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On Some Approximation Properties for a Sequence of λ-Bernstein Type Operators

Ali Jassim Muhammad*, Asma Jaber

[1]*Department of Mathematics, College of Education for Pure Sciences, University of Basrah, Basrah, IRAQ.*

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Abstract

In 2010, Long and Zeng introduced a new generalization of the Bernstein polynomials that depends on a parameter λ and called λ-Bernstein polynomials. After that, in 2018, Lain and Zhou studied the uniform convergence for these λ-polynomials and obtained a Voronovaskaja-type asymptotic formula in ordinary approximation. This paper studies the convergence theorem and gives two Voronovaskaja-type asymptotic formulas of the sequence of λ-Bernstein polynomials in both ordinary and simultaneous approximations. For this purpose, we discuss the possibility of finding the recurrence relations of the m-th order moment for these polynomials and evaluate the values of λ-Bernstein for the functions t^m, where m is a non-negative integer.

Keywords: λ-Bernstein polynomials, Voronovaskaja type asymptotic formula, the uniform convergence, ordinary and simultaneous approximations.

بعض خواص التقريب لمتتابعة مؤثرات من النمط λ−Bernstein

علي جاسم محمد* ،اسماء جابر

قسم الرياضيات، كلية التربية للعلوم الصرفة، جامعة البصرة، البصرة، العراق

الخلاصه

في عام 2010، قدم Long وZeng تعميما جديدا لمتعددة الحدود Bernstein يعتمد على معلمة λ سمي متعددة حدود $\lambda -$ Bernstein. بعد ذلك، في 2018 درس Lain و Zhou مبرهنة التقارب المنتظم التقريب العادي لهذه المتعددات من النمط λ وحصل على صيغة مشابهة لـ Voronovaskaja في التقريب العادي. هذا البحث يدرس مبرهنة التقارب ويعط صيغتين متشابهتين لصيغة Voronovaskaja لهذه المتتابعة من النمط $\lambda -$ Bernstein ولكلا التقريبين العادي والمتعدد. لهذا الغرض ناقشنا إمكانية ايجاد علاقات تكرار العزم من الرتبة m لهذه المتتابعات وحساب قيم $\lambda -$ Bernstein للدوال t^m، m هو عدد صحيح غير سالب.

1. Introduction

Let S be the linear space of all real functions acting on a set $X \neq \phi$. The operator $M: S \to S$ is linear and positive if it satisfies:

i) $\forall \alpha, \beta \in \mathbb{R}, M(\alpha f + \beta g) = \alpha M(f) + \beta M(g)$, where $f, g \in S$;

ii) $\forall f \in S: f \geq 0$, we have $M(f) \geq 0$.

*Email: alijasmoh@gmail.com

Bernstein, in 2012, [1] introduced another proof of the Weierstrass approximation theorem by using a sequence of positive linear operators, named the classical Bernstein polynomials, as

$$\Gamma_n: C[0,1] \to C[0,1], \quad \Gamma_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k}(x), f \in C[0,1].$$

Where

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, x \in [0,1]$$

In 1932 [2], Voronovaskaja showed that for $f \in C^2[0,1]$, the term of n^{-1} in $\{B_n(f;x) - f(x)\}$ exists and equals to $\frac{x(1-x)}{2} f''(x)$. This discovery appeared upon the evaluation of the limit $\lim_{n\to\infty} n\{B_n(f;x) - f(x)\}$. So, the order of approximation by using Bernstein polynomials is $O(n^{-1})$. This phenomenon, in general, is valid for most sequences of positive linear operators [2]. The evaluation of the approximation order for the different sequences is called Voronovaskaja-type asymptotic formulas. The order of $B_n(f;x)$, $O(n^{-1})$ shows that the convergence of $B_n(f;x)$ to the function f as n tends to infinity is very slow.

In 1953, Korovkin [3] introduced a simple tool to decide that, for a sequence of linear positive operators, M_n is converges to the function $f \in C[a,b]$, by checking the sequence's values of $M_n(t^m;x) \to x^m$ uniformly as $n \to \infty, m = 0,1,2$. These are called Korovkin's conditions.

Many generalizations of Korovkin's theorem to a compact subset of the real numbers $\mathbb{R}$ or the interval $[0,\infty)$ were introduced and studied. We refer here to Bohman [4, 1953] and Baskakov [5, 1957].

In 1962 [6], Schurer introduced a sequence based on a parameter and proved that the sequence has an approximation order depending on the parameter. After that, many kinds of research were developed and studied sequences depending on parameters; here we refer to [7, 8, 9, 10, 11,12].

In 2010 [13], Long and Zeng introduced a new generalization of the classical Bernstein sequence that depends on a parameter λ, as follows:

$$\Upsilon_{n,\lambda}(f;x) = \sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda;x) f\left(\frac{k}{n}\right),$$

$$\tilde{b}_{n,k}(\lambda;x) = \begin{cases} b_{n,0}(x) - \frac{\lambda}{n+1} b_{n+1,1}(x) & ; \quad k=0 \\ b_{n,k}(x) + \lambda\left(\frac{n-2k+1}{n^2-1} b_{n+1,k}(x) - \frac{n-2k-1}{n^2-1} b_{n+1,k+1}(x)\right) & ; \quad (1 \le k \le n-1) \\ b_{n,n}(x) - \frac{\lambda}{n+1} b_{n+1,n}(x) & ; \quad k=n, \end{cases}$$

where $\lambda \in [-1,1]$.

When $\lambda = 0$, the function $\tilde{b}_{n,k}(\lambda;x)$ is reduced to $b_{n,k}(x)$.

In 2018 [14], Lain and Zhou studied the uniform convergence and obtained a Voronovaskaja-type asymptotic formula for the sequence $\Upsilon_{n,\lambda}(f;x)$ in ordinary approximation.

2. Preliminary Results

Some preliminaries for the sequence $\Upsilon_{n,\lambda}(f;x)$ are introduced in this section and will be used to achieve the main results.

We assume that $m \in N^0 = \{0,1,\ldots\}$, $\phi_{n,m}(x) = \sum_{k=0}^{n} k^m\, b_{n,k}(x)$, $\tilde{\phi}_{n,m,\lambda}(x) = \sum_{k=0}^{n} k^m\, \tilde{b}_{n,k}(\lambda;x)$, and $TLP(x)$ mean terms in lower powers of x.

Lemma 2.1

The following properties are held:

(i) $x(1-x)\tilde{b}'_{n,k}(\lambda;x) =$
$(k-nx)\tilde{b}_{n,k}(\lambda;x) - x\tilde{b}_{n,k}(\lambda;x) + xb_{n,k}(x) - \frac{n-2k-1}{n^2-1}\lambda b_{n+1,k+1}(x)$.

(ii) $\tilde{\phi}_{n,0,\lambda}(x) = 1, \phi_{n,0}(x) = 1, \phi_{n+1,0}(x) = 1,$ and $\tilde{\phi}_{n,m+1,\lambda}(x) + \frac{2}{n^2-1}\lambda\phi_{n+1,m+1}(x) =$
$x(1-x)\tilde{\phi}'_{n,m+1,\lambda}(x) + (n+1)x\tilde{\phi}_{n,m+1,\lambda}(x) - x\phi_{n,m}(x) + \frac{1}{n+1}\lambda\phi_{n+1,m}(x), m \geq 1.$

Proof

By direct evaluation, the property (i) follows quickly.
Using the fact that $x(1-x)b_{n,k}(x) = (k-nx)b'_{n,k}(x)$, thus

$$\tilde{\phi}_{n,m+1,\lambda}(x) + \frac{2}{n^2-1}\lambda\phi_{n+1,m+1}(x) = x(1-x)\tilde{\phi}'_{n,m+1,\lambda}(x) + (n+1)x\tilde{\phi}_{n,m+1,\lambda}(x) - x\phi_{n,m}(x) + \frac{1}{n+1}\lambda\phi_{n+1,m}(x), m \geq 1.$$

Then, the property (ii) holds. ■

Lemma 2.2

The sequence $\Upsilon_{n,\lambda}$ satisfies:

(i) $\Upsilon_{n,\lambda}(1;x) = 1$;

(ii) $\Upsilon_{n,\lambda}(t;x) = x + \frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n(n-1)}\lambda$;

(iii) $\Upsilon_{n,\lambda}(t^2;x) = x^2 + \frac{x(1-x)}{n} + \lambda\left\{\frac{(2x-4x^2+2x^{n+1})}{n(n-1)} + \frac{x^{n+1}+(1-x)^{n+1}-1}{n^2\,(n-1)}\right\}$;

(iv) $\Upsilon_{n,\lambda}(t^m;x) =$
$\frac{1}{n^m}\left\{\frac{n!}{(n-m)!}x^m + \frac{m(m-1)n!}{2(n-m+1)!}x^{m-1} + TLP(x)\right\} + \frac{\lambda}{n^m}\left\{\frac{-2m(n+1)!}{(n^2-1)(n-m+1)!}x^m\,(1-x^{n-m+1}) + \left(\frac{nm(m-1)(n+1)!}{2(n^2-1)(n-m+2)!} + \frac{m(m-1)(n+1)!}{2(n^2-1)(n-m+1)!}\right)x^{m-1}\right.$
$$\left.(1-x^{n-m+2}) + TLP(x) - \frac{(-1)^m}{(n-1)}(1-(1-x)^{n+1}-x^{n+1})\right\}.$$

The consequences (i)-(iii) is proved in [5]. The proof of consequence (iv) is given as follows

$$\Upsilon_{n,\lambda}(t^m;x) = \sum_{k=0}^{n}\tilde{b}_{n,k}\,(\lambda;x).t^m = \sum_{k=0}^{n}t^m\left(b_{n,k}(x) + \lambda\left\{\left(\frac{1}{n-1} - \frac{2k}{n^2-1}\right)b_{n+1,k}(x) - \left(\frac{1}{n+1} - \frac{2k}{n^2-1}\right)b_{n+1,k+1}(x)\right\}\right)$$

$$= \sum_{k=0}^{n}t^m\,b_{n,k}(x) + \lambda\left(\sum_{k=0}^{n}t^m\left(\frac{1}{n-1} - \frac{2k}{n^2-1}\right)b_{n+1,k}(x) - \sum_{k=0}^{n}t^m\left(\frac{1}{n+1} - \frac{2k}{n^2-1}\right)b_{n+1,k+1}(x)\right)$$

$$= \frac{1}{n^m}\left(\frac{n!}{(n-m)!}x^m + \frac{m(m-1)n!}{2(n-m+1)!}x^{m-1} + TLP(x)\right)$$

$+\frac{\lambda}{n^m}\left\{\frac{-2m(n+1)!}{(n^2-1)(n-m+1)!}x^m(1-x^{n-m+1}) + \left(\frac{nm(m-1)(n+1)!}{2(n^2-1)(n-m+2)!} + \frac{m(m-1)(n+1)!}{2(n^2-1)(n-m+1)!}\right)x^{m-1}(1-x^{n-m+2}) + TLP(x) - \frac{(-1)^m}{(n-1)}(1-(1-x)^{n+1}-x^{n+1})\right\}$. Hence, the property (iv) is held.

For $f \in C[0,1]$, using the Lemma 2.2 and applying the Korovkin theorem [8], we have that $\Upsilon_{n,\lambda}(f;x) \to f(x)$ uniformly on [0,1] as $n \to \infty$.

For $m \in N^0$, the moment of order r, $T_{n,m,\lambda}(x)$, for the sequence $\Upsilon_{n,\lambda}(.;x)$, is defined by

$$T_{n,m,\lambda}(x) = \Upsilon_{n,\lambda}((t-x)^m;x) = \sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda;x)(t-x)^m.$$

Lemma 2.3

The sequence $T_{n,m,\lambda}(x)$ has the properties

(i) $T_{n,0,\lambda}(x) = 1;$

(ii) $T_{n,1,\lambda}(x) = \lambda \frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n^2(n-1)};$

(iii) $T_{n,2,\lambda}(x) = \frac{x(1-x)}{n} + \lambda\left\{\frac{(2n+1)x^{n+1}-2x^{n+2}-1+(2n+1)(1-x)^{n+1}}{n^2(n-1)}\right\};$

(iv) $nT_{n,m+1,\lambda}(x) + \frac{2n}{n^2-1}\lambda T_{n+1,m+1}(x) = x(1-x)T'_{n,m,\lambda}(x) + mx(1-x)T_{n,m-1,\lambda}(x) + xT_{n,m,\lambda}(x) - xT_{n,m}(x) + \left(\frac{2nx+n-1}{n^2-1}\right)\lambda T_{n+1,m}(x).$

(v) $T_{n,m,\lambda}(x)$ is polynomial in x degree at most m;

(vi) $\forall x \in [0,1], T_{n,m,\lambda}(x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right)$, where $\left[\frac{m+1}{2}\right]$ means the integer part of $\frac{m+1}{2}$.

Proof

By the direct evaluations, the proof of consequences (i-iii) is found immediately.
The proof of the consequence (iv) is going as:

$$T_{n,m,\lambda}(x) = \sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda;x)(t-x)^m = \sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda;x)\left(\frac{k}{n}-x\right)^m$$

Then,

$$nT_{n,m+1,\lambda}(x) + \frac{2n}{n^2-1}\lambda T_{n+1,m+1}(x) = x(1-x)T'_{n,m,\lambda}(x) + mx(1-x)T_{n,m-1,\lambda}(x) + xT_{n,m,\lambda}(x) - xT_{n,m}(x) + \left(\frac{2nx+n-1}{n^2-1}\right)\lambda T_{n+1,m}(x).$$

From above, the consequence (iv) is held.
For $m = 0,1$ and 2, the consequence (v) holds clearly. Now, suppose that (v) is valid for m. We show that (v) is valid for $(m+1)$. Since $x(1-x)T'_{n,m\lambda}(x)$,$mx(1-x)T_{n,m-1,\lambda}(x)$ are polynomials in x of degree$(m+1)$, hence $T_{n,m+1,\lambda}(x)$ is polynomial in x of degree$(m+1)$. Then, the consequence (v) is valid for all $\in N^0$.
Finally, from the values of $T_{n,0,\lambda}(x), T_{n,1,\lambda}(x)$ and $T_{n,2,\lambda}(x)$, the consequence (vi) is held. Suppose that the result is valid for m, then by (iv), we have

$$nT_{n,m+1,\lambda}(x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right) + O\left(n^{-\left[\frac{m}{2}\right]}\right) = \begin{cases} O\left(n^{-\left(\frac{m+1}{2}\right)}\right); \text{ if } m \text{ is odd} \\ O\left(n^{-\left(\frac{m}{2}\right)}\right); \text{ if } m \text{ is even.} \end{cases}$$

Then,

$$nT_{n,m+1,\lambda}(x) = \begin{cases} O\left(n^{-\left(\frac{m+3}{2}\right)}\right) ; \text{ if } m \text{ is odd} \\ O\left(n^{-\left(\frac{m+2}{2}\right)}\right) ; \text{ if } m \text{ is even} \end{cases} = O\left(n^{-\left[\frac{m+2}{2}\right]}\right).$$

So, the relation is valid for $m+1$. Hence, the consequence (vi) is valid for every $x \in [0,1]$.

The next result is the Lorenz-type Lemma for derivatives of the functions $\tilde{b}_{n,k}(\lambda;x)$.

Lemma 2.4 [15]

The following equality holds

$$x^r(1-x)^r\tilde{b}_{n,k}^{(r)}(\lambda;x)=\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i(k-nx)^j\tilde{b}_{n,k}\,(\lambda;x)\,Q_{i,j,r}(x),$$

where $Q_{i,j,r}(x)$ are polynomials in x independent of n and k.

3. Main Results

In a previous study [5], the authors proved a Voronovaskaja-type asymptotic formula for the λ-Bernstein sequence and expressed it as

$$\lim_{n\to\infty} n\{\Upsilon_{n,\lambda}(f;x)-f(x)\}=x(1-x)\frac{f''(x)}{2}. \tag{1}$$

This formula is the same as the classical Bernstein sequence. The authors have not explained the effect of λ in this formula. So, we give a modification of the Voronovaskaja formula for the λ-Bernstein sequence in the ordinary approximation.

Theorem 3.1

For $f(x)\in C^4[0,1]$ and $\lambda\in[-1,1]$, we have

$$\lim_{n\to\infty} n^2\left(n\{\Upsilon_{n,\lambda}(f;x)-f(x)\}-x(1-x)\frac{f''(x)}{2}\right)=-\frac{f''(x)}{2}\lambda+\frac{f^{(3)}(x)\,x(1-3x)}{2}\lambda$$
$$+\frac{f^{(4)}(x)}{24}\{-6x^4+12x^3-7x^2+x+(4x^3+16x^4)\lambda\}. \tag{2}$$

Proof

By Taylor's expansion of $f(t)$,

$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+\frac{1}{6}f^{(3)}(x((t-x)^3+\frac{1}{24}f^{(4)}(x((t-x)^4+\alpha(t,x)(t-x)^4$, $t\in[0,1]$

where $\alpha(t,x)\to 0$ as $t\to x$. Using (1),

$$\lim_{n\to\infty} n^2\left(n\{\Upsilon_{n,\lambda}(f;x)-f(x)\}-x(1-x)\frac{f''(x)}{2}\right)$$
$$=-\frac{f''(x)}{2}\lambda+\frac{f^{(3)}(x)x(1-3x)}{2}\lambda+\frac{f^{(4)}(x)}{24}\{-6x^4+12x^3-7x^2+x+(4x^3+16x^4)\lambda\}$$
$$+\lim_{n\to\infty} E$$

where $E=n^3\Upsilon_{n,\lambda}(\alpha(t,x)(t-x)^4;x)$.

Now, to show $\lim_{n\to\infty} E=0$,

$$|E|\le\ n^3\sum_{k=0}^{n}\tilde{b}_{n,k}(\lambda;x)\,|\alpha(t,x)(t-x)^4|$$
$$=n^3\sum_{|t-x|<\delta}\tilde{b}_{n,k}(\lambda;x)\,|\alpha(t,x)(t-x)^4|$$
$$+n^3\sum_{|t-x|\ge\delta}\tilde{b}_{n,k}(\lambda;x)\,|\alpha(t,x)(t-x)^4|\coloneqq I_1+I_2.$$

Since $\alpha(t,x)\to 0$ as $t\to x$ for given $\varepsilon>0\ \exists\ \delta\ >0$ such that $|t-x|<\delta\to|\alpha(t,x)|<\varepsilon$, then,

$$I_1=n^3\sum_{|t-x|<\delta}\tilde{b}_{n,k}(\lambda;x)\,|\alpha(t,x)(t-x)^4|$$
$$\le\alpha n^3\sum_{|t-x|<\delta}\tilde{b}_{n,k}(\lambda;x)\,|\alpha(t,x)(t-x)^4|\le\alpha n^3T_{n,4,\lambda}(x)=\alpha O(1).$$

Since α is arbitrary, it follows that $I_1\to 0$ as $n\to\infty$.

For $|t-x|\ge\delta\ \exists\ C>0$ such that $\alpha(t,x)(t-x)^2\le Ct^{\sigma}, C$ is a constant; therefore

$$I_2=n^3\sum_{|t-x|\ge\delta}\tilde{b}_{n,k}(\lambda;x)\,|\alpha(t,x)(t-x)^4|$$

$$\leq \sup_{x\in[0,1]}\left|n^3\sum_{|t-x|\geq\delta}\tilde{b}_{n,k}(\lambda;x)\,Ct^{\sigma}\right|$$

By using Cauchy Schwarz inequality, we get

$$\leq Mn^3\sum_{i=0}^{\infty}\left(\sum_{|t-x|\geq\delta}\tilde{b}_{n,k}(\lambda;x)\right)^{\frac{1}{2}}\left(\sum_{|t-x|\geq\delta}\tilde{b}_{n,k}(\lambda;x)\,(t-x)^{4i}\right)^{\frac{1}{2}} = Mn^3\left(T_{n,4i,\lambda}(x)\right)^{\frac{1}{2}}.$$

$$= Mn^3\left(O\left(n^{-i}\right)\right)^{\frac{1}{2}} = O(n^{-s})\quad s>0.$$

Hence, $I_2 = 0\ as\ n\to\infty$, From which (2) is held.

Next, we show that $\frac{d^r}{dx^r}\Upsilon_{n,\lambda}(f;x)$ is an approximation for the function $f^{(r)}(x)$.

Theorem 3.2

Suppose that $r\in N$, $f\in C[0,1]$ and $f^{(r)}$ exists and continuous at $x\in(0,1)$, then the following limit holds

$$\lim_{n\to\infty}\Upsilon_{n,\lambda}^{(r)}(f;x)\to f^{(r)}(x). \qquad (3)$$

Proof

By using Taylor's expansion,

$$f(t) = \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}(t-x)^i + \alpha(t,x)(t-x)^r, t\in[0,1],$$

where $\alpha(t;x)\to 0$ as $t\to x$. So,

$$\Upsilon_{n,\lambda}^{(r)}(f(t);x) = \frac{d^r}{dx^r}\left\{\Upsilon_{n,\lambda}\ \left(\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}(t-x)^i + \Upsilon_{n,\lambda}\ \ (\alpha(t,x)(t-x)^r;x)\right)\right\}$$

$$= \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\Upsilon_{n,\lambda}^{(r)}\left((t-x)^i;x\right) + \Upsilon(\alpha(t,x)(t-x)^r;x) := I_1 + I_2.$$

$$I_1 = \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\Upsilon_{n,\lambda}^{(r)}\left((t-x)^i;x\right) = \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\Upsilon_{n,\lambda}^{(r)}\left(\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}t^j\ ;x\right)$$

$$= \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\Upsilon_{n,\lambda}^{(r)}\left(t^j;x\right) = \frac{f^{(r)}(x)}{r!}\Upsilon_{n,\lambda}^{(r)}(t^r;x)$$

Because $\Upsilon_{n,\lambda}^{(r)}\left(t^j;x\right)$ is polynomial in x of degree j. Then,

$$I_1 = \left\{\frac{n!}{n^r(n-r)!} + \frac{-2r(n+1)!}{n^r(n^2-1)(n-r+1)!}\lambda\right\}r!$$

Then, $I_1 = f^{(r)}(x)$ as $n\to\infty$.

The treatment of I_2 is given below.

$$I_2 = \Upsilon_{n,\lambda}^{(r)}(\alpha(t,x)(t-x)^r;x) = \sum_{k=0}^{n}\tilde{b}_{n,k}^{(r)}(\lambda;x)\,\alpha\left(\frac{k}{n},x\right)\left(\frac{k}{n}-x\right)^r.$$

From Lemma 2.4,

$$x^r(1-x)^r\,\tilde{b}_{n,k}^{(r)}(\lambda;x) = \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i(k-nx)^j\tilde{b}_{n,k}(\lambda;x)\,Q_{i,j,r}(x).$$

Hence,

$$I_2 = \sum_{k=0}^{n} \frac{1}{x^r(1-x)^r} \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i+j} \left(\frac{k}{n}-x\right)^j \tilde{b}_{n,k}(\lambda;x)\, Q_{i,j,r}(x)\, \alpha\left(\frac{k}{n},x\right)\left(\frac{k}{n}-x\right)^r.$$

$$|I_2| \le \sum_{\substack{2i+j\le r\\ i,j\ge 0}} \frac{|Q_{i,j,r}(x)|}{\left(x(1-x)\right)^r} n^{i+j} \left\{\sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda;x) \left|\frac{k}{n}-x\right|^{j+r} \alpha\left|\left(\frac{k}{n},x\right)\right|\right\}$$

Since $\alpha(t,x)\to 0$ as $t\to x$, then $\forall\varepsilon>0$ and there exists $\delta>0$ such that $\left|\alpha\left(\frac{k}{n},x\right)\right| < \varepsilon$, whenever $0<\left|\frac{k}{n}-x\right|<\delta$. For $\left|\frac{k}{n}-x\right|\ge\delta$, then we have $\left|\alpha(\frac{k}{n},x)(\frac{k}{n}-x)\right| \le \beta\left(\frac{k}{n}\right)^{\rho}$, for some $\beta>0$. Thus,

$$|I_2| = \beta_1 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i+j} \left\{\alpha \sum_{\left|\frac{k}{n}-x\right|<\delta} \tilde{b}_{n,k}(\lambda;x)\left|\frac{k}{n}-x\right|^{j+r} + \beta \sum_{\left|\frac{k}{n}-x\right|\ge\delta} \tilde{b}_{n,k}(\lambda;x)\left(\frac{k}{n}\right)^{\rho}\right\}$$

$:= I_3 + I_4$.

Where $\beta_1 = \sup_{\substack{2i+j\le r\\ i,j\ge 0}} \frac{|Q_{i,j,r}(x)|}{\left(x(1-x)\right)^r}$, $x\in(0,1)$ is fixed.

Using Schwarz inequality, we conclude that

$$I_3 = \alpha\beta_1 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i+j} \left(\sum_{\left|\frac{k}{n}-x\right|<\delta} \tilde{b}_{n,k}(\lambda;x)\right)^{\frac{1}{2}} \left(\sum_{\left|\frac{k}{n}-x\right|<\delta} \tilde{b}_{n,k}(\lambda;x)\left|\frac{k}{n}-x\right|^{2(j+r)}\right)^{\frac{1}{2}}$$

$$\le \alpha\beta_1 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i+j}\ O\left(n^{\frac{-(r+j)}{2}}\right) = \alpha\beta_1 O(n^{-s}).$$

Since $\alpha>0$ is arbitrary, then $I_3\to 0$ as $n\to\infty$.

$$I_4 = \beta_1 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i+j} \sum_{\left|\frac{k}{n}-x\right|\ge\delta} \tilde{b}_{n,k}(\lambda;x)\left(\frac{k}{n}\right)^{\rho}$$

$$I_4 \le \beta_1 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i+j} \left(\sum_{\left|\frac{k}{n}-x\right|\ge\delta} \tilde{b}_{n,k}(\lambda;x)\right)^{\frac{1}{2}} \left(\sum_{\left|\frac{k}{n}-x\right|\ge\delta} \tilde{b}_{n,k}(\lambda;x)\left(\frac{k}{n}-x\right)^{2r}\right)^{\frac{1}{2}}$$

$$= \beta_1 \sum_{\substack{2i+j\le r\\ i,j\ge 0}} O\left(n^{i+j-s}\right) = O(1). \qquad s>i$$

$I_4\to\infty$ as $n\to\infty$. Hence, $I_2 = O(1)$ as $n\to\infty$.

By combining the estimates of I_1 and I_2, we get (3).

Theorem 3.3

Let $f\in C[0,1]$. If $f^{(r+2)}$ exists at $x\in(0,1)$, then,

$$\lim_{n\to\infty}\left\{\Upsilon_{n,\lambda}^{(r)}(f;x) - f^{(r)}(x)\right\} = -\frac{r(r-1)}{2} f^{(r)}(x) + \frac{-2rx+r}{2} f^{(r+1)}(x) + \frac{rx(x-1)}{2} f^{(r+2)}(x). \qquad (4)$$

Proof

By Taylor's expansion,

$$f(t)=\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}(t-x)^{i}+\alpha(t,x)(t-x)^{r+2}.$$

where $\alpha(t,x)\to 0$ as $t\to x$ hence,

$$n\left\{\Upsilon_{n,\lambda}^{(r)}(f;x)-f^{(r)}(x)\right\}$$
$$=n\left\{\sum_{k=0}^{r+2}\frac{f^{(i)}(x)}{i!}\Upsilon_{n,\lambda}^{(r)}\big((t-x)^{i};x\big)+\Upsilon_{n,\lambda}^{(r)}(\alpha(t,x)(t-x)^{r+2};x)-f^{(r)}(x)\right\}.$$

$$\lim_{n\to\infty} n\left\{\Upsilon_{n,\lambda}^{(r)}(f;x)-f^{(r)}(x)\right\}$$
$$=\lim_{n\to\infty} n\left\{\frac{f^{(r)}(x)}{r!}\Upsilon_{n,\lambda}^{(r)}((t-x)^{r};x)+\frac{f^{(r+1)}(x)}{(r+1)!}\Upsilon_{n,\lambda}^{(r)}((t-x)^{r+1};x)\right.$$
$$\left.+\frac{f^{(r+2)}(x)}{(r+2)!}\Upsilon_{n,\lambda}^{(r)}((t-x)^{r+2};x)-f^{(r)}(x)+\Upsilon_{n,\lambda}^{(r)}(\alpha(t,x)(t-x)^{r+2};x)\right\}$$
$$:=I_1+I_2$$

Using Lemma 2.2 (iv), then

$$I_1=\lim_{n\to\infty} n\left\{\frac{f^{(r)}(x)}{r!}\Upsilon_{n,\lambda}^{(r)}(t^{r};x)+\frac{f^{(r+1)}(x)}{(r+1)!}\Big((r+1)(-x)\Upsilon_{n,\lambda}^{(r)}(t^{r};x)+\Upsilon_{n,\lambda}^{(r)}(t^{r+1};x)\Big)\right.$$
$$+\frac{f^{(r+2)}(x)}{(r+2)!}\left(\frac{(r+2)(r+1)}{2}x^{2}\Upsilon_{n,\lambda}^{(r)}(t^{r};x)\right.$$
$$\left.\left.+(r+2)(-x)\,\Upsilon_{n,\lambda}^{(r)}(t^{r+1};x)+\Upsilon_{n,\lambda}^{(r)}(t^{r+2};x)\right)-f^{r}(x)\right\}$$
$$=\lim_{n\to\infty}\left\{nf^{(r)}(x)\left(\frac{n!}{n^{r}(n-r)!}+\frac{-2r(n+1)!}{n^{r}(n^{2}-1)(n-r+1)!}\lambda-1\right)\right.$$
$$+n\frac{f^{(r+1)}(x)}{(r+1)!}\left((r+1)!\,(-x)\left(\frac{n!}{n^{r}(n-r)!}+\frac{-2r(n+1)!}{n^{r}(n^{2}-1)(n-r+1)!}\lambda\right)\right.$$
$$+\left(\frac{n!}{n^{r+1}(n-r-1)!}+\frac{-2(r+1)(n+1)!}{n^{r+1}(n^{2}-1)(n-r)!}\lambda\right)(r+1)!\,x$$
$$\left.+\left(\frac{rn!}{2n^{r+1}(n-r)!}+\left\{\frac{nr(n+1)!}{2n^{r+1}(n^{2}-1)(n-r+1)!}+\frac{r(n+1)!}{2n^{r+1}(n^{2}-1)(n-r)!}\right\}\lambda\right)(r+1)!\right)$$
$$+n\frac{f^{(r+2)}(x)}{(r+2)!}\left(\frac{(r+2)!}{2}x^{2}\left(\frac{n!}{n^{r}(n-r)!}+\frac{-2r(n+1)!}{n^{r}(n^{2}-1)(n-r+1)!}\lambda\right)\right.$$
$$+(r+2)!\,(-x^{2})\left(\left(\frac{n!}{n^{r+1}(n-r-1)!}+\frac{-2(r+1)(n+1)!}{n^{r+1}(n^{2}-1)(n-r)!}\lambda\right)\right.$$
$$\left(\frac{rn!}{2n^{r+1}(n-r)!}+\left\{\frac{nr(n+1)!}{2n^{r+1}(n^{2}-1)(n-r+1)!}+\frac{r(n+1)!}{n^{r+1}(n^{2}-1)(n-r)!}\right\}\lambda\right)(r+2)!\,(-x)$$
$$+\frac{(r+2)!}{2}x^{2}\left(\frac{n!}{n^{r+2}(n-r-2)!}+\frac{-2(r+2)(n+1)!}{n^{r+2}(n^{2}-1)(n-r-1)!}\lambda\right)$$
$$+(r+2)!\,x\left(\frac{(r+1)n!}{2n^{r+2}(n-r-1)!}\right.$$
$$\left.\left.\left.+\left\{\frac{n(r+1)(n+1)!}{2n^{r+2}(n^{2}-1)(n-r)!}+\frac{(r+1)(n+1)!}{2n^{r+2}(n^{2}-1)(n-r-1)!}\right\}\lambda\right)\right)\right\}$$
$$:=E_1+E_2+E_3.$$

By combining the estimates of E_1,E_2,E_3, as $n\to\infty$, the required is immediate, and we get

$E_1 = -\frac{r(r-1)}{2} f^{(r)}(x);$
$E_2 = \frac{-2rx + r}{2} f^{(r+1)}(x);$
$E_3 = \frac{rx(x-1)}{2} f^{(r+2)}(x).$

Since$I_2 \to 0$ as $n \to \infty$, thus, we obtain (4).

Theorem 3.4

Let $(x) \in C[0,1]$. Then, for any $x \in (0,1)$ at which $f^{(r+4)}(x)$ exists, $\lambda \in [-1,1]$

$$\lim_{n\to\infty} n^2 \left(n\{\Upsilon_{n,\lambda}(f;x) - f(x)\} + \frac{r(r-1)}{2} f^{(r)}(x) - \frac{(-2rx+r)}{2} f^{(r+1)}(x) - \frac{rx(x-1)}{2} f^{(r+2)}(x) \right)$$
$$= \frac{-4x(r^2+1) - (r^2+r)}{2} \lambda f^{(r+1)}(x) + \{-2(r-1)x^2 + r(r-1)x\}\lambda f^{(r+2)}(x)$$
$$+ \left\{(-r^2 - 3r + 2)x^3 + (r+1)x^2 + \left(-2(r+3)x^3 - \frac{(r^2+5r+5)}{2} x^2\right)\lambda\right\} f^{(r+3)}(x)$$
$$+ \left\{-(r^2+3r+2)\,x^4 + \frac{7r^2-43r+60}{12} x^3 + \left(2(r+3)x^4 + \frac{-17r^2-55r-84}{12} x^3\right)\lambda\right\} f^{(r+4)}(x). \quad (5)$$

Proof

By Taylor's expansion,

$$f(t) = \sum_{i=0}^{r+4} \frac{f^{(i)}(x)}{i!}(t-x)^i + \alpha(t,x)(t-x)^{r+4}.$$

where $\alpha(t,x) \to 0$ as $t \to x$, hence

$$\left\{\Upsilon_{n,\lambda}^{(r)}(f;x) - f^{(r)}(x)\right\}$$
$$= \left\{\sum_{i=0}^{r+4} \frac{f^{(i)}(x)}{i!} \Upsilon_{n,\lambda}^{(r)}\left((t-x)^i;x\right) + \Upsilon_{n,\lambda}^{(r)}\left(\alpha(t,x)(t-x)^{r+4};x\right) - f^{(r)}(x)\right\}.$$

$$\lim_{n\to\infty} n^2 \left\{ n\left(\Upsilon_{n,\lambda}^{(r)}(f;x) - f^{(r)}(x)\right) + \frac{r(r+1)}{2} f^{(r)}(x) - \frac{(-2rx+r)}{2} f^{(r+1)}(x) - \frac{(-2(r+1)x^2 - (r+1)x)}{2} f^{(r+2)}(x) \right\}$$
$$+ \frac{r(r+1)}{2} f^{(r)}(x) - \frac{(-2rx+r)}{2} f^{(r+1)}(x) - \frac{(-2(r+1)x^2 - (r+1)x)}{2} f^{(r+2)}(x) \Big\}$$
$$= \lim_{n\to\infty} n^2 \left\{ n\left(\frac{f^{(r)}(x)}{r!} \Upsilon_{n,\lambda}^{(r)}((t-x)^r;x) + \frac{f^{(r+1)}(x)}{(r+1)!} \Upsilon_{n,\lambda}^{(r)}((t-x)^{r+1};x) + \frac{f^{(r+2)}(x)}{(r+2)!} \Upsilon_{n,\lambda}^{(r)}((t-x)^{r+2};x) + \frac{f^{(r+3)}(x)}{(r+3)!} \Upsilon_{n,\lambda}^{(r)}((t-x)^{r+3};x) + \frac{f^{(r+4)}(x)}{(r+4)!} \Upsilon_{n,\lambda}^{(r)}((t-x)^{r+4};x) - f^{(r)}(x) + \Upsilon_{n,\lambda}^{(r)}(\alpha(t,x)(t-x)^{r+4};x) \right) \right. := I_1 + I_2.$$

Using Lemma 1.2 (iv), then

$$
\begin{aligned}
I_1 = \lim_{n\to\infty} n^2 \Bigg(\quad n\Bigg\{ \quad & \frac{f^{(r)}(x)}{r!}\Upsilon_{n,\lambda}^{(r)}(t^r;x) \\
& + \frac{f^{(r+1)}(x)}{(r+1)!}\Big((r+1)(-x)\Upsilon_{n,\lambda}^{(r)}(t^r;x) + \Upsilon_{n,\lambda}^{(r)}(t^{r+1};x)\Big) \\
+ \frac{f^{(r+2)}(x)}{(r+2)!}&\left(\frac{(r+2)(r+1)}{2}x^2\Upsilon_{n,\lambda}^{(r)}(t^r;x) + (r+2)(-x)\,\Upsilon_{n,\lambda}^{(r)}(t^{r+1};x) + \Upsilon_{n,\lambda}^{(r)}(t^{r+2};x)\right) \\
+ \frac{f^{(r+3)}(x)}{(r+3)!}&\left(\frac{-(r+3)(r+2)(r+1)}{6}x^3\Upsilon_{n,\lambda}^{(r)}(t^r;x) + \frac{(r+3)(r+2)}{2}(x^2)\,\Upsilon_{n,\lambda}^{(r)}(t^{r+1};x)\right. \\
& \left. - (r+3)x\Upsilon_{n,\lambda}^{(r)}(t^{r+2};x) + \Upsilon_{n,\lambda}^{(r)}(t^{r+3};x)\right) \\
+ \frac{f^{(r+4)}(x)}{(r+4)!}&\left(\frac{(r+4)(r+3)(r+2)(r+1)}{24}x^4\Upsilon_{n,\lambda}^{(r)}(t^r;x)\right. \\
& - \frac{(r+4)(r+3)(r+2)}{6}(x^3)\,\Upsilon_{n,\lambda}^{(r)}(t^{r+1};x) + \frac{(r+4)(r+3)}{2}x^2\Upsilon_{n,\lambda}^{(r)}(t^{r+2};x) \\
& \left. - (r+4)\Upsilon_{n,\lambda}^{(r)}(t^{r+3};x) + \Upsilon_{n,\lambda}^{(r)}(t^{r+4};x)\right) - f^{r}(x) \\
+ \frac{r(r-1)}{2}f^{(r)}(x) - &\frac{(-2rx+r)}{2}f^{(r+1)}(x) - \frac{rx(x-1)}{2}f^{(r+2)}(x) \quad \Bigg\} \quad \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
= \lim_{n\to\infty} n^2 \Bigg\{ & n f^{(r)}(x)\left(\frac{n!}{n^r(n-r)!} + \frac{-2r(n+1)!}{n^r(n^2-1)(n-r+1)!}\lambda - 1\right) \\
& + n\frac{f^{(r+1)}(x)}{(r+1)!}\Bigg((r+1)!\,(-x)\left(\frac{n!}{n^r(n-r)!} + \frac{-2r(n+1)!}{n^r(n^2-1)(n-r+1)!}\lambda\right) \\
& + \left(\frac{n!}{n^{r+1}(n-r-1)!} + \frac{-2(r+1)(n+1)!}{n^{r+1}(n^2-1)(n-r)!}\lambda\right)(r \\
& + 1)!\,x + \Bigg(\frac{rn!}{2n^{r+1}(n-r)!} \\
& + \left\{\frac{nr(n+1)!}{n^{r+1}(n^2-1)(n-r+1)!} + \frac{r(n+1)!}{n^{r+1}(n^2-1)(n-r)!}\right\}\lambda\Bigg)(r+1)!\Bigg) \\
& + \frac{f^{(r+2)}(x)}{(r+2)!}\Bigg(\frac{(r+2)!}{2}x^2\left(\frac{n!}{n^r(n-r)!} + \frac{-2r(n+1)!}{n^r(n^2-1)(n-r+1)!}\lambda\right) \\
& + (r+2)!\,(-x^2)\Bigg(\left(\frac{n!}{n^{r+1}(n-r-1)!} + \frac{-2(r+1)(n+1)!}{n^{r+1}(n^2-1)(n-r)!}\lambda\right) \\
& + \Bigg(\frac{rn!}{2n^{r+1}(n-r)!} \\
& + \left\{\frac{nr(n+1)!}{2n^{r+1}(n^2-1)(n-r+1)!} + \frac{r(n+1)!}{2n^{r+1}(n^2-1)(n-r)!}\right\}\lambda\Bigg)(r+2)!\,(-x) \\
& + (r+2)!\,x^2\left(\frac{n!}{n^{r+2}(n-r-2)!} + \frac{-2(r+2)(n+1)!}{n^{r+2}(n^2-1)(n-r-1)!}\lambda\right) \\
& + (r+2)!\,x\Bigg(\frac{(r+1)n!}{2n^{r+2}(n-r-1)!} \\
& + \left\{\frac{n(r+1)(n+1)!}{2n^{r+2}(n^2-1)(n-r)!} + \frac{(r+1)(n+1)!}{2n^{r+2}(n^2-1)(n-r-1)!}\right\}\lambda \\
& + \frac{f^{(r+3)}(x)}{(r+3)!}\Bigg(\frac{-(r+3)!}{6}x^3\left(\frac{n!}{n^r(n-r)!} + \frac{-2r(n+1)!}{n^r(n^2-1)(n-r+1)!}\lambda\right) \\
& + \frac{(r+3)!}{2}(x^3)\left(\frac{n!}{n^{r+1}(n-r-1)!} + \frac{-2(r+1)(n+1)!}{n^{r+1}(n^2-1)(n-r)!}\lambda\right) \\
& + \Bigg(\frac{rn!}{2n^{r+1}(n-r)!} \\
& + \left\{\frac{nr(n+1)!}{2n^{r+1}(n^2-1)(n-r+1)!} + \frac{r(n+1)!}{2n^{r+1}(n^2-1)(n-r)!}\right\}\lambda\Bigg)\frac{(r+3)!}{2}(x^2) \\
-(r+3)!\,x^3 & \left(\frac{n!}{n^{r+2}(n-r-2)!} + \frac{-2(r+2)(n+1)!}{n^{r+2}(n^2-1)(n-r-1)!}\lambda\right)
\end{aligned}
$$

$$-(r+3)!\,x^2\left(\frac{(r+1)n!}{2n^{r+2}(n-r-1)!}\right.$$
$$\left.+\left\{\frac{n(r+1)(n+1)!}{2n^{r+2}(n^2-1)(n-r)!}+\frac{(r+1)(n+1)!}{2n^{r+2}(n^2-1)(n-r-1)!}\right\}\lambda\right)$$
$$+(r+3)!\,x^3\left(\frac{n!}{n^{r+3}(n-r-3)!}+\frac{-2(r+3)(n+1)!}{n^{r+3}(n^2-1)(n-r-1)!}\lambda\right)$$
$$+(r+3)!\,x^2\left(\frac{(r+2)n!}{2n^{r+3}(n-r-2)!}\right.$$
$$\left.+\left\{\frac{n(r+2)(n+1)!}{2n^{r+3}(n^2-1)(n-r-1)!}+\frac{(r+2)(n+1)!}{2n^{r+3}(n^2-1)(n-r-2)!}\right\}\lambda\right)$$
$$+\frac{f^{(r+4)}(x)}{(r+4)!}\left(\frac{(r+4)!}{24}x^4\left(\frac{n!}{n^r(n-r)!}+\frac{-2r(n+1)!}{n^r(n^2-1)(n-r+1)!}\lambda\right)\right.$$
$$-\frac{(r+4)!}{6}x^4\left(\frac{n!}{n^{r+1}(n-r-1)!}+\frac{-2(r+1)(n+1)!}{n^{r+1}(n^2-1)(n-r)!}\lambda\right)$$
$$-\left(\frac{rn!}{2n^{r+1}(n-r)!}\right.$$
$$\left.+\left\{\frac{nr(n+1)!}{2n^{r+1}(n^2-1)(n-r+1)!}+\frac{r(n+1)!}{2n^{r+1}(n^2-1)(n-r)!}\right\}\lambda\right)\frac{(r+4)!}{6}x^3$$
$$+\frac{(r+4)!}{2}x^4\left(\frac{n!}{n^{r+2}(n-r-2)!}+\frac{-2(r+2)(n+1)!}{n^{r+2}(n^2-1)(n-r-1)!}\lambda\right)$$
$$+\frac{(r+4)!}{2}x^3\left(\frac{(r+1)n!}{2n^{r+2}(n-r-1)!}\right.$$
$$\left.+\left\{\frac{n(r+1)(n+1)!}{2n^{r+2}(n^2-1)(n-r)!}+\frac{(r+1)(n+1)!}{2n^{r+2}(n^2-1)(n-r-1)!}\right\}\lambda\right)$$
$$-(r+4)!\,x^4\left(\frac{n!}{n^{r+3}(n-r-3)!}+\frac{-2(r+3)(n+1)!}{n^{r+3}(n^2-1)(n-r-1)!}\lambda\right)$$
$$-(r+4)!\,x^3\left(\frac{(r+2)n!}{2n^{r+3}(n-r-2)!}\right.$$
$$\left.+\left\{\frac{n(r+2)(n+1)!}{2n^{r+3}(n^2-1)(n-r-1)!}+\frac{(r+2)(n+1)!}{2n^{r+3}(n^2-1)(n-r-2)!}\right\}\lambda\right)$$
$$+(r+4)!\,x^4\left(\frac{n!}{n^{r+4}(n-r-4)!}\right.$$
$$\left.+\frac{-2(r+4)(n+1)!}{n^{r+3}(n^2-1)(n-r-3)!}\lambda\right)+(r+4)!\,x^3\left(\frac{(r+3)n!}{2n^{r+4}(n-r-3)!}\right.$$
$$\left.\left.+\left\{\frac{n(r+3)(n+1)!}{2n^{r+4}(n^2-1)(n-r-2)!}+\frac{(r+3)(n+1)!}{2n^{r+4}(n^2-1)(n-r-3)!}\right\}\lambda\right)\right)$$
$$+\frac{r(r-1)}{2}f^{(r)}(x)-\frac{(-2rx+r)}{2}f^{(r+1)}(x)-\frac{rx(x-1)}{2}f^{(r+2)}(x).$$
$:= E_1+E_2+E_3+E_4+E_5.$

By combining the estimates of E_1, E_2, E_3, E_4, E_5 as $n\to\infty$, we get

$E_1 = 0;$

$$E_2=\frac{-4x(r^2+1)-(r^2+r)}{2}\lambda f^{(r+1)}(x);$$

$E_3 = \{-2(r-1)x^2 + r(r-1)x\}\lambda f^{(r+2)}(x);$

$$E_4 = \left\{(-r^2 - 3r + 2)x^3 + (r+1)x^2 + \left(-2(r+3)x^3 - \frac{(r^2+5r+5)}{2}x^2\right)\lambda\right\} f^{(r+3)}(x);$$

$$E_5 = \left\{-(r^2 + 3r + 2)x^4 + \frac{7r^2 - 43r + 60}{12}x^3 + \left(2(r+3)x^4 + \frac{-17r^2 - 55r - 84}{12}x^3\right)\lambda\right\} f^{(r+4)}(x).$$

Since$I_2 \to 0$ as $n \to \infty$, thus, we obtain (5).

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