On Some Approximation Properties for a Sequence of $\lambda$-Bernstein Type Operators

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Abstract

In 2010, Long and Zeng introduced a new generalization of the Bernstein polynomials that depends on a parameter $\lambda$ and called $\lambda$-Bernstein polynomials. After that, in 2018, Lain and Zhou studied the uniform convergence for these $\lambda$-polynomials and obtained a Voronovskaja-type asymptotic formula in ordinary approximation. This paper studies the convergence theorem and gives two Voronovskaja-type asymptotic formulas of the sequence of $\lambda$-Bernstein polynomials in both ordinary and simultaneous approximations. For this purpose, we discuss the possibility of finding the recurrence relations of the $m$-th order moment for these polynomials and evaluate the values of $\lambda$-Bernstein for the functions $t^m$, where $m$ is a non-negative integer.

Keywords: $\lambda$-Bernstein polynomials, Voronovskaja type asymptotic formula, the uniform convergence, ordinary and simultaneous approximations.

1. Introduction

Let $S$ be the linear space of all real functions acting on a set $X \neq \phi$. The operator $M: S \rightarrow S$ is linear and positive if it satisfies:

i) $\forall \alpha, \beta \in \mathbb{R}, M(\alpha f + \beta g) = \alpha M(f) + \beta M(g)$, where $f, g \in S$;

ii) $\forall f \in S: f \geq 0$, we have $M(f) \geq 0.$

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Bernstein, in 2012, [1] introduced another proof of the Weierstrass approximation theorem by using a sequence of positive linear operators, named the classical Bernstein polynomials, as

\[ I_n : C[0,1] \rightarrow C[0,1], \quad I_n(f;x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) b_{n,k}(x), f \in C[0,1]. \]

Where

\[ b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, x \in [0,1] \]

In 1932 [2], Voronovaskaja showed that for \( f \in C^2[0,1] \), the term of \( n^{-1} \) in \( \{ B_n(f;x) - f(x) \} \) exists and equals to \( \frac{x(1-x)}{2} f''(x) \). This discovery appeared upon the evaluation of the limit \( \lim_{n \to \infty} n \{ B_n(f;x) - f(x) \} \). So, the order of approximation by using Bernstein polynomials is \( O(n^{-1}) \). This phenomenon, in general, is valid for most sequences of positive linear operators [2]. The evaluation of the approximation order for the different sequences is called Voronovaskaja-type asymptotic formulas. The order of \( B_n(f;x) \) to the function \( f \) as \( n \) tends to infinity is very slow.

In 1953, Korovkin [3] introduced a simple tool to decide that, for a sequence of linear positive operators, \( M_n \) is converges to the function \( f \in C[a,b] \), by checking the sequence's values of \( M_n(t^n;x) \to x^n \) uniformly as \( n \to \infty, m = 0,1,2 \). These are called Korovkin's conditions.

Many generalizations of Korovkin's theorem to a compact subset of the real numbers \( \mathbb{R} \) or the interval \( [0,\infty) \) were introduced and studied. We refer here to Bohman [4, 1953] and Baskakov [5, 1957].

In 1962 [6], Schurer introduced a sequence based on a parameter and proved that the sequence has an approximation order depending on the parameter. After that, many kinds of research were developed and studied sequences depending on parameters; here we refer to [7, 8, 9, 10, 11, 12].

In 2010 [13], Long and Zeng introduced a new generalization of the classical Bernstein sequence that depends on a parameter \( \lambda \), as follows:

\[ Y_{n,\lambda}(f;x) = \sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda;x) f \left( \frac{k}{n} \right), \]

where \( \tilde{b}_{n,k}(\lambda;x) = \left\{ \begin{array}{ll}
 b_{n,0}(x) - \frac{\lambda}{n+1} b_{n+1,1}(x) & ; \quad k = 0 \\
 b_{n,k}(x) + \lambda \left( \frac{n-2k+1}{n^2-1} b_{n+1,k}(x) - \frac{n-2k-1}{n^2-1} b_{n+1,k+1}(x) \right) & ; \quad 1 \leq k \leq n-1 \\
 b_{n,n}(x) - \frac{\lambda}{n+1} b_{n+1,n}(x) & ; \quad k = n,
\end{array} \right. \]

where \( \lambda \in [-1,1] \).

When \( \lambda = 0 \), the function \( \tilde{b}_{n,k}(\lambda;x) \) is reduced to \( b_{n,k}(x) \).

In 2018 [14], Lain and Zhou studied the uniform convergence and obtained a Voronovaskaja-type asymptotic formula for the sequence \( Y_{n,\lambda}(f;x) \) in ordinary approximation.

2. Preliminary Results

Some preliminaries for the sequence \( Y_{n,\lambda}(f;x) \) are introduced in this section and will be used to achieve the main results.

We assume that \( m \in N^0 = \{0,1, \ldots \}, \phi_{n,m}(x) = \sum_{k=0}^{n} k^m b_{n,k}(x), \)

\( \tilde{\phi}_{n,m,\lambda}(x) = \sum_{k=0}^{n} k^m \tilde{b}_{n,k}(\lambda;x), \) and \( TLP(x) \) mean terms in lower powers of \( x \).
Lemma 2.1
The following properties are held:

(i) \( x(1-x)\dot{b}_{n,k}(\lambda; x) = \)

\[(k-nx)\ddot{b}_{n,k}(\lambda; x) - x\dot{b}_{n,k}(\lambda; x) + x b_{n,k}(x) - \frac{n-2k-1}{n^2-1} \lambda b_{n+1,k+1}(x). \]

(ii) \( \phi_{n,0}(x) = 1, \phi_{n,0}(x) = 1, \phi_{n+1,0}(x) = 1, \) and \( \phi_{n,m+1,1}(x) + \frac{2}{n^2-1} \lambda \phi_{n+1,m+1}(x) = x(1-x)\ddot{\phi}_{n,m+1,1}(x) + (n+1)x\phi_{n,m+1,1}(x) - x\phi_{n,m}(x) + \frac{1}{n+1} \lambda \phi_{n+1,m}(x), m \geq 1. \)

Proof
By direct evaluation, the property (i) follows quickly.
Using the fact that \( x(1-x)\dot{b}_{n,k}(x) = (k-nx)b_{n,k}'(x), \) thus

\[\phi_{n,m+1,1}(x) + \frac{2}{n^2-1} \lambda \phi_{n+1,m+1}(x) = x(1-x)\ddot{\phi}_{n,m+1,1}(x) + (n+1)x\phi_{n,m+1,1}(x) - x\phi_{n,m}(x) + \frac{1}{n+1} \lambda \phi_{n+1,m}(x), m \geq 1.\]

Then, the property (ii) holds.

Lemma 2.2
The sequence \( Y_{n,\lambda} \) satisfies:

(i) \( Y_{n,\lambda}(1; x) = 1; \)

(ii) \( Y_{n,\lambda}(t; x) = x + \frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n(n-1)} \lambda; \)

(iii) \( Y_{n,\lambda}(t^2; x) = x^2 + \frac{\lambda}{n} \left\{ \frac{2x-4x^2+2x^{n+1}}{n} + \frac{x^{n+1}+(1-x)^{n+1}-1}{n^2} \right\}; \)

(iv) \( Y_{n,\lambda}(t^m; x) = \frac{n!}{(n-m)!} x^m + \frac{m(m-1)n!}{2(n-m+1)!} x^{m-1} + TLP(x) \) \( + \frac{\lambda}{n^m} \left\{ \frac{-2m(n+1)!}{(n^2-1)(n-m+1)!} x^m - x^{n-m+1} + \frac{m(m-1)(n-1)!}{2(n^2-1)(n-m)!} x^{m-1} \right\}. \)

The consequences (i)-(iii) is proved in [5]. The proof of consequence (iv) is given as follows

\[Y_{n,\lambda}(t^m; x) = \sum_{k=0}^{n} b_{n,k}(\lambda; x) + \frac{1}{n-1} - \frac{2k}{n^2-1} b_{n+1,k}(x) \]

\[= \sum_{k=0}^{n} t^m b_{n,k}(x) + \lambda \left( \sum_{k=0}^{n} t^m \left( \frac{1}{n-1} - \frac{2k}{n^2-1} \right) b_{n+1,k}(x) \right) \]

\[= \frac{1}{n^m} \left\{ \frac{n!}{(n-m)!} x^m + \frac{m(m-1)n!}{2(n-m+1)!} x^{m-1} + TLP(x) \right\} \]

Hence, the property (iv) is held.
For $f \in C[0,1]$, using the Lemma 2.2 and applying the Korovkin theorem [8], we have that $Y_{n,\lambda}(f; x) \to f(x)$ uniformly on $[0,1]$ as $n \to \infty$.

For $m \in \mathbb{N}^0$, the moment of order $r$, $T_{n,m,\lambda}(x)$, for the sequence $Y_{n,\lambda}(\cdot; x)$, is defined by

$$T_{n,m,\lambda}(x) = Y_{n,\lambda}((t-x)^m; x) = \sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda; x)(t-x)^m.$$ 

**Lemma 2.3**

The sequence $T_{n,m,\lambda}(x)$ has the properties

(i) $T_{n,0,\lambda}(x) = 1$;

(ii) $T_{n,1,\lambda}(x) = \frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n^2(n-1)}$;

(iii) $T_{n,2,\lambda}(x) = \frac{x(1-x)}{n} + \lambda \left( \frac{(2n+1)x^{n+1}-2x^{n+2}-1+(2n+1)(1-x)^{n+1}}{n^2(n-1)} \right)$;

(iv) $nT_{n,m+1,\lambda}(x) + \frac{2n}{n^2-1} \lambda T_{n+1,m+1}(x) = x(1-x)T'_{n,m,\lambda}(x) + mx(1-x)T_{n,m-1,\lambda}(x) + xT_{n,m,\lambda}(x) - xT_{n,m}(x) + \left( \frac{2nx+n-1}{n^2-1} \right) \lambda T_{n+1,m}(x)$.

(v) $T_{n,m,\lambda}(x)$ is polynomial in $x$ degree at most $m$;

(vi) $\forall x \in [0,1], T_{n,m,\lambda}(x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right)$, where $\left[\frac{m+1}{2}\right]$ means the integer part of $\frac{m+1}{2}$.

**Proof**

By the direct evaluations, the proof of consequences (i-iii) is found immediately. The proof of the consequence (iv) is going as:

$$T_{n,m,\lambda}(x) = \sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda; x)(t-x)^m = \sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda; x) \left( \frac{k}{n} - x \right)^m$$

Then,

$$nT_{n,m+1,\lambda}(x) + \frac{2n}{n^2-1} \lambda T_{n+1,m+1}(x) = x(1-x)T'_{n,m,\lambda}(x) + mx(1-x)T_{n,m-1,\lambda}(x) + xT_{n,m,\lambda}(x) - xT_{n,m}(x) + \left( \frac{2nx+n-1}{n^2-1} \right) \lambda T_{n+1,m}(x).$$

From above, the consequence (iv) is held.

For $m = 0, 1$ and 2, the consequence (v) holds clearly. Now, suppose that (v) is valid for $m$. We show that (v) is valid for $(m + 1)$. Since $x(1-x)T'_{n,m,\lambda}(x), mx(1-x)T_{n,m-1,\lambda}(x)$ are polynomials in $x$ of degree $(m + 1)$, hence $T_{n,m+1,\lambda}(x)$ is polynomial in $x$ of degree $(m + 1)$. Then, the consequence (v) is valid for all $x \in N^0$.

Finally, from the values of $T_{n,0,\lambda}(x), T_{n,1,\lambda}(x)$ and $T_{n,2,\lambda}(x)$, the consequence (vi) is held. Suppose that the result is valid for $m$, then by (iv), we have

$$nT_{n,m+1,\lambda}(x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right) + O\left(n^{-\left[\frac{m}{2}\right]}\right) = \begin{cases} O\left(n^{-\left(\frac{m+1}{2}\right)}\right); & \text{if } m \text{ is odd} \\ O\left(n^{-\left(\frac{m}{2}\right)}\right); & \text{if } m \text{ is even} \end{cases}.$$

Then,

$$nT_{n,m+1,\lambda}(x) = \begin{cases} O\left(n^{-\left(\frac{m+1}{2}\right)}\right); & \text{if } m \text{ is odd} \\ O\left(n^{-\left(\frac{m}{2}\right)}\right); & \text{if } m \text{ is even} \end{cases} = O\left(n^{-\left(\frac{m+2}{2}\right)}\right).$$

So, the relation is valid for $m + 1$. Hence, the consequence (vi) is valid for every $x \in [0,1]$.

The next result is the Lorenz-type Lemma for derivatives of the functions $\tilde{b}_{n,k}(\lambda; x)$.

**Lemma 2.4** [15]

The following equality holds
\[ x^r (1 - x)^r \tilde{b}_{n,k}^{(r)}(\lambda; x) = \sum_{\substack{i+j \leq r \geq 0}} n^i (k - nx)^j Q_{i,j,r}(x), \]

where \( Q_{i,j,r}(x) \) are polynomials in \( x \) independent of \( n \) and \( k \).

### 3. Main Results

In a previous study [5], the authors proved a Voronovaskaja-type asymptotic formula for the \( \lambda \)-Bernstein sequence and expressed it as

\[
\lim_{n \to \infty} n \{ Y_{n,\lambda}(f; x) - f(x) \} = x (1 - x) \frac{f''(x)}{2}. \tag{1}
\]

This formula is the same as the classical Bernstein sequence. The authors have not explained the effect of \( \lambda \) in this formula. So, we give a modification of the Voronovaskaja formula for the \( \lambda \)-Bernstein sequence in the ordinary approximation.

**Theorem 3.1**

For \( f(x) \in C^4[0,1] \) and \( \lambda \in [-1,1] \), we have

\[
\text{lim}_{n \to \infty} n^2 \left( n \left\{ Y_{n,\lambda}(f; x) - f(x) \right\} - x (1 - x) \frac{f''(x)}{2} \right) = - \frac{f''(x)}{2} \lambda + \frac{f^{(3)}(x) x (1 - 3x)}{2} \left\{ -6x^4 + 12x^3 - 7x^2 + x + (4x^3 + 16x^4)\lambda \right\}. \tag{2}
\]

**Proof**

By Taylor's expansion of \( f(t) \),

\[
f(t) = f(x) + f'(x)(t - x) + \frac{1}{2} f''(x)(t - x)^2 + \frac{1}{6} f^{(3)}(x)((t - x)^3 + \frac{1}{24} f^{(4)}(x)((t - x)^4 + \alpha(t, x)(t - x)^4, \ t \in [0,1]
\]

where \( \alpha(t, x) \to 0 \) as \( t \to x \). Using (1),

\[
\text{lim}_{n \to \infty} n^2 \left( n \left\{ Y_{n,\lambda}(f; x) - f(x) \right\} - x (1 - x) \frac{f''(x)}{2} \right) = - \frac{f''(x)}{2} \lambda + \frac{f^{(3)}(x) x (1 - 3x)}{2} \left\{ -6x^4 + 12x^3 - 7x^2 + x + (4x^3 + 16x^4)\lambda \right\} + \text{lim}_{n \to \infty} E
\]

where \( E = n^2 Y_{n,\lambda}(\alpha(t, x)(t - x)^4; x) \).

Now, to show \( \text{lim}_{n \to \infty} E = 0 \),

\[
|E| \leq n^2 \sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda; x) |\alpha(t, x)(t - x)^4|
\]

\[
= n^3 \sum_{|t - x| < \delta} \tilde{b}_{n,k}(\lambda; x) |\alpha(t, x)(t - x)^4| + n^3 \sum_{|t - x| \geq \delta} \tilde{b}_{n,k}(\lambda; x) |\alpha(t, x)(t - x)^4| := I_1 + I_2.
\]

Since \( \alpha(t, x) \to 0 \) as \( t \to x \) for given \( \varepsilon > 0 \) \( \exists \delta > 0 \) such that \( |t - x| < \delta \to |\alpha(t, x)| < \varepsilon \), then,

\[
I_1 = n^3 \sum_{|t - x| < \delta} \tilde{b}_{n,k}(\lambda; x) |\alpha(t, x)(t - x)^4| 
\]

\[
\leq \alpha n^3 \sum_{|t - x| < \delta} \tilde{b}_{n,k}(\lambda; x) |\alpha(t, x)(t - x)^4| \leq \alpha n^3 T_{n,\lambda}(x) = \alpha O(1).
\]

Since \( \alpha \) is arbitrary, it follows that \( I_1 \to 0 \) as \( n \to \infty \).

For \( |t - x| \geq \delta \\exists C > 0 \) such that \( \alpha(t, x)(t - x)^2 \leq Ct^\sigma, C \) is a constant; therefore

\[
I_2 = n^3 \sum_{|t - x| \geq \delta} \tilde{b}_{n,k}(\lambda; x) |\alpha(t, x)(t - x)^4| 
\]
By using Cauchy Schwarz inequality, we get
\[ \leq M n^3 \left( \sum_{|t-x| \geq \delta} b_{n,h}(\lambda; x) C t^\sigma \right) \left( \sum_{|t-x| \geq \delta} b_{n,h}(\lambda; x) (t-x)^{4i} \right)^{1/2} = M n^3 \left( T_{n,4i,\lambda}(x) \right)^{1/2}. \]
Hence, \( I_2 = 0 \) as \( n \to \infty \), From which (2) is held.
Next, we show that \( \frac{d^r}{dx^r} Y_{n,\lambda}(f; x) \) is an approximation for the function \( f^{(r)}(x) \).

**Theorem 3.2**

Suppose that \( r \in \mathbb{N} \), \( f \in C[0,1] \) and \( f^{(r)} \) exists and continuous at \( x \in (0,1) \), then the following limit holds
\[ \lim_{n \to \infty} Y_{n,\lambda}^{(r)}(f; x) = f^{(r)}(x). \]  

**Proof**

By using Taylor's expansion,
\[ f(t) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} (t-x)^i + \alpha(t,x)(t-x)^r, t \in [0,1], \]
where \( \alpha(t; x) \to 0 \) as \( t \to x \). So,
\[ Y_{n,\lambda}^{(r)}(f(t); x) = \frac{d^r}{dx^r} \left[ Y_{n,\lambda} \left( \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} (t-x)^i + Y_{n,\lambda}(\alpha(t,x)(t-x)^r; x) \right) \right] \]
\[ = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} Y_{n,\lambda}^{(r)}((t-x)^i; x) + Y(\alpha(t,x)(t-x)^r; x) := I_1 + I_2. \]
\[ I_1 = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} Y_{n,\lambda}^{(r)}((t-x)^i; x) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} Y_{n,\lambda}^{(r)} \left( \sum_{j=0}^{i} \frac{(i-j)!}{j!} (-x)^{i-j} t^j; x \right) \]
\[ = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \frac{(i-j)!}{j!} (-x)^{i-j} Y_{n,\lambda}^{(r)}(t^j; x) = \frac{f^{(r)}(x)}{r!} Y_{n,\lambda}^{(r)}(t^r; x) \]
Because \( Y_{n,\lambda}^{(r)}(t^j; x) \) is polynomial in \( x \) of degree \( j \). Then,
\[ I_1 = \left\{ \frac{n!}{n^n(n-r)!} \right\}^{r} \frac{-2r(n+1)!}{n^n(n^2-1)(n-r+1)!} \]
Then, \( I_1 = f^{(r)}(x) \) as \( n \to \infty \).

The treatment of \( I_2 \) is given below.

\[ I_2 = Y_{n,\lambda}^{(r)}(\alpha(t,x)(t-x)^r; x) = \sum_{k=0}^{n} b_{n,k}^{(r)}(\lambda; x) a \left( \frac{k}{n}, x \right) \left( \frac{k}{n} - x \right)^r. \]
From Lemma 2.4,
\[ x^r (1-x)^r b_{n,k}(\lambda; x) = \sum_{2i+j \geq r \atop \sum_{i,j \geq 0}} n^i (k-nx)^j b_{n,k}(\lambda; x) Q_{i,j,r}(x). \]
Hence,
\[ I_2 = \sum_{k=0}^{n} \frac{1}{x^r (1-x)^r} \sum_{2i+j \leq r \atop i,j \geq 0} n^{i+j} \left( \frac{k}{n} - x \right)^j \tilde{b}_{n,k}(\lambda; x) Q_{i,j,r}(x) \alpha \left( \frac{k}{n}, x \right) \left( \frac{k}{n} - x \right)^r. \]

\[ |I_2| \leq \sum_{2i+j \leq r \atop i,j \geq 0} \left| \frac{Q_{i,j,r}(x)}{(x(1-x))^r} \right| n^{i+j} \left( \sum_{k=0}^{n} \tilde{b}_{n,k}(\lambda; x) \left| \frac{k}{n} - x \right|^{j+r} \alpha \left| \left( \frac{k}{n}, x \right) \right| \right) \]

Since \( \alpha(t, x) \to 0 \) as \( t \to x \), then \( \forall \varepsilon > 0 \) and there exists \( \delta > 0 \) such that \( |\alpha \left( \frac{k}{n}, x \right)| < \varepsilon \), whenever \( 0 < \left| \frac{k}{n} - x \right| < \delta \). For \( \left| \frac{k}{n} - x \right| \geq \delta \), then we have \( |\alpha \left( \frac{k}{n}, x \right)(\frac{k}{n} - x)| \leq \beta \left( \frac{k}{n} \right)^{\rho} \), for some \( \beta > 0 \). Thus,

\[ |I_2| = \beta_1 \sum_{2i+j \leq r \atop i,j \geq 0} n^{i+j} \left\{ \sum_{\left| \frac{k}{n} - x \right| < \delta} \tilde{b}_{n,k}(\lambda; x) \left| \frac{k}{n} - x \right|^{j+r} + \beta \sum_{\left| \frac{k}{n} - x \right| \geq \delta} \tilde{b}_{n,k}(\lambda; x) \left( \frac{k}{n} \right)^{\rho} \right\} \]

\[ := I_3 + I_4. \]

Where \( \beta_1 = \sup_{2i+j \leq r \atop i,j \geq 0} \left| \frac{Q_{i,j,r}(x)}{(x(1-x))^r} \right|, x \in (0,1) \) is fixed.

Using Schwarz inequality, we conclude that

\[ I_3 = \alpha \beta_1 \sum_{2i+j \leq r \atop i,j \geq 0} n^{i+j} \left( \sum_{\left| \frac{k}{n} - x \right| < \delta} \tilde{b}_{n,k}(\lambda; x) \right) \left( \sum_{\left| \frac{k}{n} - x \right| < \delta} \tilde{b}_{n,k}(\lambda; x) \right)^{\frac{1}{2}} \left( \sum_{\left( \left| \frac{k}{n} - x \right|^{2(i+j+r)} \right) \left( \frac{k}{n} \right)^{\rho} \right)^{\frac{1}{2}} \]

\[ \leq \alpha \beta_1 \sum_{2i+j \leq r \atop i,j \geq 0} n^{i+j} O \left( \frac{n^{-(r+1)}}{2} \right) = \alpha \beta_1 O(n^{-s}). \]

Since \( \alpha > 0 \) is arbitrary, then \( I_3 \to 0 \) as \( n \to \infty \).

\[ I_4 = \beta_1 \sum_{2i+j \leq r \atop i,j \geq 0} n^{i+j} \sum_{\left| \frac{k}{n} - x \right| \geq \delta} \tilde{b}_{n,k}(\lambda; x) \left( \frac{k}{n} \right)^{\rho} \]

\[ I_4 \leq \beta_1 \sum_{2i+j \leq r \atop i,j \geq 0} n^{i+j} \left( \sum_{\left| \frac{k}{n} - x \right| \geq \delta} \tilde{b}_{n,k}(\lambda; x) \right) \left( \sum_{\left| \frac{k}{n} - x \right| \geq \delta} \tilde{b}_{n,k}(\lambda; x) \left( \frac{k}{n} - x \right)^{2r} \right)^{\frac{1}{2}} \]

\[ = \beta_1 \sum_{2i+j \leq r \atop i,j \geq 0} O(n^{i+j-s}) = O(1). \]

\( I_4 \to \infty \) as \( n \to \infty \). Hence, \( I_2 = O(1) \) as \( n \to \infty \).

By combining the estimates of \( I_1 \) and \( I_2 \), we get (3).

**Theorem 3.3**

Let \( f \in C[0,1] \). If \( f^{(r+2)} \) exists at \( x \in (0,1) \), then,

\[
\lim_{n \to \infty} \left\{ \gamma_{n,\lambda}^{(r)}(f; x) - f^{(r)}(x) \right\} = - \frac{r(r-1)}{2} f^{(r)}(x) + \frac{2r x + r}{2} f^{(r+1)}(x) + \frac{r x (x-1)}{2} f^{(r+2)}(x).
\]

**Proof**

By Taylor's expansion.
\[ f(t) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} (t - x)^i + \alpha(t, x)(t - x)^{r+2}, \]

where \( \alpha(t, x) \to 0 \) as \( t \to x \) hence,

\[
\left\{ Y_{n, \lambda}^{(r)}(f; x) - f^{(r)}(x) \right\} = \left\{ \sum_{k=0}^{r+2} \frac{f^{(i)}(x)}{i!} Y_{n, \lambda}^{(r)}((t - x)^i; x) + Y_{n, \lambda}^{(r)}(\alpha(t, x)(t - x)^{r+2}; x) - f^{(r)}(x) \right\}.
\]

\[
\lim_{n \to \infty} n \left\{ Y_{n, \lambda}^{(r)}(f; x) - f^{(r)}(x) \right\}
= \lim_{n \to \infty} n \left\{ \frac{f^{(r)}(x)}{r!} Y_{n, \lambda}^{(r)}((t - x)^r; x) + \frac{f^{(r+1)}(x)}{(r + 1)!} Y_{n, \lambda}^{(r)}((t - x)^{r+1}; x) + \frac{f^{(r+2)}(x)}{(r + 2)!} Y_{n, \lambda}^{(r)}((t - x)^{r+2}; x) - f^{(r)}(x) + Y_{n, \lambda}^{(r)}(\alpha(t, x)(t - x)^{r+2}; x) \right\}
= : I_1 + I_2
\]

Using Lemma 2.2 (iv), then

\[
I_1 = \lim_{n \to \infty} n \left\{ \frac{f^{(r)}(x)}{r!} Y_{n, \lambda}^{(r)}((t - x)^r; x) + \frac{f^{(r+1)}(x)}{(r + 1)!} ((r + 1)(-x) Y_{n, \lambda}^{(r)}((t - x)^r; x) + Y_{n, \lambda}^{(r)}((t - x)^{r+1}; x)) \right\}
\]

\[
+ \frac{f^{(r+2)}(x)}{(r + 2)!} \left( \frac{1}{2} x^2 Y_{n, \lambda}^{(r)}((t - x)^r; x) \right)
+ (r + 2)(-x) Y_{n, \lambda}^{(r)}((t - x)^{r+1}; x) + Y_{n, \lambda}^{(r)}((t - x)^{r+2}; x) - f^{(r)}(x) \right\}
= \lim_{n \to \infty} \left\{ \frac{n f^{(r)}(x)}{n r^r(n - r)!} + \frac{-2r(n + 1)!}{r!} \frac{n!}{n r^r(n - r)!} + \frac{-2r(n + 1)!}{r!} \frac{n!}{n r^r(n - r)!} \right\}
\]

\[
\frac{(r + 1)!}{n!} \left( \frac{n r(n + 1)!}{n r^r(n - r)!} + \frac{r(n + 1)!}{n r^r(n - r)!} \right) (r + 1)! x
\]

\[
+ \left( \frac{2n^r+1(n - r)!}{n r^r(n - r)!} + \frac{2n^r+1(n - r)!}{n r^r(n - r)!} \right) \left( \frac{-2r(n + 1)!}{n!} \frac{n!}{n r^r(n - r)!} + \frac{-2r(n + 1)!}{n!} \frac{n!}{n r^r(n - r)!} \right) (r + 2)! (-x)
\]

\[
+ \frac{2n^r+2(n - r)!}{n!} \left( \frac{2n^r+2(n - r)!}{n r^r(n - r)!} + \frac{2n^r+2(n - r)!}{n r^r(n - r)!} \right) \right\}
\]

\[
= E_1 + E_2 + E_3.
\]

By combining the estimates of \( E_1, E_2, E_3, \) as \( n \to \infty, \) the required is immediate, and we get
\[ E_1 = -\frac{r(r - 1)}{2} f^{(r)}(x); \]
\[ E_2 = -\frac{2rx + r}{2} f^{(r+1)}(x); \]
\[ E_3 = \frac{rx(x - 1)}{2} f^{(r+2)}(x). \]
Since \( l_2 \to 0 \) as \( n \to \infty \), thus, we obtain (4).

**Theorem 3.4**

Let \( (x) \in C[0,1] \). Then, for any \( x \in (0,1) \) at which \( f^{(r+4)}(x) \) exists, \( \lambda \in [-1,1] \)
\[
\lim_{n \to \infty} n^2 \left( n \left\{ Y_{n,\lambda}(f; x) - f(x) \right\} + \frac{r(r - 1)}{2} f^{(r)}(x) - \frac{(-2rx + r)}{2} f^{(r+1)}(x) \right.
\]
\[
- \frac{rx(x - 1)}{2} f^{(r+2)}(x) \right)
\]
\[
= -\frac{4x(r^2 + 1) - (r^2 + r)}{2} \lambda f^{(r+1)}(x) + \left\{ -2(r - 1)x^2 + r(r - 1)x \right\} \lambda f^{(r+2)}(x)
\]
\[
+ \left\{ (-r^2 - 3r + 2)x^3 + (r + 1)x^2 + (-2r + 3)x^3 - \frac{(r^2 + 5r + 5)}{2} x^2 \right\} \lambda f^{(r+3)}(x)
\]
\[
+ \left\{ -(r^2 + 3r + 2)x^4 + \frac{7r^2 - 43r + 60}{12} x^3 + (2r + 3)x^4 + \frac{-17r^2 - 55r - 84}{12} x^3 \right\} \lambda f^{(r+4)}(x). \]
(5)

**Proof**

By Taylor’s expansion,
\[
f(t) = \sum_{i=0}^{r+4} \frac{f^{(i)}(x)}{i!} (t - x)^i + \alpha(t, x)(t - x)^{r+4}.
\]
where \( \alpha(t, x) \to 0 \) as \( t \to x \), hence
\[
\left\{ Y_{n,\lambda}(f; x) - f^{(r)}(x) \right\}
\]
\[
= \sum_{i=0}^{r+4} \frac{f^{(i)}(x)}{i!} Y_{n,\lambda}((t - x)^i; x) + Y_{n,\lambda}^{(r)}(\alpha(t, x)(t - x)^{r+4}, x) - f^{(r)}(x).
\]
\[
\lim_{n \to \infty} n^2 \left\{ n \left( Y_{n,\lambda}(f; x) - f^{(r)}(x) \right) + \frac{r(r + 1)}{2} f^{(r)}(x) - \frac{(-2rx + r)}{2} f^{(r+1)}(x)
\right.
\]
\[
- \frac{2}{2} f^{(r)}(x) - \frac{(-2rx + r)}{2} f^{(r+1)}(x) - \frac{2}{2} f^{(r+2)}(x) \right)
\]
\[
= \lim_{n \to \infty} n^2 \left\{ n \left( \frac{f^{(r)}(x)}{r!} Y_{n,\lambda}^{(r)}((t - x)^r; x) + \frac{f^{(r+1)}(x)}{(r + 1)!} Y_{n,\lambda}^{(r)}((t - x)^{r+1}; x)
\right.
\]
\[
+ \frac{f^{(r+2)}(x)}{(r + 2)!} Y_{n,\lambda}^{(r)}((t - x)^{r+2}; x)
\]
\[
+ \frac{f^{(r+3)}(x)}{(r + 3)!} Y_{n,\lambda}^{(r)}((t - x)^{r+3}; x) + \frac{f^{(r+4)}(x)}{(r + 4)!} Y_{n,\lambda}^{(r)}((t - x)^{r+4}; x) - f^{(r)}(x)
\right.
\]
\[
+ \left. Y_{n,\lambda}^{(r)}(\alpha(t, x)(t - x)^{r+4}, x) \right) = I_1 + I_2.
\]

Using Lemma 1.2 (iv), then
\[ I_1 = \lim_{n \to \infty} n^2 \left( n \left\{ \frac{f^{(r)}(x)}{r!} \mathcal{Y}_{n,\lambda}^{(r)}(t^r; x)ight. \\
+ \frac{f^{(r+1)}(x)}{(r+1)!} \left((r+1)(-x)\mathcal{Y}_{n,\lambda}^{(r)}(t^{r+1}; x) + \mathcal{Y}_{n,\lambda}^{(r)}(t^{r+1}; x)\right) \\
+ \frac{f^{(r+2)}(x)}{(r+2)!} \left((r+2)(r+1) \frac{x^2\mathcal{Y}_{n,\lambda}^{(r)}(t^{r+2}; x) + (r+2)(-x)\mathcal{Y}_{n,\lambda}^{(r)}(t^{r+1}; x) + \mathcal{Y}_{n,\lambda}^{(r)}(t^{r+2}; x)\right) \\
\left. + \frac{f^{(r+3)}(x)}{(r+3)!} \left(-\frac{(r+3)(r+2)(r+1)}{6} x^3\mathcal{Y}_{n,\lambda}^{(r)}(t^{r+3}; x) + \frac{(r+3)(r+2)}{2} x^2\mathcal{Y}_{n,\lambda}^{(r)}(t^{r+2}; x) \right) \\
\right. - (r+3)x\mathcal{Y}_{n,\lambda}^{(r)}(t^{r+2}; x) + \mathcal{Y}_{n,\lambda}^{(r)}(t^{r+3}; x) \right) \\
\right. + \frac{f^{(r+4)}(x)}{(r+4)!} \left(\frac{(r+4)(r+3)(r+2)(r+1)}{24} x^4\mathcal{Y}_{n,\lambda}^{(r)}(t^{r+4}; x) \\
- \frac{(r+4)(r+3)(r+2)}{6} x^3 \mathcal{Y}_{n,\lambda}^{(r)}(t^{r+2}; x) + \frac{(r+4)(r+3)}{2} x^2 \mathcal{Y}_{n,\lambda}^{(r)}(t^{r+2}; x) \\
- (r+4)\mathcal{Y}_{n,\lambda}^{(r)}(t^{r+3}; x) + \mathcal{Y}_{n,\lambda}^{(r)}(t^{r+4}; x) \right) - f^{(r)}(x) \\
+ \frac{r(r-1)}{2} f^{(r)}(x) - \frac{(-2rx+r)}{2} f^{(r+1)}(x) - \frac{rx(x-1)}{2} f^{(r+2)}(x) \right\} \) \]
\[
\begin{align*}
&= \lim_{n \to \infty} n^2 \left\{ \sum_{r=0}^{n} \left( \frac{n!}{n^r(n-r)!} + \frac{-2r(n+1)!}{n^r(n^2-1)(n-r+1)!} \lambda - 1 \right)
\right. \\
&\quad \left. + n \frac{f^{(r+1)}(x)}{(r+1)!} \left( (r+1)! \left( n^r(n-r)! + \frac{-2r(n+1)!}{n^r(n^2-1)(n-r+1)!} \lambda \right) 
\right.
\right. \\
&\quad \left. + \left( \frac{n!}{n^{r+1}(n-r-1)!} + \frac{-2(r+1)(n+1)!}{n^{r+1}(n^2-1)(n-r-1)!} \lambda \right) \right) (r+1)
\right.
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\[-(r + 3)! x^2 \left( \frac{(r + 1)n!}{2n^{r+2}(n - r - 1)!} + \frac{n(r + 1)(n + 1)!}{2n^{r+2}(n^2 - 1)(n - r)!} + \frac{(r + 1)(n + 1)!}{2n^{r+2}(n^2 - 1)(n - r - 1)!} \right) \]

\[+ (r + 3)! x^2 \left( \frac{n!}{n^{r+3}(n - r - 3)!} - \frac{2(r + 3)(n + 1)!}{n^{r+3}(n^2 - 1)(n - r - 1)!} \right) \]

\[+ (r + 3)! x^2 \left( \frac{n(r + 2)(n + 1)!}{2n^{r+3}(n^2 - 1)(n - r - 1)!} + \frac{(r + 2)(n + 1)!}{2n^{r+3}(n^2 - 1)(n - r - 2)!} \right) \]

\[+ \frac{f^{(r+4)}(x)}{(x + 4)!} \left( \frac{n!}{24} x^4 \left( \frac{n!}{n^{r+1}(n - r)!} + \frac{-2(r + 1)(n + 1)!}{n^{r+1}(n^2 - 1)(n - r)!} \right) \right) \]

\[= E_1 + E_2 + E_3 + E_4 + E_5. \]

By combining the estimates of $E_1, E_2, E_3, E_4, E_5$ as $n \to \infty$, we get

$E_1 = 0$;

$E_2 = \frac{-4x(r^2 + 1) - (r^2 + r)}{2} \lambda f^{(r+1)}(x)$;

$E_3 = \{-2(r - 1)x^2 + r(r - 1)x\} \lambda f^{(r+2)}(x)$;
\[ E_4 = \left\{ (-r^2 - 3r + 2)x^3 + (r + 1)x^2 + (-2(r + 3)x^3 - \frac{(r^2 + 5r + 5)}{2}x^2) \right\} f^{(r+3)}(x); \]

\[ E_5 = \left\{ -(r^2 + 3r \right. \\
+ 2)x^4 + \frac{7r^2 - 43r + 60}{12}x^3 + (2(r + 3)x^4 \\
+ \frac{-17r^2 - 55r - 84}{12}x^3) \right\} f^{(r+4)}(x). \]

Since \( I_2 \to 0 \) as \( n \to \infty \), thus, we obtain (5).

References


