Iraqi Journal of Science, 2021, Vol. 62, No. 9, pp: 3102-3113
DOI: 10.24996/ijs.2021.62.9.26

ISSN: 0067-2904

Commutativity Results for Multiplicative (Generalized) (α,β) Reverse Derivations on Prime Rings

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Received: 7/1/2021 Accepted: 2/4/2021

Abstract

Let R be a prime ring, I be a non-zero ideal of R, and α,β be automorphisms on R. A mapping $F: R \to R$ is called a multiplicative (generalized) (α,β) reverse derivation if $F(xy) = F(y)\alpha(x) + \beta(y)d(x)$ for all $x, y \in R$, where $d: R \to R$ is any map (not necessarily additive). In this paper, we proved the commutativity of a prime ring R admitting a multiplicative (generalized) (α,β) reverse derivation F satisfying any one of the properties:
(i) $a\,(F(xy) \pm \alpha(xy)) = 0$ (ii) $a\,(F(x)F(y) \pm \alpha(xy)) = 0$
(iii) $a\left(F(xy) \pm F(y)F(x)\right) = 0$ (iv) $a\left(F(xy) \pm F(x)F(y)\right) = 0$
(v) $a(F(x)F(y) \pm \alpha(yx)) = 0$ (vi) $a(F(xy) \pm \alpha(yx)) = 0$ for all $x, y \in I$ and for some $0 \neq a \in R$.

Keywords: Prime Ring, Reverse Derivation, Multiplicative (Generalized) (α,β) Reverse Derivation, Generalized Multiplicative (α,β) Reverse Derivation.

نتائج الابدالية للمشتقات المعكوسة (α, β) الضربية المعممة على الحلقات الأولية

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الخلاصة

لتكن R حلقة اولية, I مثالي غير صفري من R و α,β التشكل على R. التطبيق $F: R \to R$ يدعى مشتقة معكوسة (α,β) الضربية المعممة اذا $F(xy) = F(y)\alpha(x) + \beta(y)d(x)$ for all $x, y \in R$ عندما $d: R \to R$ تكون أي تطبيق (ليس بالضرورة جمعي). في هذا البحث سنبرهن الابدالية للحلقة الاولية R *التي تمتلك* مشتقة معكوسة (α,β) الضربية المعممة و التي تحقق أحد هذه الخواص
(i) $a\,(F(xy) \pm \alpha(xy)) = 0$ (ii) $a\,(F(x)F(y) \pm \alpha(xy)) = 0$
(iii) $a\left(F(xy) \pm F(y)F(x)\right) = 0$ (iv) $a\left(F(xy) \pm F(x)F(y)\right) = 0$
(v) $a(F(x)F(y) \pm \alpha(yx)) = 0$ (vi) $a(F(xy) \pm \alpha(yx)) = 0$ for all $x, y \in I$
and for some $0 \neq a \in R$.

1. Introduction

Let R be an associative ring with the center $Z(R)$ and $\alpha,\beta: R \to R$ denote automorphisms. For all $x, y \in R$, we write down for commutator $[x, y] = xy - yx$. For any $a, b \in R$, a ring R is called prime ring if $a\,R\,b = 0$ then either $a = 0$ or $b = 0$ and is called semiprime if $aRa = 0$ where $a \in R$, then $a = 0$. A ring R is called 2-torsion free if $2a = 0$, implies that $a = 0$, for all $a \in R$. Over the past

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forty years, many results concerning derivations of rings have been obtained. An additive mapping $d: R \to R$ is said to be a derivation of R if $d(xy) = d(x)y + xd(y)$ where $x, y \in R$. Recall that an additive mapping d on R is said to be left multiplier if $d(xy) = d(x)y$ for all $x, y \in R$. The concept of Left α-multipliers (centralizers) was initiated by Albash [1], an additive mapping $d: R \to R$ is called left α-multipliers (centralizers) of R if $d(xy) = d(x)\alpha(y)$ for all $x,y \in R$, where α is an endomorphism of R.
An additive mapping $d: R \to R$ is called a (α, β) derivation if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$, where $x, y \in R$. Brešar [2] expanded the concept of derivation to generalized derivation. An additive map $F: R \to R$ associated with a derivation of $d: R \to R$ is called a generalized derivation of R if $F(xy) = F(x)y + xd(y)$ holds, where $x, y \in R$. It is clear that every derivation is a generalized derivation, but the converse needs not to be true in general. Hence, generalized derivation covers both the concepts of derivation and left multiplier maps. In [3], an additive mapping $F: R \to R$ is said to be a generalized (α, β) derivation associated with a map $d: R \to R$ such that d is a (α, β) derivation of R if $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$, where $x, y \in R$. Let H be a non-empty subset of R. We call the map $f: R \to R$ as centralizing on H if $[f(x), x] \in Z(R)$, where $x \in H$ and commuting on H if $[f(x), x] = 0$, where $x \in H$.

Posner [4] was the first to study the commutativity of rings in this way. He showed that if R is a prime ring with a non-zero derivation d on R and d is centralizing on R, then R is commutative.

The concept of multiplicative derivation was first introduced by Daif [5], inspired by the work of Martindale [6]. He has asked question of when is a multiplicative mapping additive? He answered his question for a multiplicative isomorphism of a ring R.

A mapping $d: R \to R$ is called a multiplicative derivation if it satisfies $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Of course, these maps need not to be additive. Daif and El-Sayiad [7] extended the concept of multiplicative derivation to a multiplicative generalized derivation. A map $F: R \to R$ is called a multiplicative generalized derivation if there exists a derivation $d: R \to R$ such that $F(xy) = F(x)y + xd(y)$, where $x, y \in R$, where maps need not to be additive. In this definition, if we take d to be a mapping that is not necessarily a derivation or an additive map, then F is called a multiplicative (generalized) derivation, which was introduced by Dhara and Ali [8]. Thus, multiplicative (generalized) derivation covers both the concepts of a multiplicative derivation and a multiplicative generalized derivation. In [9], a mapping $F: R \to R$ (not necessarily additive) is said to be a multiplicative left multiplier (centralizer) $F(xy) = F(x)y$ that holds for all $x, y \in R$. In this paper, we define a multiplicative left α- centralizer for a map $d: R \to R$ (not necessarily additive), which satisfies that $d(xy) = d(x)\alpha(y)$ holds for all $x, y \in R$, where α is an automorphism of R. A multiplicative (generalized) derivation associated with mapping $d = 0$ covers the concept of multiplicative left centralizer.

In [10], the authors generalized the concept of a multiplicative (generalized) derivation to a multiplicative (generalized) (α, β) derivation of R, if $F(xy) = F(x)\,\alpha(y) + \beta(x)d(y)$ for any $x, y \in R$, where $d: R \to R$ is any map (not necessarily additive) and $\alpha, \beta: R \to R$ are automorphisms of R. The authors investigated the commutativity of a prime ring satisfying the following algebraic identities:
(i)$F(xy) + \alpha(xy) = 0$ (ii) $F(xy) + \alpha(yx) = 0$ (iii)$F(xy) + F(x)F(y) = 0$ (iv) $F(xy) = \alpha(y) o\, H(x)$ and (v)$F(xy) = [\alpha(y), H(x)]$, for all x, y in an appropriate subset of R, where H is a multiplicative (generalized) (α, β) derivation. Herstein was the first to introduce the concept of reverse derivation [11]; a reverse derivation is an additive mapping $d: R \to R$ if $d(xy) = d(y)x + y\,d(x)$ that holds for all $x, y \in R$. He showed that if R is a prime ring and d is a nonzero reverse derivation of R, then R is a commutative integral domain and d is a derivation. Aboubakr and Gonzalez [12] generalized the notion of reverse derivation to generalized reverse derivation; an additive map $F: R \to R$ is called a generalized reverse derivation if $F(xy) = F(y)x + yd(x)$ for all x, $y \in R$, where d is a reverse derivation of R. Other authors [13, 14] extended the concept of reverse derivation to those of (α, β) reverse derivation and generalized (α, β) reverse derivation; an additive mapping $d: R \to R$ is called a (α, β) reverse derivation of R if $d(xy) = d(y)\alpha(x) + \beta(y)d(x)$ for all $x, y \in R$, where $\alpha, \beta: R \to R$ are two mappings. An additive mapping $F: R \to R$ is called a generalized (α, β) reverse derivation associated with $d: R \to R$ if $F(xy) = F(y)\alpha(x) + \beta(y)d(x)$, for all $x, y \in R$ there exists d be a (α, β) reverse derivation.

Another work [15] gave the concept of a multiplicative (generalized) reverse derivation; a map $F: R \to R$ is called a multiplicative (generalized) reverse derivation if $F(xy) = F(y)x + y\,d(x)$ holds for all $x, y \in R$, where d is any map on R and F is not necessarily additive. The authors extended the concept of a multiplicative (generalized) reverse derivation to a multiplicative (generalized) (α, β) reverse derivation. A mapping $F: R \to R$ is called a multiplicative (generalized) (α, β) reverse derivation of R associated with a mapping d on R if $F(xy) = F(y)\alpha(x) + \beta(y)d(x)$ for all $x, y \in R$, where α, β are automorphisms on R. Gurninder and Deepak [16] proved several results of multiplicative (generalized) reverse derivations.

In this paper, we proved the commutativity of a prime ring admitting a multiplicative (generalized) (α, β) reverse derivation satisfying any one of the following identities:
(i) $a\,(F(xy) \pm \alpha(xy)) = 0$ (ii)$a\,(F(x)F(y) \pm \alpha(xy)) = 0$ (iii)$a\left(F(xy) \pm F(y)F(x)\right) = 0$
(iv)$a\left(F(xy) \pm F(x)F(y)\right) = 0$ (v) $a(F(x)F(y) \pm \alpha(yx)) = 0$ (vi) $a(F(xy) \pm \alpha(yx)) = 0$, for all $x, y \in I$ and for some $0 \neq a \in R$, where I is a nonzero ideal in a prime ring R, and α, β are automorphisms of R.
The following basic identities are useful in the proof of our results:

$$[x, yz] = y[x, z] + [x, y]z\,, \quad [xy, z] = x[y, z] + [x, z]y.$$

We need the following lemma for the proof of our main results.

Lemma 1.1. [17]

(i) The center of a nonzero ideal is contained in the center of semi prime ring R. In particular, any commutative one-side ideal is contained in the center of R.

(ii) R is commutative if it is a prime ring with a nonzero central ideal.

2. Main Results

Lemma 2.1

Let R be a prime ring, I be a nonzero ideal of R, and $a \neq 0 \in R$ such that $a[x, a] = 0$ for all $x \in I$, then $a \in Z(R)$.

Proof

Suppose that

$$a[x, a] = 0 \text{ for all } x \in I. \tag{1}$$

By substituting xr in the place of x in equation (1), where $r \in R$, we get

$$ax[r, a] + a[x, a]r = 0\,.$$

By using equation (1), we have, $ax[r, a] = 0$ for all $x \in I, r \in R$.
By primness of R and $0 \neq a \in R$, it implies that $x[r, a] = 0$. Again, by primness of R, with I is a nonzero ideal of R, we get $[r, a] = 0$ for all $r \in R$, implies that $a \in Z(R)$.

Lemma 2.2

Let R be a prime ring, I be a nonzero ideal of R and $0 \neq a \in R$ such that $[x, a]x = 0$ for all $x \in I$, then R is commutative or $a \in Z(R)$.

Proof

We suppose that

$$[x, a]x = 0 \text{ for all } x \in I. \tag{2}$$

By linearizing equation (2) on x, we infer that

$$[x + y, a](x + y) = 0 \text{ for all } x, y \in I\,.$$

This means that

$$[x, a]x + [x, a]y + [y, a]x + [y, a]y = 0.$$

By using equation (2), we get

$$[x, a]y + [y, a]x = 0 \text{ for all } x, y \in I. \tag{3}$$

By exchanging x by xr in equation (3), where $r \in R$, we obtain

$$x[r, a]y + [x, a]ry + [y, a]xr = 0 \text{ for all } x, y \in I, r \in R\,. \tag{4}$$

By multiplying equation (3) by r on the right, we find

$$[x, a]yr + [y, a]xr = 0 \quad \text{for all } x, y \in I, r \in R. \tag{5}$$

Comparing equation (4) and equation (5), gives

$$[x, a][r, y] + x[r, a]y = 0 \text{ for all } x, y \in I, r \in R. \tag{6}$$

By taking yz in place of y in equation (6), where $z \in I$, it becomes

$$[x,a]y[r,z]+[x,a][r,y]z+x[r,a]yz=0 \text{ for all } x,y,z\in I, r\in R. \quad (7)$$

By using equation (6), we get $[x,a]y[r,z]=0$ for all $x,y,z\in I, r\in R$.
By primness of R, we get either $[x,a]y=0$ or $[r,z]=0$.
If $[x,a]y=0$ for all $x,y\in I$ then by primness of R, with I is a nonzero ideal of R, gives $[I,a]=0$. So, we get $a\in Z(I)$, by Lemma (1.1), implies that $a\in Z(R)$.
On the other hand, if $[R,I]=0$, implies that R contains a nonzero central ideal by Lemma (1.1), then we get R is commutative.

Lemma 2.3

Let R be a prime ring and I be a nonzero ideal of R. If $[yz,t]=0$ for all $y,z,t\in I$, then R is commutative.

Proof

We suppose that $[yz,t]=0$ for all $y,z,t\in I$.
Which means that

$$y[z,t]+[y,t]z=0. \quad (8)$$

By putting $z=zr$ in equation (8), where $r\in R$, we have $yz[r,t]+y[z,t]r+[y,t]zr=0$.
By using equation (8), we have $yz[r,t]=0$ for all $y,z,t\in I, r\in R$.
Once more, by primness of R, we get that R contains a nonzero central ideal. By Lemma (1.1), we get that R is commutative.

Lemma 2.4

Let R be a prime ring and $F:R\to R$ be a multiplicative (generalized)(α,β) reverse derivation of R associated with a map $d:R\to R$, then either R is commutative or d is the multiplicative left α-centralizer.

Proof

Since F is a multiplicative generalized (α,β) reverse derivation, then

$$F(xy)=F(y)\alpha(x)+\beta(y)d(x) \text{ for all } x,y\in R. \quad (9)$$

By putting $y=zy$ in equation (9), where $z\in R$, we get

$$\begin{aligned} F(xzy) &= F(zy)\alpha(x)+\beta(zy)d(x) \\ &= F(y)\alpha(z)\alpha(x)+\beta(y)d(z)\alpha(x)+\beta(z)\beta(y)d(x) \end{aligned} \quad (10)$$

On the other hand, we have

$$F(xzy)=F(y)\alpha(xz)+\beta(y)d(xz) \quad (11)$$

Comparing equation (10) and equation (11), we find that

$$F(y)\alpha[z,x]+\beta(y)d(z)\alpha(x)+\beta(z)\beta(y)d(x)-\beta(y)d(xz)=0. \quad (12)$$

By replacing x by xz in equation (12), this gives

$$F(y)\alpha([z,x]z)+\beta(y)d(z)\alpha(x)\alpha(z)+\beta(z)\beta(y)d(xz)-\beta(y)d(xz^2)=0. \quad (13)$$

We right multiply equation (12) by $\alpha(z)$, then we get

$$F(y)\alpha[z,x]\alpha(z)+\beta(y)d(z)\alpha(x)\alpha(z)+\beta(z)\beta(y)d(x)\alpha(z)-\beta(y)d(xz)\alpha(z)=0. \quad (14)$$

We subtract equation (14) from equation (13), we have

$$\beta(z)\beta(y)d(xz)-\beta(y)d(xz^2)-\beta(z)\beta(y)d(x)\alpha(z)+\beta(y)d(xz)\alpha(z)=0. \quad (15)$$

Putting $ty=y$ in equation (15), where $t\in R$, gives

$$\beta(z)\beta(ty)d(xz)-\beta(ty)d(xz^2)-\beta(z)\beta(ty)d(x)\alpha(z)+\beta(ty)d(xz)\alpha(z)=0. \quad (16)$$

We left multiply equation (15), by $\beta(t)$ to obtain

$$\beta(t)\beta(z)\beta(y)d(xz)-\beta(t)\beta(y)d(xz^2)-\beta(t)\beta(z)\beta(y)d(x)\alpha(z)+\beta(t)\beta(y)d(xz)\alpha(z)=0 \quad (17)$$

By subtracting equation (17) from equation (16), we have

$$[\beta(z),\beta(t)]\beta(y)\big(d(xz)-d(x)\alpha(z)\big)=0 \text{ for all } x,y,z,t\in R. \quad (18)$$

Let $y=yr$ in equation (18), where $y,r\in R$, then we find that

$$\beta[z,t]\beta(y)\beta(r)\,(d(xz)-d(x)z)=0.$$

$$\beta([z,t]y)R\big(d(xz)-d(x)\alpha(z)\big)=0.$$

By primness of R, we get either $\beta[z,t]\beta(y)=0$ for all $z,t,y\in R$, or $d(xz)-d(x)\alpha(z)=0$.
If $d(xz)-d(x)\alpha(z)=0$. This means that $d(xz)=d(x)\alpha(z)$, implies that d is the multiplicative left α −centralizer.
On the other hand, if $\beta[z,t]\beta(y)=0$ for all $z,t,y\in R$, we have

$$\beta^{-1}(\beta[z,t]y)=0 \text{ for all } z,y,t\in R.$$

That is, $[z,t]y=0$ for all $z,t,y\in R$.

By primness of *R*, this implies that $[z,t]=0$ for all $z,t \in R$. Then *R* is commutative.

Proposition 2.5

Let *R* be a nocommutative prime ring and $F: R \to R$ be a mapping of *R* satisfying $F(x+y)=F(x)+F(y)$ and $F(xy)=F(y)\alpha(x)+\beta(y)d(x)$ for all $x,y \in R$, with d is a map on *R*, then d is left α-centerlizer of *R*.

Proof

By the hypothesis, we have

$$F(xy)=F(y)\alpha(x)+\beta(y)d(x) \quad (19)$$

Putting $x=x+y$ and $y=z$, where $x,y,z \in R$, in equation (19), yields

$$F\big((x+y)z\big)=F(xz)+F(yz)$$
$$=F(z)\alpha(x)+\beta(z)d(x)+F(z)\alpha(y)+\beta(z)d(y) \quad (20)$$

On the other hand,

$$F\big((x+y)z\big)=F(z)\alpha(x+y)+\beta(z)d(x+y)$$
$$=F(z)\alpha(x)+F(z)\alpha(y)+\beta(z)d(x+y) \quad (21)$$

Comparing equation (21) and equation (20) gives $\beta(z)\big(d(x+y)-d(x)-d(y)\big)=0$.

By putting $z=zr$ in the above equation and using *R* as a prime ring, we find that $d(x+y)=d(x)+d(y)$.

Then, d is additive and, by using Lemma (2.4), we get that d is left α-centerlizer.

Lemma 2.6

Let R be a prime ring, I be a nonzero ideal of R, and $F: R \to R$ be a multiplicative (generalized) (α,β) reverse derivation associated with a map $d: R \to R$. If for some $0 \neq a \in R, a(F(xy) \pm F(y)F(x))=0$ for all $x,y \in I$ and $d(I)=0$, then either *R* is commutative or $d(R)=0$.

Proof

Suppose that $d(I)=0$.

In this instance, $F(xy)=F(y)\alpha(x)$ for all $x,y \in I$.

By our hypothesis, we have

$$a\,F(y)\big(\alpha(x) \pm F(x)\big)=0 \quad \text{for all } x,y \in I. \quad (22)$$

Now, we substitute ty for y in equation (22), where $t \in I$, to get

$$a\,F(y)\alpha(t)\big(\alpha(x) \pm F(x)\big)+a\beta(y)d(t)\big(\alpha(x) \pm F(x)\big)=0 \text{ for all } x,y,t \in I.$$

Since $d(I)=0$, we get $a\,F(y)\,\alpha(t)\,(\alpha(x) \pm F(x))=0$.

Putting $t=rt$ in the above relation, where $r \in R$, gives

$$a\,F(y)\,\alpha(r)\alpha(t)\,\big(\alpha(x) \pm F(x)\big)=0\,.$$

By primness of *R*, we have either $a\,F(y)=0$,

or $\alpha(t)\big(\alpha(x) \pm F(x)\big)=0$ for all $x,t \in I$.

Putting $t=tr$ in the above equation and using *R* as a prime ring, with *I* being a non-zero ideal of *R*, we get $\alpha(x) \pm F(x)=0$ for all $x \in I$.

At first, we suppose that

$$aF(y)=0 \quad \text{for all } y \in I. \quad (23)$$

By replacing ry by y in equation (23), we find that

$$a\,\big(F(y)\alpha(r)+\beta(y)d(r)\big)=0.$$

By using equation (23) in the above equation, we get $a\beta(y)d(r)=0$ for all $y \in I, r \in R$.

Putting $y=ry$ in the above equation gives $a\beta(r)\beta(y)d(r)=0$ for all $y \in I, r \in R$.

Since β is an automorphism of *R*, then $a\,R\,\beta(y)d(r)=0$.

By primness of *R* and $a \neq 0$, we have $\beta(y)d(r)=0$ for all $y \in I, r \in R$.

That is, $\beta(I)d(r)=0$.

Since $\beta(I)$ is an ideal, then we get $\beta(I)R\,d(r)=0$.

Since *R* is a prime ring with I is a nonzero ideal of R, then we get $d(R)=0$.

Next, we assume that

$$\alpha(x) \pm F(x)=0 \quad \text{for all } x \in I. \quad (24)$$

Assume that *R* is noncommutative and using rx instead of x in equation (24), where $r \in R$, we have

$$\alpha(r)\alpha(x) \pm F(x)\alpha(r) \pm \beta(x)d(r)+\alpha(xr)-\alpha(xr)=0\,.$$
$$\alpha[r,x] \pm \big(\alpha(x) \pm F(x)\big)\alpha(r) \pm \beta(x)\,d(r)=0.$$

By the application of equation (24), we find that

$$\alpha[r,x] \pm \beta(x)d(r) = 0 \quad \text{for all } x \in I, r \in R. \tag{25}$$

By taking rs in place of r in equation (25), where $r, s \in R$, we find that $\alpha[rs,x] \pm \beta(x\,)d(rs) = 0$.
Applying Lemma (2.4), gives

$$\alpha(r)\alpha[s,x] + \alpha[r,x]\alpha(s) \pm \beta(x)d(r)\alpha(s) = 0. \tag{26}$$

Multiplying the right side of equation (25) by $\alpha(s)$ yields

$$\alpha[r,x]\alpha(s) \pm \beta(x)d(r)\alpha(s) = 0 \quad \text{for all } x \in I, r, s \in R. \tag{27}$$

Subtract equation (27) from equation (26), gives $\alpha(r)\alpha[s,x] = 0$ for all $x \in I, r, s \in R$.
Since α is automorphism of R, then we get $r[s,x] = 0$ for all $x \in I, s, r \in R$.
By multiplying the left side of the above equation by $[s,x]$, we have $[s,x]R[s,x] = 0$.
By primness of R, we have $[R,I] = 0$ by Lemma (1.1), we get that R is commutative that is contradict with our assumption.
Using a similar approach, we can prove that the same product holds for the instance that $a(F(xy) - F(y)F(x)) = 0$ for all $x, y \in I$.

Theorem 2.7
Let R be a prime ring, I be a nonzero ideal of R, and $F: R \to R$ be a multiplicative (generalized) (α,β) reverse derivation associated with a map $d: R \to R$. If for some $0 \neq a \in R$ such that $\alpha(a) = \beta(a) = a$ and $a\big(F(xy) \pm \alpha(xy)\big) = 0$ for all $x, y \in I$, then R is commutative.
Proof
Suppose that R is noncommutative, consider the case

$$a(F(xy) + \alpha(xy)) = 0 \text{ for all } x, y \in I. \tag{28}$$

Substituting yz in the place of y in equation (28), gives

$$0 = a(F(xyz) + \alpha(xyz)) \text{ for all } x, y, z \in I,$$
$$0 = a(F(yz)\alpha(x) + \beta(yz)d(x) + \alpha(xyz)) + a\,\alpha(yzx) - a\,\alpha(yzx)$$
$$= a(F(yz) + \alpha(yz))\alpha(x) + a\,\beta(yz)\,d(x) + a\,\alpha[x,yz]$$

By using equation (28), we have

$$a\big(\beta(yz)d(x) + \alpha(y)\alpha[x,z] + \alpha[x,y]\alpha(z)\big) = 0 \text{ for all } x, y, z \in I. \tag{29}$$

By replacing y by ay in equation (29), we have

$$a\big(\beta(a)\beta(yz)d(x) + \alpha(a)\alpha(y)\alpha[x,z] + \alpha(a)\alpha[x,y]\alpha(z) + \alpha[x,a]\alpha(y)\alpha(z)\big) = 0.$$

By the assumption that $\beta(a) = a, \alpha(a) = a$, we obtain

$$a^2\beta(yz)d(x) + a^2\alpha(y)\alpha[x,z] + a^2\alpha[x,y]\alpha(z) + a\,\alpha[x,a]\alpha(y)\alpha(z) = 0. \tag{30}$$

Left multiplying equation (29), by a yields

$$a^2\big(\beta(yz)d(x) + \alpha(y)\alpha[x,z] + \alpha[x,y]\alpha(z)\big) = 0 \text{ for all } x, y, z \in I. \tag{31}$$

Subtract equation (31) from equation (30), we obtain $a\,\alpha[x,a]\alpha(y)\alpha(z) = 0$ for all $x, y, z \in I$.

$$\alpha^{-1}\big(\alpha(a[x,a]yz)\big) = 0.$$
$$a[x,a]yz = 0.$$

By primness of R with I being a nonzero ideal of R, we get $a[x,a]y = 0$ for all $x, y, z \in I$. Again, by primness of R and $I \neq 0$, we get $a[x,a] = 0$ for all $x \in I$. By Lemma (2.1), we find that $a \in Z(R)$.
Since R is a prime ring and $0 \neq a$, then equation (29) becomes

$$\beta(yz)d(x) + \alpha(y)\alpha[x,z] + \alpha[x,y]\alpha(z) = 0 \text{ for all } x, y, z \in I. \tag{32}$$

By substituting ty for y in equation (32), we find that

$$\beta(tyz)d(x) + \alpha(ty)\alpha[x,z] + \alpha[x,ty]\alpha(z) = 0.$$

This means that

$$\beta(t)\beta(yz)d(x) + \alpha(t)\alpha(y)\alpha[x,z] + \alpha(t)\alpha[x,y]\alpha(z) + \alpha[x,t]\alpha(y)\alpha(z) = 0. \tag{33}$$

By left multiplying equation (32) by $\alpha(t)$, we get

$$\alpha(t)\beta(yz)d(x) + \alpha(t)\alpha(y)\alpha[x,z] + \alpha(t)\alpha[x,y]\alpha(z) = 0, \text{ for all } x, y, z, t \in I \tag{34}$$

Comparing equation (33) and equation (34), we get

$$\big(\beta(t) - \alpha(t)\big)\beta(yz)d(x) + \alpha[x,t]\alpha(y)\alpha(z) = 0. \tag{35}$$

By putting $x = xt$ in equation (35), we have

$$\big(\beta(t) - \alpha(t)\big)\beta(yz)d(xt) + \alpha[xt,t]\alpha(y)\alpha(z) = 0.$$

By Lemma (2.4), since d is a multiplicative left α-centerlizer, then

$$\big(\beta(t) - \alpha(t)\big)\beta(yz)d(x)\alpha(t) + \alpha[x,t]\alpha(t)\alpha(y)\alpha(z) = 0. \tag{36}$$

From equation (35), we get $\big(\beta(t)-\alpha(t)\big)\beta(yz)d(x) = -\alpha[x,t]\alpha(y)\alpha(z)$.
By substituting the value $-\alpha[x,t]\alpha(y)\alpha(z)$ in equation (36), we find that $\alpha[x,t]\alpha[t,yz] = 0$.

$$\alpha^{-1}(\alpha[x,t]\alpha[t,yz]) = 0.$$

$$[x,t][t,yz] = 0 \text{ for all } x,y,z,t \in I. \quad (37)$$

By putting $z = xz$ in equation (37), we have $[x,t]yx[t,z] + [x,t][t,yx]z = 0$.
By the application of equation (37), we obtain $[x,t]yx[t,z] = 0$.
By primness of R, with I being a nonzero ideal of R, implies that $[I,I] = 0$, therefore I is commutative. By Lemma (1.1), then R is commutative, which contradicts our assumption.
Using a similar approach, we can prove that the same product holds for the instance that $a(F(xy) - \alpha(xy)) = 0$ for all $x,y \in I$.

Theorem 2.8
Let R be a prime ring, I be a nonzero ideal of R, and $F: R \to R$ be a multiplicative (generalized) (α,β) reverse derivation associated with a map $d: R \to R$. If for some $0 \neq a \in R$ such that $\alpha(a) = a$ and $a(F(x)F(y) \pm \alpha(xy)) = 0$ for all $x,y \in I$, then R is commutative.
Proof
Suppose that R is noncommutative, consider the case

$$a(F(x)F(y) + \alpha(xy)) = 0 \text{ for all } x,y \in I. \quad (38)$$

By replacing y by yt in equation (38), where $t \in I$, we obtain

$$0 = a(F(x)F(t)\alpha(y) + F(x)\beta(t)d(y) + \alpha(xyt) + \alpha(xty) - \alpha(xty))$$
$$= a(F(x)F(t) + \alpha(xt))\alpha(y) + aF(x)\beta(t)d(y) + a\alpha(x)\alpha[y,t]$$

By using equation (38), we find

$$aF(x)\beta(t)d(y) + a\alpha(x)\alpha[y,t] = 0 \text{ for all } x,y,t \in I. \quad (39)$$

By replacing t by zt in equation (39), where $z \in I$, we get

$$a(F(x)\beta(z)\beta(t)d(y) + \alpha(x)\alpha(z)\alpha[y,t] + \alpha(x)\alpha[y,z]\alpha(t)) = 0. \quad (40)$$

Now, by replacing x with zx in equation (39), we have

$$a\big(F(x)\alpha(z)\beta(t)d(y) + \beta(x)d(z)\beta(t)d(y)\big) + a\alpha(z)\alpha(x)\alpha[y,t] = 0. \quad (41)$$

By subtraction equation (40) from equation (41), we have

$$aF(x)\big(\alpha(z)-\beta(z)\big)\beta(t)d(y) + a\beta(x)d(z)\beta(t)d(y) + a\alpha[z,x]\alpha[y,t] - a\alpha(x)\alpha[y,z]\alpha(t) = 0 \text{ for all } x,y,z,t \in I. \quad (42)$$

Putting $y = yt$ in equation (42) and applying Lemma (2.4), yields

$$a(F(x)\big(\alpha(z)-\beta(z)\big)\beta(t)d(y)\alpha(t) + \beta(x)d(z)\beta(t)d(y)\alpha(t) + \alpha[z,x]\alpha[y,t]\alpha(t) - \alpha(x)\alpha(y)\alpha[t,z]\alpha(t) - \alpha(x)\alpha[y,z]\alpha(t)\alpha(t)) = 0 \text{ for all } x,y,z,t \in I. \quad (43)$$

Multiplying the right-hand side of equation (42) by (t) , gives

$$aF(x)\big(\alpha(z)-\beta(z)\big)\beta(t)d(y)\alpha(t) + a\beta(x)d(z)\beta(t)d(y)\alpha(t) + a\alpha[z,x]\alpha[y,t]\alpha(t) - a\alpha(x)\alpha[y,z]\alpha(t)\alpha(t) = 0. \quad (44)$$

Subtracting equation (43) from equation (44) gives $a\alpha(x)\alpha(y)\alpha[t,z]\alpha(t) = 0$ for all $x,y,z,t \in I$.

$$\alpha^{-1}\alpha(axy[t,z]t) = 0,$$

$$axy[t,z]t = 0.$$

By primness of R with $a \neq 0$, we get $xy[t,z]t = 0$. Once more, by primness of R, we find that $[t,z]t = 0$ for all $t,z \in I$.
Putting $z = zx$ in the last equation and using it, yield $[t,z]xt = 0$ for all $x,t,z \in I$.
By primness of R, with I being a nonzero ideal of R, we get $[t,z]x = 0$ for all $x,t,z \in I$.
Since R is a prime and $0 \neq I$, we get $[I,I] = 0$, therefore, I is commutative. By Lemma (1.1), R is commutative, which is a contradiction with our assumption.
Using a similar approach, we can prove that the same product holds for the instance

$$a(F(x)F(y) - \alpha(xy)) = 0 \text{ for all } x,y \in I.$$

Theorem 2.9
Let R be a prime ring, I be a nonzero ideal of R, and $F: R \to R$ be a multiplicative (generalized) (α,β) reverse derivation associated with a map $d: R \to R$. If for some $0 \neq a \in R$ such that $a\big(F(xy) \pm F(y)F(x)\big) = 0$ for all $x,y \in I$, then either $d = 0$ or R is commutative.
Proof
Suppose that R is noncommutative, consider the case

$$a\big(F(xy) + F(y)F(x)\big) = 0 \quad \text{for all } x, y \in I. \tag{45}$$

Substituting zx for x in equation (45), where $z \in I$, gives

$$0 = a(F(xy)\alpha(z) + \beta(xy)d(z) + F(y)F(x)\alpha(z) + F(y)\beta(x)d(z)$$
$$= a(F(y)\alpha(x)\alpha(z) + \beta(y)d(x)\alpha(z) + a\big(\beta(xy)d(z) + F(y)F(x)\alpha(z) + F(y)\beta(x)d(z)\big) \tag{46}$$

Or, $a\big(F(y)\alpha(zx) + \beta(y)d(zx) + F(y)F(x)\alpha(z) + F(y)\beta(x)d(z)\big) = 0.$

By using Lemma (2.4), we get

$$a\big(F(y)\alpha(z)\alpha(x) + \beta(y)d(z)\alpha(x) + F(y)F(x)\alpha(z) + F(y)\beta(x)d(z)\big) = 0. \tag{47}$$

By comparing equation (47) and equation (46), we get

$$a\big(F(y)\alpha[z,x] + \beta(y)d(z)\alpha(x) - \beta(y)d(x)\alpha(z) - \beta(x)\beta(y)d(z)\big) = 0. \tag{48}$$

Replacing x by xz in equation (48), yields

$$a\big(F(y)\alpha[z,xz] + \beta(y)d(z)\alpha(xz) - \beta(y)d(xz)\alpha(z) - \beta(xz)\beta(y)d(z)\big) = 0.$$

Applying Lemma (2.4), gives

$$a\big(F(y)\alpha[z,x]\alpha(z) + \beta(y)d(z)\alpha(x)\alpha(z) - \beta(y)d(x)\alpha(z)\alpha(z) - \beta(x)\beta(z)\beta(y)d(z)\big) = 0 . \tag{49}$$

Right multiplying equation (48), by $\alpha(z)$ gives

$$a(F(y)\alpha[z,x]\alpha(z) + \beta(y)d(z)\alpha(x)\alpha(z) - \beta(y)d(x)\alpha(z)\alpha(z) - \beta(x)\beta(y)d(z)\alpha(z)) = 0. \tag{50}$$

Subtracting equation (49) from equation (50), gives

$$a\beta(x)(\beta(z)\beta(y)d(z) - \beta(y)d(z)\alpha(z)) = 0 \text{ for all } x, y, z \in I. \tag{51}$$

Putting $x = rx$ in equation (51), where $r \in R$, and since β is automorphism of R, yields

$$a\, R\, \beta(x)\big(\beta(z)\beta(y)d(z) - \beta(y)d(z)\alpha(z)\big) = 0.$$

Once more, by primness of R, we arrive at

$$\beta(z)\beta(y)d(z) - \beta(y)d(z)\alpha(z) = 0 \text{ for all } y, z \in I. \tag{52}$$

Let $y = xy$ in equation (52), then we have

$$\beta(z)\beta(x)\beta(y)d(z) - \beta(x)\beta(y)d(z)\alpha(z) = 0 \quad \text{for all } x, y, z \in I. \tag{53}$$

From equation (52), we get $\beta(y)d(z)\alpha(z) = \beta(z)\beta(y)d(z)$.

By substituting the value $\beta(z)\beta(y)d(z)$ in equation (53), we obtain $\beta[z,x]\beta(y)d(z) = 0$.

$$\beta^{-1}\big(\beta[z,x]\beta(y)d(z)\big) = 0.$$
$$[z,x]y\,\beta^{-1}\big(d(z)\big) = 0 \text{ for all } x, y, z \in I.$$

By primness of R, we get either $[z,x]y = 0$ for all $x, y, z \in I$ or $\beta^{-1}\big(d(z)\big) = 0$.

If $\beta^{-1}\big(d(z)\big) = 0$ then $d(I) = 0$. By Lemma (2.6), we get $d(R) = 0$.

On the other hand, if $[z,x]y = 0$. By primness of R, with I being a nonzero ideal of R, we find that I is commutative. By Lemma (1.1), R is commutative, which contradicts our assumption.

Using a similar approach, we can prove that the same product holds for the instance $a\big(F(xy) - F(y)F(x)\big) = 0$ for all $x, y \in I$.

Theorem 2.10

Let R be a prime ring, I be a nonzero ideal of R, and $F: R \to R$ be a multiplicative (generalized) (α, β) reverse derivation associated with a map $d: R \to R$. If for some $0 \neq a \in R$, such that $a\big(F(xy) \pm F(x)F(y)\big) = 0$ for all $x, y \in I$, then either $d = 0$ or R is commutative.

Proof

Suppose that R is noncommutative, consider the case

$$a\big(F(xy) + F(x)F(y)\big) = 0 \text{ for all } x, y \in I.$$
$$a\big(F(y)\alpha(x) + \beta(y)d(x) + F(x)F(y)\big) = 0 \text{ for all } x, y \in I. \tag{54}$$

By putting $y = zy$ in equation (54), where $z \in I$, we obtain

$$a\big(F(zy)\alpha(x) + \beta(zy)d(x) + F(x)F(zy)\big) = 0,$$
$$a\big(F(y)\alpha(z)\alpha(x) + \beta(y)d(z)\alpha(x) + \beta(z)\beta(y)d(x) + F(x)F(y)\alpha(z) + F(x)\beta(y)d(z)\big) = 0. \tag{55}$$

Right multiply equation (54), by $\alpha(z)$, we have

$$a(F(y)\alpha(x)\alpha(z) + \beta(y)d(x)\alpha(z) + F(x)F(y)\alpha(z)) = 0 . \tag{56}$$

Subtract equation (56) from equation (55), gives

$$a\big(F(y)\alpha[z,x] + \beta(y)d(z)\alpha(x) + F(x)\beta(y)d(z) + \beta(z)\beta(y)d(x) - \beta(y)d(x)\alpha(z)\big) = 0. \tag{57}$$

By writing z by zx in equation (57), we find that

$$a\big(F(y)\alpha[zx,x] + \beta(y)d(zx)\alpha(x) + \beta(zx)\beta(y)d(x) + F(x)\beta(y)d(zx) - \beta(y)d(x)\alpha(zx)\big) = 0.$$

By applying Lemma (2.4), in above equation, then

$a(F(y)\alpha[z,x]\alpha(x) + \beta(y)d(z)\alpha(x)\alpha(x) + \beta(z)\beta(x)\beta(y)d(x) + F(x)\beta(y)d(z)\alpha(x) - \beta(y)d(x)\alpha(z)\alpha(x)) = 0$ for all $x,y,z \in I$. (58)

Right multiplying equation (57), by $\alpha(x)$ gives

$a\big(F(y)\alpha[z,x]\alpha(x) + \beta(y)d(z)\alpha(x)\alpha(x) + F(x)\beta(y)d(z)\alpha(x) + \beta(z)\beta(y)d(x)\alpha(x) - \beta(y)d(x)\alpha(z)\alpha(x)\big) = 0$ for all $x,y,z \in I$. (59)

Subtract equation (59) from equation (58), gives

$$a\beta(z)\big(\beta(x)\beta(y)d(x) - \beta(y)d(x)\alpha(x)\big) = 0 \text{ for all } x,y,z \in I. \quad (60)$$

Now, we replace z by rz in equation (60), where $r \in R$, to have

$$a\beta(r)\beta(z)\big(\beta(x)\beta(y)d(x) - \beta(y)d(x)\alpha(x)\big) = 0.$$

By using R as a prime ring with $a \neq 0$, we get

$$\beta(z)\big(\beta(x)\beta(y)d(x) - \beta(y)d(x)\alpha(x)\big) = 0.$$

By putting $z = zr$ in the above equation, we have

$$\beta(z)\beta(r)\big(\beta(x)\beta(y)d(x) - \beta(y)d(x)\alpha(x)\big) = 0.$$

Since R is a prime ring and I is a non-zero ideal of R, then we find that

$$\beta(x)\beta(y)d(x) - \beta(y)d(x)\alpha(x) = 0. \quad (61)$$

Let $y = zy$ in the above equation, then we have

$$\beta(x)\beta(z)\beta(y)d(x) - \beta(z)\beta(y)d(x)\alpha(x) = 0. \quad (62)$$

From equation (61), to get $\beta(y)d(x)\alpha(x) = \beta(x)\beta(y)d(x)$

Substituting the value $\beta(x)\beta(y)d(x)$ in equation (62), gives

$$\beta[x,z]\beta(y)d(x) = 0,$$

$$\beta^{-1}\big(\beta[x,z]\beta(y)d(x)\big) = 0.$$

That is,

$$[x,z]y\,\beta^{-1}\big(d(x)\big) = 0.$$

By primness of R, we get either $[x,z]y = 0$ for all $z,y,x \in I$, or $\beta^{-1}\big(d(x)\big) = 0$.

If $[x,z]y = 0$ since R is a prime ring and $I \neq 0$, then we get $[x,z] = 0$ for all $x,z \in I$. Therefore, I is commutative. By Lemma (1.1), R is commutative, which contradicts our assumption.

On the other hand, if $\beta^{-1}(\,d(x)) = 0$ since β is automorphism of R, then we find that $d(I) = 0$.

In this case, $F(xy) = F(y)\alpha(x)$ for all $x,y \in I$.

Therefore, our hypothesis implies that

$$a\big(F(y)\alpha(x) + F(x)F(y)\big) = 0 \text{ for all } x,y \in I. \quad (63)$$

By putting $y = zy$ where $z \in I$ in equation (63), we find that

$$a\big(F(y)\alpha(z)\alpha(x) + \beta(y)d(z)\alpha(x) + F(x)F(y)\alpha(z) + F(x)\beta(y)d(z)\big) = 0.$$

Since $d(I) = 0$, we get

$$a\big(F(y)\alpha(z)\alpha(x) + F(x)F(y)\alpha(z)\big) = 0 \text{ for all } x,y,z \in I. \quad (64)$$

Multiplying the right side of equation (63), by $\alpha(z)$ implies that

$$a\big(F(y)\alpha(x)\alpha(z) + F(x)F(y)\alpha(z)\big) = 0 \text{ for all } x,y,z \in I. \quad (65)$$

Subtract equation (64) from equation (65), to find that

$$aF(y)[\alpha(z),\alpha(x)] = 0 \text{ for all } x,y,z \in I. \quad (66)$$

By replacing y with yr, where $r \in R$, in equation (66), we have

$$a(F(r)\alpha(y)[\alpha(z),\alpha(x)] + \beta(r)d(y)[\alpha(z),\alpha(x)]) = 0.$$

Since $d(I) = 0$, we obtain $aF(r)\alpha(y)[\alpha(z),\alpha(x)] = 0$ for all $x,y,z \in I, r \in R$.

By putting $y = ry$ in the above relation, we get $aF(r)\alpha(r)\alpha(\,y\,)\alpha[z,x] = 0$ for all $x,y,z \in I, r \in R$.

$$a\,F(r)R\,\alpha(y)\alpha[z,x] = 0.$$

By primness of R, we get either $aF(r) = 0$, or $\alpha(y)\alpha[z,x] = 0$ for all $y,z,x \in I$.

If $\alpha(y)\alpha[z,x] = 0$ for all $x,z,y \in I$.

By taking α^{-1} in above equation then we get $y[z,x] = 0$.

By primness of R, with I be a non-zero ideal of R, we get $[I,I] = 0$, by Lemma (1.1), we conclude that R is commutative, which contradicts with our assumption.

Next, assume that

$$aF(r) = 0 \text{ for all } r \in R. \quad (67)$$

By putting $r = ry$ in equation (67), we find that $aF(ry) = 0$ for all $y \in I, r \in R$.

This implies that

$$a\big(F(y)\alpha(r) + \beta(y)d(r)\big) = 0.$$

By using equation (67) in the above equation, it becomes $a\beta(y)d(r) = 0$.
By putting $y = ry$ in the above equation, we have $a\beta(r)\beta(y)d(r) = 0$.
Since R is a prime ring and $a \neq 0$, then $\beta(y)d(r) = 0$ for all $y \in I, r \in R$.
Putting $y = yr$ in the last equation and using R as a prime ring, we obtain $d(R) = 0$.
Using a similar approach, we can prove that the same product holds for the instance

$$a\big(F(xy) - F(x)F(y)\big) = 0 \text{ for all } x, y \in I.$$

Theorem 2.11

Let R be a prime ring, I be a nonzero ideal of R and $F: R \to R$ be a multiplicative (generalized) (α, β) reverse derivation associated with a map $d: R \to R$. If for some $0 \neq a \in R$ such that $\alpha(a) = a$ and $a(F(x)F(y) \pm \alpha(yx)) = 0$ for all $x, y \in I$, then R is commutative.

Proof

Suppose that R is noncommutative, consider the case

$$a(F(x)F(y) + \alpha(yx)) = 0 \text{ for all } x, y \in I. \quad (68)$$

By substituting zy in place of y in equation (68), we obtain

$$\begin{aligned} 0 &= a(F(x)F(zy) + \alpha(zyx)) \text{ for all } x, y, z \in I, \\ &= a(F(x)F(y)\alpha(z) + F(x)\beta(y)d(z) + \alpha(zyx) + a\alpha(yxz) - a\alpha(yxz) \\ &= a(F(x)F(y) + \alpha(yx))\alpha(z) + aF(x)\beta(y)d(z) + a\alpha(zyx - yxz). \end{aligned}$$

By using equation (68), we have

$$a(F(x)\beta(y)d(z) + a\alpha[z, yx]) = 0,$$

$$a(F(x)\beta(y)d(z) + \alpha(y)\alpha[z, x] + \alpha[z, y]\alpha(x) = 0 \text{ for all } x, y, z \in I. \quad (69)$$

Now, by replacing z with zt in equation (69), where $t \in I$ we get

$$a\big(F(x)\beta(y)d(zt) + \alpha(y)\alpha[zt, x] + \alpha[zt, y]\alpha(x)\big) = 0.$$

By applying Lemma (2.4), we get

$$a(F(x)\beta(y)d(z)\alpha(t) + \alpha(y)\alpha(z)\alpha[t, x] + \alpha(y)\alpha[z, x]\alpha(t) + \alpha(z)\alpha[t, y]\alpha(x) + \alpha[z, y]\alpha(t)\alpha(x)) = 0 \text{ for all } x, y, z, t \in I. \quad (70)$$

By right multiplying equation (69), by $\alpha(t)$, we get

$$a(F(x)\beta(y)d(z)\alpha(t) + \alpha(y)\alpha[z, x]\alpha(t) + \alpha[z, y]\alpha(x)\alpha(t)) = 0. \quad (71)$$

Subtract equation (71) from equation (70), gives

$$a\big(\alpha(y)\alpha(z)\alpha[t, x] + \alpha(z)\alpha[t, y]\alpha(x) + \alpha[z, y]\alpha(t)\alpha(x) - \alpha[z, y]\alpha(x)\alpha(t)\big) = 0.$$

Since $\alpha(a) = a$, then we obtain

$$\alpha(a)\big(\alpha(y)\alpha(z)\alpha[t, x] + \alpha(z)\alpha[t, y]\alpha(x) + \alpha[z, y]\alpha(t)\alpha(x) - \alpha[z, y]\alpha(x)\alpha(t)\big) = 0.$$

$$\alpha^{-1}\alpha\big(a(yz[t, x] + z[t, y]x + [z, y][t, x])\big) = 0.$$

$$a(yztx - yzxt + ztyx - zytx + zytx - yztx - zyxt + yzxt) = 0.$$

This implies that $a\, z\, [t, yx] = 0$.
By putting $z = rz$ in the above relation, where $r \in R$, we get $a\, R\, z\, [t, yx] = 0$.
By primness of R, with $0 \neq a$, we get $z[t, yx] = 0$ for all $z, t, y, x \in I$.
Again, by primness of R, with I is a nonzero ideal of R, implies that $[t, yx] = 0$. By Lemma (2.3), we conclude that R is commutative, which contradicts our assumption.
Using a similar approach, we can prove that the same product holds for the instance

$$a\big(F(x)F(y) - \alpha(yx)\big) = 0 \text{ for all } x, y \in I.$$

Theorem 2.12

Let R be a prime ring, I be a nonzero ideal of R and $F: R \to R$ be a multiplicative (generalized) (α, β) reverse derivation associated with a map $d: R \to R$. If for some $0 \neq a \in R$, such that $\beta(a) = a$, $\alpha(a) = a$ and $a(F(xy) \pm \alpha(yx)) = 0$ for all $x, y \in I$, then R is commutative.

Proof

Suppose that R is noncommutative, consider the case

$$a(F(xy) + \alpha(yx)) = 0 \text{ for all } x, y \in I. \quad (72)$$

By writing zx by x in equation (72), where $z \in I$, we obtain

$$0 = a(F(xy)\alpha(z) + \beta(xy)d(z) + \alpha(yzx)) + a\alpha(yxz) - a\alpha(yxz)$$

By using equation (72) in the above equation, we get

$$a(\beta(x)\beta(y)d(z) + \alpha(y)\alpha\, [z, x]) = 0 \text{ for all } x, y, z \in I. \quad (73)$$

By substituting ax in place of x in equation (73), we have

$$a(\beta(ax)\beta(y)d(z)+\alpha(y)\alpha[z,ax])=0.$$

This means that

$$a(\beta(a)\beta(x)\,\beta(y)d(z)+\alpha(y)\alpha(a)\alpha[z,x]+\alpha(y)\alpha[z,a]\alpha(x))=0.$$

By the assumption that $\beta(a)=a$, $\alpha(a)=a$, we obtain

$$a^2\,\beta(x)\beta(y)d(z)+a\alpha(y)a\,\alpha[z,x]+a\alpha(y)\alpha[z,a]\alpha(x)=0. \quad (74)$$

By multiplying the left side of equation (73) by a, we get

$$a^2\,\beta(x)\beta(y)d(z)+a^2\,\alpha(y)\alpha[z,x]=0 \text{ for all } x,y,z\in I. \quad (75)$$

Subtract equation (75) from equation (74), we find that

$$a([\alpha(y),a]\alpha[z,x]+\alpha(y)\alpha[z,a]\alpha(x))=0 \text{ for all } x,y,z\in I. \quad (76)$$

By putting zx instead of z in equation (76), we get

$$a([\alpha(y),a]\alpha[zx,x]+\alpha(y)\alpha[zx,a]\alpha(x))=0,$$

Which means that

$$a([\alpha(y),a]\alpha[z,x]\alpha(x)+\alpha(y)\alpha(z)\alpha[x,a]\alpha(x)+\alpha(y)\alpha[z,a]\alpha(x)\alpha(x))=0.$$

By using equation (76), we get

$$a\,\alpha(y)\alpha(\,z)\alpha[x,a]\alpha(x)=0 \text{ for all } x,y,z\in I. \quad (77)$$

Since $\alpha(a)=a$, we obtain

$$\alpha^{-1}\alpha(ayz[x,a]x)=0.$$

Implies that

$$a\;y\;z\;[x,a]x=0.$$

Now, we replace y by ry in the above equation, where $r\in R$, to have $a\,ry\,z[x,a]x=0$ for all $x,y,z\in I, r\in R$.

By primness of R and $a\neq 0$, we find that $yz[x,a]x=0$ for all $x,y,z\in I$.

Once more, by primness of R, we get $[x,a]x=0$ for all $x\in I$.

By Lemma (2.2), we get $a\in Z(R)$ or R is commutative, which contradicts our assumption.

Suppose that $a\in Z(R)$ and R is a prime ring, because $0\neq a\in R$, so by equation (73), we find that

$$\beta(x)\beta(y)d(z)+\alpha(y)\alpha[z,x]=0 \text{ for all } x,y,z\in I. \quad (78)$$

By putting $x=tx$ in equation (78), where $t\in I$, we get

$$\beta(t)\beta(\,x\,y)\,d(z)+\alpha(y)\alpha(t)\alpha[z,x]+\alpha(y)\alpha[z,t]\alpha(x)=0\,. \quad (79)$$

By multiplying the left side of equation (78) by $\alpha(t)$, we have

$$\alpha(t)\beta(x\,y)d(z)+\alpha(t)\alpha(\,y)\alpha[z,x]=0 \;\text{ for all } x,y,z\in I, r\in R. \quad (80)$$

By combining equation (79) and equation (80), we obtain

$$\big(\beta(t)-\alpha(t)\big)\beta(xy)d(z)+\alpha[y,t]\alpha[z,x]+\alpha(y)\alpha[z,t]\alpha(x)=0. \quad (81)$$

By substituting zt in place of z in equation (81), we obtain

$$\big(\beta(t)-\alpha(t)\big)\beta(xy)d(zt)+\alpha[y,t]\alpha[zt,x]+\alpha(y)\alpha[zt,t]\alpha(x)=0.$$

By using Lemma (2.4), we get

$$\big(\beta(t)-\alpha(t)\big)\beta(xy)d(z)\alpha(t)+\alpha[y,t]\alpha(z)\alpha[t,x]+\alpha[y,t]\alpha[z,x]\alpha(t)+$$
$$\alpha(y)\alpha[z,t]\alpha(t)\alpha(x)=0 \text{ for all } x,y,z,t\in I. \quad (82)$$

Now, by multiplying equation (81), by $\alpha(t)$ from the right, implies

$$\big(\beta(t)-\alpha(t)\big)\beta(xy)d(z)\alpha(t)+\alpha[y,t]\alpha[z,x]\alpha(t)+\alpha(y)\alpha[z,t]\alpha(x)\alpha(t)=0. \quad (83)$$

Subtract equation (82) from equation (83), we get $\alpha(y)\alpha[z,t]\alpha[x,t]-\alpha[y,t]\alpha(z)\alpha[t,x]=0$.

$$\alpha^{-1}\alpha(y[z,t][x,t]-[y,t]z[t,x])=0.$$
$$y[z,t][x,t]-[y,t]z[t,x]=0.$$
$$yztxt-yzttx-ytzxt+ytztx-ytztx+ytzxt+tyztx-tyzxt=0.$$
$$[yz,t][x,t]=0 \quad \text{ for all } x,y,z,t\in I. \quad (84)$$

By putting $x=wx$ in equation (84), where $w\in I$, we have

$$[yz,t][wx,t]=0,$$
$$[yz,t]w[x,t]+[yz,t][w,t]x=0.$$

By Relation (84), we get

Since R is a prime ring, we get either $[yz,t]=0$, or $w[x,t]=0$.

If $w[x,t]=0$ for all $x,t,w\in I$. By primness of R with I be a non-zero ideal of R, we get $[I,I]=0$, which means that I is commutative, by Lemma (1.1), we conclude that R is commutative,

which contradicts our assumption. Alternatively, $[yz,t]=0$ for all $y,z,t\in I$. By Lemma (2.3), we conclude that R is commutative, which contradicts our assumption.
Using a similar approach, we can prove that the same product holds for the instance

$$a(F(xy)-\alpha(yx))=0 \text{ for all } x,y\in I.$$

Example 2.13

Consider the ring $R=\left\{\begin{bmatrix}0&a&b\\0&0&c\\0&0&0\end{bmatrix}: a,b,c\in Z\right\}$,where Z is the set of integers. Let us define $F,d,\alpha,\beta: R\to R$ by

$$d\left(\begin{bmatrix}0&a&b\\0&0&c\\0&0&0\end{bmatrix}\right)=\begin{bmatrix}0&a&bc\\0&0&c\\0&0&0\end{bmatrix},\alpha\left(\begin{bmatrix}0&a&b\\0&0&c\\0&0&0\end{bmatrix}\right)=\begin{bmatrix}0&a&b\\0&0&c\\0&0&0\end{bmatrix}$$

$$\beta\left(\begin{bmatrix}0&a&b\\0&0&c\\0&0&0\end{bmatrix}\right)=\begin{bmatrix}0&-a&b\\0&0&-c\\0&0&0\end{bmatrix}.$$

Cleary, d is a multiplicative (α,β) reverse derivation. Let $F=d$, then F is a multiplicative (generalized) (α,β) reverse derivation associated with the mapping d on R, where α and β are automorphisms of R. It is easy to see that the identities $a(F(xy)\pm\alpha(xy))=0, a(F(xy)\pm\alpha(yx))=0$, $a(F(x)F(y)\pm\alpha(xy))=0$ and $a(F(x)F(y)\pm\alpha(yx))=0$ are satisfied for some $a\in R$ and for all $x,y\in R$. Here, R is not commutative, hence the primness condition of the ring in our results is essential.

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