a function and $T, G : X \to X$. Suppose that T is a $G - \alpha$ −general admissible and triangular α −general admissible. Assume that there exists $x_0 \in X$ such that $\alpha(Gx_0, Tx_0) \geq m$ for all $m \in \mathbb{N}$. Then $\alpha(Gx_n, Gx_k) \geq m$ for all $n, k \in \mathbb{N}_0$ with $k < n$, where $Gx_{n+1} = Tx_n$.

Proof:

By assumption, there exists $x_0 \in X$ such that $\alpha(Gx_0, Tx_0) \geq m$ and T is a $G - \alpha$ −general admissible, we assume that:

$$\alpha(Gx_0, Gx_1) = \alpha(Gx_0, Tx_0) \geq m$$

Then by $Gx_{n+1} = Tx_n$, we get:

$$\alpha(Gx_1, Gx_2) = \alpha(Tx_0, Tx_1) \geq m,$$

which impales that:

$$\alpha(Gx_2, Gx_3) = \alpha(Tx_1, Tx_2) \geq m.$$

Inductively, we get:

$$\alpha(Gx_n, Gx_{n+1}) \geq m, \qquad n \in \mathbb{N}_0$$

Suppose that $k < n$. Since $\alpha(Gx_k, Gx_{k+1}) \geq m, \alpha(Gx_{k+1}, Gx_{k+2}) \geq m$
and T is triangular α −general admissible, we have:
$\alpha(Gx_k, Gx_{k+2}) \geq m.$
Inductively, we obtain:
$\alpha(Gx_k, Gx_n) \geq m.$

The following proposition is the key to show our main results.

Proposition 3.8: Let (X, d, s) be a complete b −metric space, $TX \subseteq GX$, T, G be a self- mappings on X which satisfies:
$\alpha : X \times X \to \mathbb{R}$. Suppose that GX is closed and the following condition holds:

$$\alpha(x, y)d(Tx, Ty) \leq \frac{1}{s^3} \rho \begin{pmatrix} d(Gx, Gy), d(Gx, Tx), d(Gy, Ty), \frac{d(Gx, Ty)}{2s}, \\ \frac{d(Gy, Tx)}{2s}, \frac{d(Gx, Tx) + d(Gy, Ty)}{2s} \end{pmatrix} \qquad (2)$$

for $x, y \in X$ and ρ defined by Definition (3.5). Assume also that the following conditions hold:

1. T is $G - \alpha -$general admissible and triangular $\alpha -$general admissible;
2. there exists $x_0 \in X$ such that $\alpha(Gx_0, Tx_0) \geq m$, for some $m \in \mathbb{N}$;
3. X is $\alpha -$general regular with respect to G. Then T and G have point of coincidence.

Proof:

Let $x_0 \in X$ be such that $\alpha(Gx_0, Tx_0) \geq m$ (by the condition (1)). Since $TX \subseteq GX$ and T, G are self-mappings on X, it satisfies

$$Gx_{n+1} = Tx_n, n \in \mathbb{N}_0$$

By using Lemma (3.7), we get

$\alpha(Gx_{n+1}, Gx_n) \geq m, n \in \mathbb{N}_0$, for some $m \in \mathbb{N}$.

Case (1): If $Tx_{n_0} = Tx_{n_0+1}$ for some $n_0 \in \mathbb{N}_0$, then by (1), we obtain:

$$Gx_{n_0+1} = Tx_{n_0} = Tx_{n_0+1},$$

Thus $T$and G have a coincidence point at $x = x_{n_0+1}$.

Case (2): If $Tx_n \neq Tx_{n+1}$., for all $n \in \mathbb{N}_0$, therefore by (2), we have:

$$d(Gx_n, Gx_{n+1}) \leq s^3\alpha(Gx_n, Gx_{n+1})d(Tx_{n-1}, Tx_n)$$

$$\leq \rho\begin{pmatrix} d(Gx_{n-1}, Gx_n), d(Gx_{n-1}, Tx_{n-1}), \\ d(Gx_n, Tx_n), \dfrac{d(Gx_{n-1}, Tx_n)}{2s}, \dfrac{d(Gx_n, Tx_{n-1})}{2s} \\ , \dfrac{d(Gx_{n-1}, Tx_{n-1}) + d(Gx_n, Tx_n)}{2s} \end{pmatrix}$$

Therefore,

$$d(Gx_n, Gx_{n+1}) \leq \rho\begin{pmatrix} d(Gx_{n-1}, Gx_n), d(Gx_{n-1}, Gx_n), \\ d(Gx_n, Gx_{n+1}), \dfrac{d(Gx_{n-1}, Gx_{n+1})}{2s}, \dfrac{d(Gx_n, Gx_n)}{2s}, \\ \dfrac{d(Gx_{n-1}, Gx_n) + d(Gx_n, Gx_{n+1})}{2s} \end{pmatrix}$$

Thus,

$$d(Gx_n, Gx_{n+1}) \leq \rho\begin{pmatrix} d(Gx_{n-1}, Gx_n), d(Gx_{n-1}, Gx_n), d(Gx_n, Gx_{n+1}), \\ \dfrac{d(Gx_{n-1}, Gx_{n+1})}{2s}, 0, \dfrac{d(Gx_{n-1}, Gx_n) + d(Gx_n, Gx_{n+1})}{2s} \end{pmatrix}$$

Hence,

$$d(Gx_n, Gx_{n+1}) \leq \rho\begin{pmatrix} d(Gx_{n-1}, Gx_n), d(Gx_{n-1}, Gx_n), d(Gx_n, Gx_{n+1}), \\ \dfrac{d(Gx_{n-1}, Gx_n) + d(Gx_n, Gx_{n+1})}{2}, 0, \\ \dfrac{d(Gx_{n-1}, Gx_n) + d(Gx_n, Gx_{n+1})}{2s} \end{pmatrix} \quad (3)$$

Note that $d(Gx_{n-1}, Gx_n) > d(Gx_n, Gx_{n+1})$. In fact if not, then from (2) and by Definition 3.5, we get:

$$d(Gx_n, Gx_{n+1}) \leq \rho\begin{pmatrix} d(Gx_n, Gx_{n+1}), d(Gx_n, Gx_{n+1}), d(Gx_n, Gx_{n+1}), \\ d(Gx_n, Gx_{n+1}), 0, d(Gx_n, Gx_{n+1}) \end{pmatrix}$$

Therefore,

$$d(Gx_n, Gx_{n+1}) < d(Gx_n, Gx_{n+1})$$

That is a contradiction.

So, $d(Gx_n, Gx_{n+1}) < d(Gx_{n-1}, Gx_n)$ for all $n \in \mathbb{N}$, that is, the sequence of positive numbers $\{d(Gx_n, Gx_{n+1})\}$ is decreasing. Hence it converges to r such that $r \geq 0$.

If $r > 0$, then letting $n \to \infty$ in (3) and since ρ is upper semi-continuous, then we get:

$$r \leq \rho(r, r, r, r, 0, r) < r,$$

Which is a contraction. So that,

$$\lim_{n \to \infty} d(Gx_n, Gx_{n+1}) = 0 \quad (4)$$

Now, we claim that:

$$\lim_{n,m \to \infty} d(Gx_n, Gx_m) = 0$$

Assume that it is not true, thus from lemma 1.6 there exists $\epsilon > 0$ and, subsequently, $\{Gx_{n(k)}\}, \{Gx_{m(k)}\}$ of $\{Gx_n\}$ with $n(k) > m(k) \geq k$, such that:

$$d\big(Gx_{m(k)}, Gx_{n(k)}\big) \geq \epsilon \tag{5}$$

Further, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying(5). Hence ,

$$d\big(Gx_{m(k)}, Gx_{n(k)-1}\big) < \epsilon \tag{6}$$

Then we have:

$$\epsilon \leq \limsup_{k\to\infty} d\big(Gx_{m(k)}, Gx_{n(k)}\big) \leq s\epsilon \tag{7}$$

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d\big(Gx_{m(k)}, Gx_{n(k)+1}\big) \leq s^2\epsilon \tag{8}$$

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d\big(Gx_{m(k)+1}, Gx_{n(k)}\big) \leq s^2\epsilon. \tag{9}$$

Also, by b-triangle inequality, taking limsup as $k \to \infty$, and by (9), we have:

$$d\big(Gx_{m(k)+1}, Gx_{n(k)}\big) \leq sd\big(Gx_{m(k)+1}, Gx_{n(k)+1}\big) + sd\big(Gx_{n(k)+1}, Gx_{n(k)}\big)$$

$$\frac{\epsilon}{s^2} \leq \limsup_{k\to\infty} d\big(Gx_{m(k)+1}, Gx_{n(k)+1}\big) \tag{10}$$

Now, using inequality (2), we obtain:

$$d\big(Gx_{m(k)+1}, Gx_{n(k)+1}\big) \leq s^3\alpha\big(Gx_{m(k)+1}, Gx_{n(k)+1}\big)d\big(Tx_{m(k)}, Tx_{n(k)}\big)$$

$$\leq \rho\left(\begin{array}{c} d\big(Gx_{m(k)}, Gx_{n(k)}\big), d\big(Gx_{m(k)}, Gx_{m(k)+1}\big), \\ d\big(Gx_{n(k)}, Gx_{n(k)+1}\big), \\ \dfrac{d\big(Gx_{m(k)}, Gx_{n(k)+1}\big)}{2s}, \dfrac{d\big(Gx_{n(k)}, Gx_{m(k)+1}\big)}{2s}, \\ \dfrac{d\big(Gx_{m(k)}, Gx_{m(k)+1}\big) + d\big(Gx_{n(k)}, Gx_{n(k)+1}\big)}{2s} \end{array}\right),$$

Since ρ is upper semi- continuous, then by (4), (7), (8), (9) and (10), we have:

$$\frac{\epsilon}{s^2} \leq \limsup_{k\to\infty} d\big(Gx_{m(k)+1}, Gx_{n(k)+1}\big) \leq \frac{1}{s^3}\rho\left(s\epsilon, 0,0, \frac{s\epsilon}{2}, \frac{s\epsilon}{2}, 0\right) \leq \frac{1}{s^3}\rho(s\epsilon, 0,0, s\epsilon, s\epsilon, 0) < \frac{\epsilon}{s^2}$$

This is a contradiction, so that $\{Gx_n\}$ is a Cauchy sequence. By (1), we get: $\{Tx_n\} = \{Gx_{n+1}\} \subseteq GX$ and GX is closed. There exists $x \in X$ such that:

$$\lim_{n\to\infty} Gx_n = Gx \tag{11}$$

To prove that x is a coincidence point of T and G, assume the contrary, i.e. $Tx \neq Gx$.
Since X is α −general regular with respect to G and by (11), we obtain $\alpha\big(Gx, Gx_{n(k)+1}\big) \geq m$ for all $k \in \mathbb{N}_0$.
By b −triangle inequality:

$$d(Gx, Tx) \leq sd\big(Gx, Gx_{n(k)+1}\big) + sd\big(Tx_{n(k)}, Tx\big).$$

If making $k \to \infty$, then we have:

$$d(Gx, Tx) \leq s \lim_{k\to\infty} d\big(Tx_{n(k)}, Tx\big) \tag{12}$$

By the properties of ρ, (2) and (12), we have:

$$d(Gx, Tx) \leq s \lim_{k\to\infty} d\big(Tx_{n(k)}, Tx\big)$$

$$\leq s \lim_{k\to\infty} \alpha\big(Gx, Gx_{n(k)+1}\big)\, d\big(Tx_{n(k)}, Tx\big)$$

$$\leq \frac{1}{s^2}\lim_{k\to\infty}\rho\left(\begin{array}{c} d\big(Gx_{n(k)}, Gx\big), d\big(Gx_{n(k)}, Gx_{n(k)+1}\big), d(Gx, Tx), \\ \dfrac{d\big(Gx_{n(k)}, Gx\big) + d(Gx, Tx)}{2}, \\ \dfrac{d\big(Gx, Gx_{n(k)+1}\big)}{2s}, \\ \dfrac{d\big(Gx_{n(k)}, Gx_{n(k)+1}\big) + d(Gx, Tx)}{2s} \end{array}\right)$$

$$\leq \frac{1}{s^2}\rho\big(0,0, d(Gx, Tx), d(Gx, Tx), 0, d(Gx, Tx)\big)$$

which is a contradiction. Thus, $Gx = Tx = w$ where w is the point of coincidence of T and G. ■

Theorem 3.9: Let (X, d, s) be a complete b −metric space, $TX \subseteq GX$, T, G be self- mappings on X, which satisfies Jungck Picard iterative Definition 2.7, and
$\alpha : X \times X \to \mathbb{R}$. Suppose that GX is closed and the following condition holds:

$$\alpha(x,y)d(Tx,Ty) \le \frac{1}{s^3}\rho\begin{pmatrix} d(Gx,Gy), d(Gx,Tx), d(Gy,Ty), \dfrac{d(Gx,Ty)}{2s}, \\ \dfrac{d(Gy,Tx)}{2s}, \dfrac{d(Gx,Tx)+d(Gy,Ty)}{2s} \end{pmatrix} \tag{13}$$

for $x, y \in X$ and ρ is defined by Definition 3.5 . Let the following conditions hold:
1. T is $g - \alpha$ −general admissible and triangular α −general admissible;
2. there exists $x_0 \in X$ such that $\alpha(Gx_0, Tx_0) \ge m$, for some $m \in \mathbb{N}$;
3. X is α −general regular with respect to G.

If the pair $\{T, G\}$ is weakly compatible and either $\alpha(v, w) \ge m$ or $\alpha(w, v) \ge m$ whenever $Tv = Gw$ and $Tw = Gv$, then T and G have a unique common fixed point.
Proof:
By Proposition 3.8, we get T and G have a coincidence point.
If $Tw = Gw$ and $Tv = Gv$, then $Gv = Gw$. By hypotheses,
$\alpha(v, w) \ge m$ or $\alpha(w, v) \ge m$. Suppose that $\alpha(w, v) \ge m$, then,

$$d(Gw,Gv) = d(Tw,Tv) \le s^3\alpha(w,v)d(Tw,Tv)$$

Hence,

$$d(Gw,Gv) \le \rho\begin{pmatrix} d(Gw,Gv), d(Gw,Tw), d(Gv,Tv), \dfrac{d(Gw,Tv)}{2s}, \\ \dfrac{d(Gv,Tw)}{2s}, \dfrac{d(Gw,Tw)+d(Gv,Tv)}{2s} \end{pmatrix}$$

$$\le \frac{1}{s^3}\rho\begin{pmatrix} d(Gw,Gv), 0, 0, \dfrac{d(Gw,Gv)}{2s}, \\ \dfrac{d(Gv,Gw)}{2s}, 0 \end{pmatrix}$$

$$= \frac{1}{s^3}\rho(\, d(Gw,Gv), 0,0, d(Gw,Gv), d(Gw,Gv), 0)$$

This is a contradiction, then $Gu = Gw$, similarity, $\alpha(w, v) \ge m$, implies that $Gv = Gw$. Now, we show that T and G have a common fixed point if $w = Tv = Gv$.
Since the pair $\{T, G\}$ is weakly compatible, we have $Tw = T(Gv) = G(Tv) = Gw$. Thus, w is a coincidence point of T and G. Now, we show the uniqueness of the point of coincidence of T and G. Let u be another point of coincidence of T and G, such that $w \ne u$, then we can find x in X such that $Tx = Gx = u$.
Assume that $Gv \ne Gx$, then,

$$d(Gx,Gv) = d(Tx,Tv) \le s^3\alpha(x,v)d(Tx,Tv)$$

Hence,

$$d(Gx,Gv) \le \rho\begin{pmatrix} d(Gx,Gv), d(Gx,Tx), d(Gv,Tv), \dfrac{d(Gx,Tv)}{2s}, \\ \dfrac{d(Gv,Tx)}{2s}, \dfrac{d(Gx,Tx)+d(Gv,Tv)}{2s} \end{pmatrix}$$

$$\le \rho\begin{pmatrix} d(Gx,Gv), 0, 0, \dfrac{d(Gx,Gv)}{2s}, \\ \dfrac{d(Gv,Gx)}{2s}, 0 \end{pmatrix}$$

$$\le \rho\begin{pmatrix} d(Gx,Gv), 0,0, d(Gx,Gv), \\ d(Gx,Gv), 0 \end{pmatrix} < d(Gx,Gv)$$

which is a contradiction, then $w = u$. Therefore, w is a unique point of coincidence of T and G. Since the pair is weakly compatible, and by Proposition 1.1, w is a unique common fixed point of T and G.

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