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# Properties of ~ Self-Adjoint and ~ Positive Operators in b- Hilbert Space

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**Abstract**
In this paper, we will introduce a new concept of operators in b-Hilbert space, which is respected to ~self- adjoint operator and ~positive operator. Moreover we will show some of their properties as well as the relation between them.

**Keywords**: self- adjoint operator, positive operator, b-Hilbert space.

# خواص المؤثرات ~ الذاتية المرافق والمؤثرات~ الموجبة في فضاء هلبرت من النمط "b"

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**الخلاصة**
في هذا البحث، سوف نقدم مفاهيم جديدة لانواع المؤثرات في فضاء هلبرت من النمط "b" المتعلقة بالمؤثر ~ الذاتية المرافق والمؤثر ~ الموجبة. اضافة الى ذلك سوف نقوم ببرهان بعض خواصهما والعلاقة بينهما.

## Introduction:

The concept of n-normed spaces was introduced at first in the 1960′s by Gähler [1], as a generalization of usual normed space, where initially suggested by the area function of parallelogram spanned by the two associated vectors as follows:

Let X be a vector space over a field $\mathbb{F}$ either real or complex space with finite or infinite dimension d≥n; $n \in \mathbb{N}$ and a real-valued function $\|.,\dots,.\| : X^n \to \mathbb{R}$ satisfying the following properties for all $x_1,\dots,x_n \in X^n$ and $\alpha \in \mathbb{F}$.

1. $\|x_1,\dots,x_n\| = 0$ if and only if $x_1,\dots,x_n$ are linear dependent.
2. $\|x_1,\dots,x_n\|$ is invariant under permutation of $x_1,\dots,x_n$.
3. $\|x_1,\dots,\alpha x_n\| = |\alpha|\|x_1,\dots,x_n\|$.
4. $\|x+y,\ x_2,\dots,x_n\| \le \|x,\ x_2,\dots,x_n\| + \|y,\ x_2,\dots,x_n\|$.

$\|.,\dots,.\|$is called n-norm on X and the pair $(X, \|.,\dots,.\|)$ is called an n-normed space [2].

The related concept of 2-normed space is a real 2-inner product space, which is appeared in [3] by Diminnie. Later Gordjiet.al. in [2] extended the concept as a complex 2-inner product space, which is later called a generalized 2-inner product space [4] by the next according:

A complex valued space X is said to be a generalized 2- inner product space if there is $<(.,.),(.,.)>: X^2 \times X^2 \to \mathbb{C}$, such that for all a, b, c, d, e ∈ X and α ∈ ℂ.

1. If a and b are linear independent in X, then $<(a,b),(a,b)> > 0$.
2. $<(a,b),(c,d)> = \overline{<(c,d),(a,b)>}$
3. $<(a,b),(c,d)> = -<(b,a),(c,d)>$
4. $<(\alpha a+e,b),(c,d)> = \alpha<(a,b),(c,d)> + <(e,b),(c,d)>$.

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Mazaheriand Kazemiin [5] were introduced concept of b-Cauchy sequence, where b is a non-zero vector in generalized 2-inner product space$\mathrm{X}$, then the sequence $\{x_{\mathrm{n}}\}$ is called b-Cauchy sequence in $\mathrm{X}$, if $\lim_{\mathrm{n,m}\to\infty}\|x_{n-}x_m,\mathrm{b}\| = 0$**.**

With respected to Khan and Siddiqui in 1982 [6] were introduced many types of orthogonal, that lead Mazaheri in [7] to derived the concept of b-orthogonal in 2-normed space X which contained non-zero vector b and $x, y \in \mathrm{X}\backslash < \mathrm{b} >$, then $x$ is b- orthogonal to $y$ if $\|x,\mathrm{b}\| \le \|x+\alpha y,\mathrm{b}\|$ for all scalar $\alpha \in \mathbb{R}$and denoted by $x \perp^b y$.The same authors also proved theorem which is given a characterization to the concept; Let X be a generalized 2-inner product space and $x, y \in \mathrm{X}\backslash < \mathrm{b} >$, then x is b-orthogonal to y if and only if $< (x,\mathrm{b}),(\mathrm{y},\mathrm{b}) >= 0$.

Gordji et.al in [2] defined the concept of complex b-Hilbert space dependent on that every b-Cauchy sequence is convergence in semi normed space$(\mathrm{X},\|.,\mathrm{b}\|)$.

Corresponding to complex b-Hilbert space, Riyas and Ravindran in [4] introduced the concept of $\sim$ adjoint operator: Let X be a b-Hilbert space and let T be alinear bounded operator onX such that$< \mathrm{b} >$ is a T-invariant subspace of X, then there exist a unique linear bounded operator $\mathrm{T}'$on$\widetilde{\mathrm{X}}$satisfying that$< (\mathrm{T}(x),\mathrm{b}),(\mathrm{y},\mathrm{b}) >=< (x,\mathrm{b}),(\mathrm{T}'(\mathrm{y}),\mathrm{b}) >$, where $\widetilde{\mathrm{X}}$ is a completion of $\frac{\mathrm{X}}{<\mathrm{b}>}$ which is a Hilbert space.

In this paper we will introduce some new concepts about $\sim$ self adjoint and $\sim$ positive operators in b-Hilbert space as well as give many properties. This paper consist two sections; in section one we will introduce the concept of $\sim$self –adjoint operator (1.1), and study some of their properties. In section two, we will introduce the definition of $\sim$ positive operator (2.1) and discuss its properties, as well as show the relation with $\sim$self –adjoint operator.

Finally, in this paper we will use the notion $< x, y >_{\mathrm{b}}$ instead $< (x,\mathrm{b}),(y,\mathrm{b}) >_{\mathrm{b}}$ and denoted the set of all bounded linear operator on X by $\mathrm{B}(\mathrm{X})$.

## 1. Properties of$\sim$Self –Adjoint Operator

In this section, we give some properties of $\sim$self – adjoint operator in b-Hilbert space.

**Definition 1.1:**

Let X be a b-Hilbert space, $\mathrm{T} \in \mathrm{B}(\mathrm{X})$ and $\mathrm{T}' \in \mathrm{B}\left(\tilde{\mathrm{X}}\right)$, then T is said to be $\sim$ **self- adjoint operator**if $\mathrm{T}|_{\tilde{\mathrm{X}}} = \mathrm{T}'$, , where $\widetilde{\mathrm{X}}$ is a completion of $\frac{\mathrm{X}}{<\mathrm{b}>}$ which is a Hilbert space.

By using the definition (1.1), we immediately get the following properties:

**Proposition 1.2:**

Let X be a b-Hilbert space and $\mathrm{T} \in \mathrm{B}(\mathrm{X})$. If T is $\sim$ self-adjoint, then

1. $\mathrm{T}\,\mathrm{T}'$is$\sim$ self-adjoint
2. $\mathrm{T}+ \mathrm{T}'$is$\sim$ self-adjoint when T is restricting on $\tilde{\mathrm{X}}$.
3. $\mathrm{T}\,\mathrm{T}' = \mathrm{T}^2|_{\tilde{\mathrm{X}}}$

Now we give the characterization of$\sim$ self-adjoint operators.

**Proposition 1.3:**

Let X be a b-Hilbert space,$\mathrm{T} \in \mathrm{B}(\mathrm{X})$ and $\mathrm{T}' \in \mathrm{B}\left(\tilde{\mathrm{X}}\right)$, then the following conditions are equivalent for all $x$ and$y \in \tilde{\mathrm{X}}$.

1. T is$\sim$ self-adjoint.
2. $< \mathrm{T}(x),\mathrm{y} >_{\mathrm{b}}=< \mathrm{T}'(x), y >_{\mathrm{b}}$
3. $< \mathrm{T}(x) , y >_{\mathrm{b}}=< x , \mathrm{T}(y) >_{\mathrm{b}}$
4. $< \mathrm{T}(x) , x >_{\mathrm{b}}=< x , \mathrm{T}(x) >_{\mathrm{b}}$
5. $< \mathrm{T}(x) , x >_{\mathrm{b}}$ is real.

**Proof: (1⇒ 2)**

By definition (1.1), $\mathrm{T}|_{\tilde{\mathrm{X}}} = \mathrm{T}'$, then$< \mathrm{T}(x),\mathrm{y} >_{\mathrm{b}}=< x, \mathrm{T}'(y) >_{\mathrm{b}}=< x, \mathrm{T}(y) >=< \mathrm{T}'(x), y >$

**(2⇒ 3)**

By (2), $< \mathrm{T}(x) , y >_{\mathrm{b}}=< \mathrm{T}'(x) , \mathrm{y} >_{\mathrm{b}}=< x , \mathrm{T}(y) >_{\mathrm{b}}$.

**(3⇒ 4)**

Let $y = x$, since $< \mathrm{T}(x) , y >_{\mathrm{b}}=< x , \mathrm{T}(y) >_{\mathrm{b}}$, therefore $< \mathrm{T}(x) , y >_{\mathrm{b}}=< x , \mathrm{T}(x) >_{\mathrm{b}}$

**(4⇒ 5)**
By (part (2) of generalize 2-inner product space),
$< \mathrm{T}(x)\,, x >_{\mathrm{b}} = < x\,, \mathrm{T}(x) >_{\mathrm{b}} = \overline{< \mathrm{T}(x\,), x >_{\mathrm{b}}}$.Therefore$< T(x)\,, x >_{\mathrm{b}}$ is real.
**(5⇒ 1)**
Since $< \mathrm{T}(x)\,, x >_{\mathrm{b}}$ is real, then$< \mathrm{T}(x)\,, x >_{\mathrm{b}} = \overline{< \mathrm{T}(x)\,, x >_{\mathrm{b}}} = < \mathrm{T}'(x)\,, x >_{\mathrm{b}}$, therefore T is $\sim$ self-adjoint.
The next results give us more properties of composition operators.
**Proposition 1.4:**
Let X be a b-Hilbert space, $\mathrm{T}, \mathrm{S} \in \mathrm{B(X)}$ and $\mathrm{T}', \mathrm{S}' \in \mathrm{B}\left(\tilde{\mathrm{X}}\right)$. IfT is $\sim$ self –adjoint, then $\mathrm{S}'\mathrm{TS}$ is$\sim$self-adjoint.
**Proof:**
Let $x, \mathrm{y} \in \tilde{\mathrm{X}}$. Then$< \mathrm{S}'\mathrm{TS}(x), \mathrm{y} >_{\mathrm{b}} = < x, \mathrm{S}'\mathrm{T}'\mathrm{S}(\mathrm{y}) >_{\mathrm{b}} = < (\mathrm{S}'\mathrm{T}'\mathrm{S})'(x), \mathrm{y} >_{\mathrm{b}}$.But T is $\sim$ self –adjoint operator, so $< \mathrm{S}'\mathrm{TS}(x), \mathrm{y} >_{\mathrm{b}} = < (\mathrm{S}'\mathrm{TS})'(x), \mathrm{y} >_{\mathrm{b}}$.Therefore by proposition (1.3), $\mathrm{S}'\mathrm{TS}$ is a $\sim$ self –adjoint operator.
**Proposition 1.5:**
Let X be a b-Hilbert space and $\mathrm{T}, \mathrm{S} \in \mathrm{B(X)}$, such that they are$\sim$ self-adjoint operators.Then ST is$\sim$ self-adjoint if and only if $\mathrm{ST}|_{\tilde{\mathrm{X}}} = \mathrm{TS}|_{\tilde{\mathrm{X}}}$.
**Proof:**(⇒) Let x and $\mathrm{y} \in \tilde{\mathrm{X}}$. Since $\mathrm{ST}, S$ and Tare$\sim$ self-adjoint operators, then by proposition (1.3), $< \mathrm{ST}(x), y >_{\mathrm{b}} = < \mathrm{T}'\mathrm{S}'(x), \mathrm{y} >_{\mathrm{b}} = < \mathrm{TS}(x), \mathrm{y} >_{\mathrm{b}}$.Thus$< ST(x), \mathrm{y} >_{\mathrm{b}} - < \mathrm{TS}(x), \mathrm{y} >_{\mathrm{b}} = 0$,
for all $x, \mathrm{y} \in \tilde{\mathrm{X}}$. Therefore $\mathrm{ST}|_{\tilde{\mathrm{X}}} = \mathrm{TS}|_{\tilde{\mathrm{X}}}$.
(⇐) :Now we have $< ST(x), \mathrm{y} >_{\mathrm{b}} = < x, (\mathrm{ST})'(\mathrm{y}) >_{\mathrm{b}}$ . Since $\mathrm{ST}|_{\tilde{\mathrm{X}}} = \mathrm{TS}|_{\tilde{\mathrm{X}}}$, then

$$< \mathrm{ST}(x), \mathrm{y} >_{\mathrm{b}} = < x, (\mathrm{TS})'(\mathrm{y}) >_{\mathrm{b}} = < x, \mathrm{S}'\mathrm{T}'(\mathrm{y}) >_{\mathrm{b}} = < x, \mathrm{ST}(\mathrm{y}) >_{\mathrm{b}} = < (\mathrm{ST})'(x), \mathrm{y} >_{\mathrm{b}}.$$

Therefore by proposition (1.3), ST is $\sim$ self-adjoint operator.
The next proposition shows the linear combination of $\sim$ self-adjoint operators is also $\sim$ self-adjoint.
**Proposition 1.6:**
Let X be a b-Hilbert space, T and $\mathrm{S} \in \mathrm{B(X)}$. If T and S are $\sim$ self-adjoint operators, then for all αand $\beta \in \mathbb{R}$, $\alpha\mathrm{T} + \beta\mathrm{S}$ is$\sim$ self-adjoint operator.
**Proof:**
Let $x \in \tilde{\mathrm{X}}$.Since $\mathrm{T}, \mathrm{S}$ are $\sim$ self-adjoint operators, then

$$\begin{aligned} < (\alpha\mathrm{T} + \beta\mathrm{S})(x)\,, \mathrm{y} >_{\mathrm{b}} &= < (\alpha\mathrm{T}' + \beta\mathrm{S}')(x), \mathrm{y} >_{\mathrm{b}} = < \alpha\mathrm{T}'(x), \mathrm{y} >_{\mathrm{b}} + < \beta\mathrm{S}'(x), \mathrm{y} >_{\mathrm{b}} \\ &= < x, \alpha\mathrm{T}(\mathrm{y}) >_{\mathrm{b}} + < x, \beta\mathrm{S}(\mathrm{y}) >_{\mathrm{b}} = < (\alpha\mathrm{T} + \beta\mathrm{S})'(x), \mathrm{y} >_{\mathrm{b}} \end{aligned}$$

Therefore by proposition (1.3), $\alpha\mathrm{T} + \beta\mathrm{S}$ is$\sim$ self-adjoint operator.
Turning to eigenvalues and eigenvectors, the next proposition shows that eigenvalues of$\sim$self-adjoint operator is real and their corresponding eigenvectors are b-orthogonal.
**Proposition 1.7:**
LetT: $\mathrm{X} \rightarrow X$ be a linear bounded $\sim$self –adjoint operator on complex b-Hilbert spaceX, then:
1. All the eigenvalues of T (if exist) are real.
2. Eigenvectors$x, y \in \mathrm{X} \setminus < \mathrm{b} >$, which are corresponding to different eigen values of T are b- orthogonal.
**Proof:**
(1):Let λ be an eigenvalue of$\sim$self –adjoint operator T, and let$x \in \mathrm{X} \setminus < \mathrm{b} >$. Then we have, $\lambda < x, x >_{\mathrm{b}} = < \lambda x, x >_{\mathrm{b}} = < \mathrm{T}x, x >_{\mathrm{b}} = < x, \mathrm{T}x >_{\mathrm{b}} = \bar{\lambda} < x, x >_{\mathrm{b}}$,since $x \in \mathrm{X} \setminus < \mathrm{b} >$,then
$< x, x >_{\mathrm{b}} \neq 0$, thus $\lambda = \bar{\lambda}$. Therefore λ is real.
(2): Let λ and μ be eigenvalues of T, and let $x, y \in \mathrm{X} \setminus < \mathrm{b} >$. be corresponding eigenvectors to λ andμ respectively. Since T is $\sim$self –adjoint, then
$\lambda < x, y >_{\mathrm{b}} = < \mathrm{T}x, y >_{\mathrm{b}} = < x, \mathrm{T}y >_{\mathrm{b}} = \mu < x, y >_{\mathrm{b}}$, so $(\lambda - \mu) < x, y >_{\mathrm{b}} = 0$.
But $\lambda - \mu \neq 0$,, hence $< x, y >_{\mathrm{b}} = 0$. Therefore $x \perp^{\mathrm{b}} y$
**Properties of ~Positive Operator**
In this section we discuss another kind of operator in b-Hilbert space which is $\sim$positive operator.

**Definition 2.1:**
Let X be a b-Hlibert spaceand $T \in B(X)$, then T is said to be~**positive operator**, denoted by $T \geq 0$ if and only if $< T(x), x >_b \geq 0$ for all $x \in X$.
The following explains the relation between~positive operator and~self-adjoint operator in b-Hilbert space.
**Proposition 2.2:**
Let X be a b-Hilbert space, $T \in B(X)$ and $T' \in B\left(\tilde{X}\right)$. If T is ~positive operator, thenT is ~self-adjoint operator.
**Proof:**
Since T is ~positive operator, then $< T(x), x >_b \geq 0$ for all $x \in \tilde{X}$, thus $< T(x), x >_b$ is real,
so $< T(x), x >_b = < x, T(x) >_b = < T'(x), x >_b$. Therefore by proposition (1.3), T is ~ self-adjoint.
The linear combination of ~positive operatorsis ~ positive operator. As the following shows.
**Proposition 2.3:**
Let X be a b-Hilbert space, T and $S \in B(X)$. If $T \geq 0$ and $S \geq 0$,then $\alpha T + \beta S \geq 0$ for all non negative scalars$\alpha$and $\beta$.
**Proof:**
Since T and S are ~ positive operators, and $\alpha, \beta \geq 0$, then
$\alpha < T(x), x >_b + \beta < S(x), x >_b \geq 0$.
Hence $< (\alpha T + \beta S), x >_b \geq 0$, therefore$\alpha T + \beta S$ is ~ positive operator.
To show the relation between two operators have the same b-congruence class, we need first the following definition.
**Definition 2.4:**
Let X be any linear space over a field $\mathbb{F}$, $0 \neq b \in X$, T and $S \in B(X)$, we say that T is b-congruent to $S$ means $T(x) - S(x) \in < b >$ for all $x \in X$.
**Proposition 2.5:**
Let X be a b-Hilbert space, T and $S \in B(X)$. If T and S have the same b-congruence class, then T - S is~ positive operator.
**Proof:**
Let $x \in X$, Since T and S have the same b-congruence class, then $T - S \in < b >$, so $< (T - S)(x), x >_b = 0$.Therefore by definition (2.1), $T - S$ is ~ positive operator.

We will generalize~ positive operators to the following:
**Definition 2.6:**
Let T and $S \in B(X)$ are operators, where X is b-Hilbert space,then $T \geq S$ if and only if $T - S \geq 0$.
**Proposition 2.7:**
Let X be a b-Hilbert space,$T_1$, $T_2$, $T_3$and $T_4$ are bounded linear operators from Xonto X, then we have the following properties:
1. $T_1 \leq T_2$ if and only if $< T_1(x), x >_b \leq < T_2(x), x >_b$ for all $x \in X$.
2. If $T_1 \leq T_2$ and $T_2 \leq T_1$, then $T_1 = T_2$.
3. If $T_1 \leq T_2$ and $T_2 \leq T_3$, then $T_1 \leq T_3$.
4. If $T_1 \leq T_2$ and $T_3 \leq T_4$, then $T_1 + T_3 \leq T_2 + T_4$.
5. If $T_1 \leq T_2$ and $\alpha \geq 0$, then $\alpha T_1 \leq \alpha T_2$.
6. If $T_1 \leq T_2$, then $-T_1 \geq -T_2$.

**Proof:**
1): $T_1 \leq T_2$ if and only if $T_2 - T_1 \geq 0$ if and only if $< (T_2 - T_1)x, x >_b \geq 0$if and only if$< T_1(x), x >_b \leq < T_2(x), x >_b$.
(2): Since $T_1 \leq T_2$and $T_2 \leq T_1$, then from (1), we get
$< T_1(x), x >_b \leq < T_2(x), x >_b \leq < T_1(x), x >_b$.
Hence$< T_1(x), x >_b = < T_2(x), x >_b$ for all$x \in X$, therefore $T_1 = T_2$
(3): Since $T_1 \leq T_2$ and $T_2 \leq T_3$, then
$< T_1(x), x >_b \leq < T_2(x), x >_b \leq < T_3(x), x >_b$.
Hence $< T_1(x), x >_b \leq < T_3(x), x >_b$. Therefore $T_1 \leq T_3$
(4): Since $T_1 \leq T_2$and $T_3 \leq T_4$, then

$< T_1(x), x >_b \leq < T_2(x), x >_b$ (1)
and
$< T_3(x), x >_b \leq < T_4(x), x >_b$ (2)
Hence $< T_1(x), x >_b + < T_3(x), x >_b \leq < T_2(x), x >_b + < T_4(x), x >_b$.
Therefore $T_1 + T_3 \leq T_2 + T_4$
(5):Since $T_1 \leq T_2$, then $< T_1(x), x >_b \leq < T_2(x), x >_b$, hence
$\alpha < T_1(x), x >_b \leq \alpha < T_2(x), x >_b$; $\alpha \geq 0$, therefore $\alpha T_1 \leq \alpha T_2$.
(6): Since $T_1 \leq T_2$, then $< T_1(x), x >_b \leq < T_2(x), x >_b$, thus
$-< T_1(x), x >_b \geq -< T_2(x), x >_b$, therefore $-T_1 \geq -T_2$.

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