DOI:10.24996/ijs.2018.59.2C.13

ISSN: 0067-2904

Generalized Spline Method for Integro-Differential Equations of Fractional Order

Nabaa N. Hasan*, Doaa A. Hussien
Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq

Abstract
In This paper the generalized spline method and Caputo differential operator are applied to solve linear fractional integro-differential equations of the second kind. Comparison of the applied method with exact solutions reveals that the method is tremendously effective.

Keywords: Caputo differential operator, Fractional integro-differential equations, generalized spline.

طريقة السبلاين العامة للمعادلات التفاضلية التكاملية من الرتبة الكسرية

نبأ نجدي حسن*، دعاء احمد حسين
قسم الرياضيات، كلية العلوم، الجامعة المستنصرية، بغداد، العراق

الخلاصة
في هذا البحث طريقة السبلاين العامة و المؤثر التفاضلي Caputo طبق لحل المعادلات التفاضلية التكاملية الكسرية من النوع الثاني. مقارنة الطريقة مع الحل المضبوط يكشف ان هذه الطريقة فعالة بشكل كبير.

1. Introduction

The analytical solution of fractional differential equations, in general, has many difficulties, therefore numerical methods may be suitable for approximating the solution. Polynomial splines have been extensively used in several areas of applied mathematics such as computer graphics and approximation theory. Therefore, one of the first generalizations in this direction are the so called generalized splines which were introduced in the 50's of the 20th century by Ahlberg, Nilson and Walsh, [1]. In [2] generalized splines in R^n is used to solve problems of optimal control. In [3] cubic spline and collection method is used to solve the integral equations of second kind. In [4] the soluton of fractional differential equations is approximated using linear multi-step methods with the cooperation of G-spline interpolation.

generalized spline function is applied to solve fractional integro-differential equations .We are concerned with the numerical solution of the following linear fractional integro differential equation:

$$D^{\alpha}u(x) = f(x) + \int_0^1 k(x,t)u(t)dt \quad 0<x,t\leq 1,\ n-1< \alpha \leq n,\ n\in N \qquad \ldots\ldots(1)$$

With initial conditions: $u^{(i)} = \delta_i$, $i=1,\ldots,n-1$.

This section define some basic definitions of generalized spline and fractional calculus.

where D^{α} u(x) indicates the fractional derivative of $u(x)$.

$f(x)$ and $k(x,t)$ are given continuous function.

x and t are real variables varying in the interval [0,1].

*Email: alzaer1972@uomustansiriyah.edu.iq

2. Preliminaries

Definition (1), [5]: The linear differential operator L of order n , defined by :

$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)$

Where $a_j(x) \in C^n[a,b]$, C class of all functions which are n continuously differentiable defined on [a,b]; j=0,1,…,n and $a_n(x)\neq 0$ on [a,b] , $D=d/dx$ and associated with L its formal adjoints operator:

$L^* = (-1)^n D^n\{a_n(x)\} + (-1)^{n-1}D^{n-1}\{a_{n-1}(x)\} + \cdots + (-1)D\{a_1(x)\} + a_0(x)$

Definition (2), [5]: Let $\Delta : a = x_0 < x_1 < \ldots < x_N = b$, $N \in \mathbb{N}$ be a partition on [a,b]. A real function S, defined on [a,b] is said to be generalized spline with partition Δ if the following holds simultaneously:

1. $S \in \mathcal{K}^{2n}[x_{i-1},x_i]$; i=1,2, …, N.
2. $L^* LS(x) = 0$; $\forall x \in [x_{i-1},x_i]$; i=1,2, …, N.
3. $S \in C^{2n-2}[a, b]$

where $\mathcal{K}^{2n}[x_{i-1},x_i]$ class of all functions defined on $[x_{i-1},x_i]$ has derivative of order 2n, and LS(x) linear genealized spline operator.

Definition(3), [6]: Suppose that $\alpha > 0$, $t > a, \alpha, t \in R$.then

$$D^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dt^n} \displaystyle\int_a^t \dfrac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n \in N, \\ \dfrac{d^n}{dt^n} f(t), & \alpha = n \in N, \end{cases}$$

Is called the Riemann-Liouville fractional derivative or the Riemann-Liouville fractional differential operator of order α.

Definition (4), [6]: Suppose that $\alpha > 0, t > a, \alpha \text{ and } a, t \in R$. The fractional operator

$$D_*{}^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t \dfrac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau , & n-1 < \alpha < n \in N, \\ \dfrac{d^n}{dt^n} f(t), & \alpha = n \in N, \end{cases}$$

is called the Caputo fractional derivative or Caputo fractional differential operator of order α.

Theorem,[6]: Let $\alpha \in R, n-1 < \alpha < n , n \in N, \lambda \in \mathbb{C}$ (set of complex numbers), then the Caputo fractional derivative of the exponential function has the form:

$$D_*{}^\alpha e^{\lambda t} = \sum_{k=0}^{\infty} \frac{\lambda^{k+n} t^{k+n-\alpha}}{\Gamma(k+1+n-\alpha)} = \lambda^n t^{n-\alpha} E_{1,n-\alpha+1}(\lambda t)$$

where $E_{\alpha,\beta}(z)$ is the two-parameter function of Mittag-leffler type.

3. Aproximate Solution of The Fractional Integro-Differential Equation:

Generalized spline function is applied to find the approximate solution of the fractional integro-differential equation given in (1),Let:

$$S(x) = \sum_{j=1}^{2n} c_j\, q_j(x) \;, 0 \le x \le 1 \quad (3)$$

Be the generalized spline function of order 2n will be used to approximate eq.(1).

Where q_j , $j=1,2,\ldots,2n$ be the basis function of generalized spline $S(x)$,$2n$ is order of $L^*Lx=0$ and $c_1,c_2,\ldots,c_{2n}$ are constants to be found.

Substituting (3) in (1) we obtain :

$$D^\alpha\left(\sum_{j=1}^{2n} c_j\, q_j(x)\right) = f(x) + \int_0^1 k(x,t)\left[\sum_{j=1}^{2n} c_j\, q_j(t)\right]dt \quad (4)$$

$$\sum_{j=1}^{2n} c_j\, D^\alpha q_j(x) - \int_0^1 k(x,t) \sum_{j=1}^{2n} c_j\, q_j(t)dt = f(x) \quad (5)$$

$$\sum_{j=1}^{2n} c_j \left[D^\alpha\left(q_j(x)\right) - \int_0^1 k(x,t)\, q_j(t)dt\right] = f(x)$$

Let $M_j(x) = D^\alpha q_j(x) - \int_0^1 k(x,t)\, q_j(t)dt$, $j = 1,2,\ldots,2n$

Adding the initial conditions of eq.(1) as a new raw in the following matrices:

$$M=\begin{pmatrix} M_1(x_0) & M_2(x_0) & \dots & M_{2n}(x_0) \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ M_1(x_N) & M_2(x_N) & \dots & M_{2n}(x_N) \\ u'_1(0) & u'_2(0) & \dots & u'_{2n}(0) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ u^{(n-1)}{}_1(0) & u^{(n-1)}{}_2(0) & \dots & u^{(n-1)}{}_{2n}(0) \end{pmatrix}, C=\begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_{2n} \end{pmatrix}, F=\begin{pmatrix} f(x_0) \\ . \\ . \\ . \\ f(x_N) \\ \delta_1 \\ . \\ . \\ . \\ \delta_{n-1} \end{pmatrix} \quad \dots(6)$$

Or in the system form:

$$MC = F \quad \dots(7)$$

M and F are constant matrices with dimensions *(N+2)*×*2n* and *(N+2)*×*1* C of dimension *2n*× 1
The system will construct has *n* equations and *m* coefficients s.t *n* > *m* therefore ,
Calculate ,$M^T MC = M^T F$
to find the C_j , $j = 1,2,\dots 2n$ and substitute this solution in eq.(6) to get the approximate solution of equation (1).

4-Generalized Spline to Solve Practical Examples:

Example (1): Consider the Fredholm fractional integro-differential equation :

$$D^{\frac{1}{2}}u(x) = \frac{\left(\frac{8}{3}\right)x^{\frac{3}{2}} - 2x^{1/2}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 x\, tu(t)dt \text{ , } 0 \le x, t \le 1 \quad \dots(8)$$

with initial condition :

$$u(0) = 0 \quad \dots(9)$$

The exact solution is, [7]:

$u(x) = x^2 - x$ Let Δ be a partition for the x-axis , s.t : ; $\Delta: 0 = x_0 < x_1 < x_2 < x_3 < x_4 < x_5 = 1$
where h=0.2,then $x_0 = 0, x_1 = 0.2, x_2 = 0.4\ , x_3 = 0.6\ , x_4 = 0.8, x_5 = 1$
Applying the generalized spline function to Fredholm fractional integro-differential equation (8)
Let $L = D^2 - 4$,be the differential operator of order 2,
Then it is adjoint is $L^*=D^2-4$ by using the homogeneous differential equation
$L^*Lu = D^4u - 8D^2u + 16u$, we have the solutions:
$u_1(t) = e^{2t}\ , u_2(t) = e^{-2t}\ , u_3(t) = te^{2t}\ , u_4(t) = te^{-2t}$
So that the generalized spline function in each $[t_i , t_{i+1}]$ is :

$$S(t) = c_1e^{2t} + c_2e^{-2t} + c_3te^{2t} + c_4te^{-2t} \quad \dots(10)$$

In eq.(10) the coefficients c_1 , c_2 , c_3 , c_4 are unknown four algebraic equations are needed substituting eq.(10) in the initial condition eq.(9)
, yield:

$$c_1 + c_2 = 0 \quad \dots(11)$$

Now for applying eq.(10) in eq.(8) we get

$$D^{\frac{1}{2}}(c_1e^{2x} + c_2e^{-2x} + c_3xe^{2x} + c_4xe^{-2x}) - \int_0^1 x\, t(c_1e^{2t} + c_2e^{-2t} + c_3te^{2t} + c_4te^{-2t})dt = f(x) \quad (12)$$

where $f(x) = \frac{\left(\frac{8}{3}\right)x^{3/2} - 2x^{1/2}}{\sqrt{\pi}} + \frac{x}{12}$

The system will construct from eq.(11) and eq.(12) has 5 equations and 4 coefficients ,
Therefore , calculate:

$$M^T MC = M^T F \quad \dots(13)$$

Where M is constant matrix of dimension (5× 4) gain from eq.(12)
$C=[c_1\ \ c_2\ \ c_3\ \ c_4]^T$
$F=[u(0)\ \ f(x_0)\ \ f(x_1)\ \ f(x_2)\ \ f(x_3)]^T$
Finally , Gauss elimination method is used to solve system (13) to find :
so the approximate solution S(x) is :
c_1 =-0.086 ,c_2 =0.087 , c_3 =0.102 , c_4 =-0.99
$S(x) = -0.086e^{2x} + 0.087e^{-2x} + 0.102xe^{2x} - 0.99xe^{-2x}$

Table (1), presents a comparison between the exact and numerical solution

Table 1-Numerical results of example (1)

x	u(x)	s(x)	$\lvert u(x)-s(x)\rvert$
0	0	1×10^{-3}	1×10^{-3}
0.1	-0.09	-0.102	0.012
0.2	-0.16	-0.172	0.012
0.3	-0.21	-0.216	6.196×10^{-3}
0.4	-0.24	-0.239	5.629×10^{-4}
0.5	-0.25	-0.245	4.765×10^{-3}
0.6	-0.24	-0.235	4.956×10^{-3}
0.7	-0.21	-0.209	1.356×10^{-3}
0.8	-0.16	-0.164	4.13×10^{-3}
0.9	-0.09	-0.098	7.812×10^{-3}
1	0	-3.983×10^{-3}	3.983×10^{-3}

Figure-1, give the exact solution u(x) and its approximation S(x) for the considered example

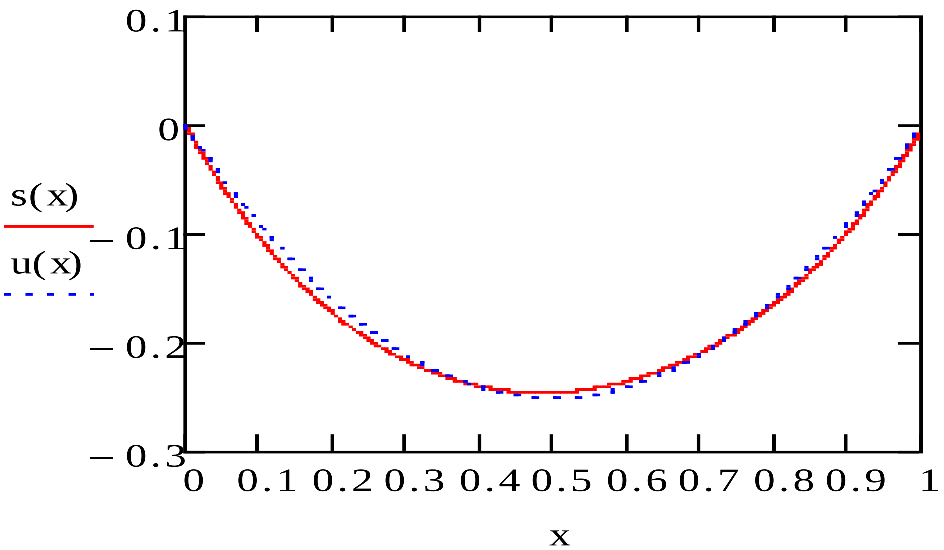

Figure 1-Exact and approximate solutions of example(1)

Example(2):

$$D^2y(x)+D^{\frac{1}{2}}y(x)+y(x)=\frac{9}{4}-\frac{x}{3}+\frac{2}{\Gamma\left(\frac{5}{3}\right)}x^{\frac{3}{2}}+x^2+\int_0^1(x-t)y(t)dt \quad ...(14)$$

with the initial conditions

$$y(0)=y'(0)=0 \quad ...(15)$$

the exact solution is, [8]:

$y(x)=x^2$

let Δ be a partition for the x-axis , s.t : $\Delta: 0=x_0<x_1<x_2<x_3=1$

where h=1/3 , then $x_0=0\,, x_1=\frac{1}{3}\,, x_2=\frac{2}{3}\,, x_3=1$

Applying the generalized spline function to Fredholm fractional integro-differential equation

Let $L=D^2$, then $L^*Lu=D^4u$

We have the solutions: $u_1(t)=1\,, u_2(t)=t\,, u_3(t)=\frac{t^2}{2}\,, u_4(t)=\frac{t^3}{6}$

Which gives the generalized spline polynomial

$$y(t)=c_1+c_2t+c_3\frac{t^2}{2}+c_4\frac{t^3}{6} \quad ...(16)$$

The coefficients c_1,c_2,c_3 and c_4 in eq.(16) are unknown four algebraic equations

Now substituting eq.(16) in the initial conditions eq.(15) ,we have:

$c_1 = 0$ …(17)
$c_2 = 0$ …(18)
Now ,for applying eq.(16)in eq.(14)

$$D^2\left(c_1 + xc_2 + \frac{x^2}{2}c_3 + \frac{x^3}{6}c_4\right) + D^{\frac{1}{2}}\left(c_1 + xc_2 + \frac{x^2}{2}c_3 + \frac{x^3}{6}c_4\right) + \left(c_1 + xc_2 + \frac{x^2}{2}c_3 + \frac{x^3}{6}c_4\right) - \int_0^1 (x-t)\left(c_1 + tc_2 + \frac{t^2}{2}c_3 + \frac{t^3}{6}c_4\right)dt = f(x)$$

$$c_3 + xc_4 + \frac{\Gamma(2)}{\Gamma(\frac{3}{2})}x^{1/2}c_2 + \frac{\Gamma(3)}{\Gamma(\frac{5}{2})}x^{3/2}c_3 + \frac{\Gamma(4)}{\Gamma(\frac{7}{2})}x^{5/2}c_4 + \left(c_1 + xc_2 + \frac{x^2}{2}c_3 + \frac{x^3}{6}c_4\right) - xc_1 - \frac{1}{2}xc_2 - \frac{1}{6}xc_3 - \frac{1}{24}xc_4 + \frac{1}{2}c_1 + \frac{1}{3}c_2 + \frac{1}{8}c_3 + \frac{1}{30}c_4 = f(x) \quad \dots(19)$$

where $f(x) = \frac{9}{4} - \frac{x}{3} + \frac{2}{\Gamma(\frac{5}{3})}x^{3/2} + x^2$

The system will construct from eq.(17),eq.(18)and eq.(19) has 6 equations and 4 coefficients , therefore ,calculate :

$M^T MC = M^T F$ …(20)

Where M is constant matrix of dimension (6×4) then:
C=[c_1 c_2 c_3 c_4]^T
F=[y(0) y′(0) f(x_0) f(x_1) f(x_2) f(x_3)]^T
Finally, Gauss elimination method may be used to solve system (20) to find: c_1 =-1.599×10^{-3} , c_2 =-0.01 , c_3 =1.972 , c_4 =0.539
So the approximate solution y(x):

$$y(x) = -1.599 - 0.01x + \frac{1.972}{2}x^2 + \frac{0.539}{6}x^3$$

Table-2 present a comparison between the exact and numerical solution:
Table 2-Numerical results of example (2):

x	Exact	approximate	Absolute error
0	0	-1.599×10^{-3}	1.599×10^{-3}
0.1	0.01	7.351×10^{-3}	2.649×10^{-3}
0.2	0.04	0.037	3.44×10^{-3}
0.3	0.09	0.087	3.434×10
0.4	0.16	0.158	2.09×10^{-3}
0.5	0.25	0.251	1.13×10^{-3}
0.6	0.36	0.367	6.765×10^{-3}
0.7	0.49	0.505	0.015
0.8	0.64	0.667	0.027
0.9	0.81	0.854	0.044
1	1	1.064	0.064

Figure-2, give the exact solution ex(x) and its approximation u(x) for the considered example, in which can see the accuracy of the obtained result and the applicability of the method.

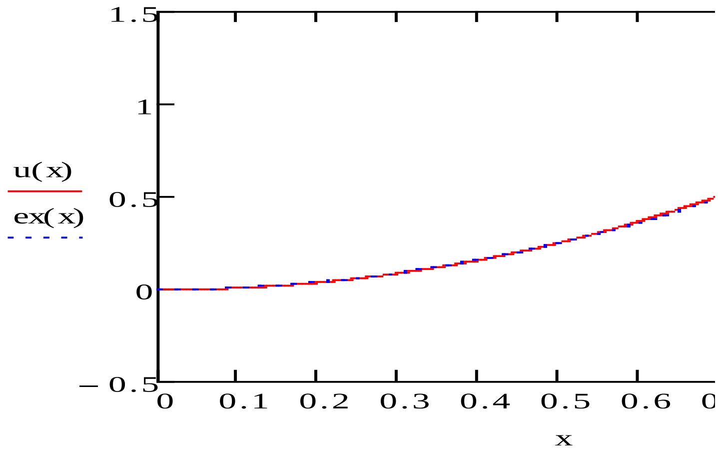

Figure 2-Exact and approximate solution of example (2).

Example (3): consider the Volterra linear fractional voltera integro-differential equation

$D^{\frac{3}{4}}y(t) = \frac{6t^{9/4}}{\Gamma(\frac{13}{4})} + \left(\frac{-t^2e^t}{5}\right)y(t) + \int_0^t e^t s y(s)ds$ …(21)

With the initial condition :

$y(0) = 0$ …(22)

and the exact solution is, [9]:

$y(t) = t^3$

let Δ be a partition of the x-axis , s.t $\Delta: 0 = x_0 < x_1 < x_2 < x_3 < x_4 < x_5 = 1$

where $h = 0.2$, then $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_4 = 0.6$, $x_4 = 0.8$, $x_5 = 1$

Applying the generalized spline function to the Volterra linear fractional integro-differential equation

Let $LL^*u = D^4u - 13D^2u + 36u$, with basis functions:

$u_1(t) = e^{3t}$, $u_2(t) = e^{-3t}$, $u_3(t) = e^{2t}$, $u_4(t) = e^{-2t}$

Which give the generalized spline function:

$y(t) = c_1e^{3t} + c_2e^{-3t} + c_3e^{2t} + c_4e^{-2t}$ …(23)

The coefficients c_1, c_2, c_3 *and* c_4 in eq.(23) are unknown four algebraic equations

Now substituting eq.(23) in the initial condition eq.(22), we get:

$c_1 + c_2 + c_3 + c_4 = 0$ …(24)

Now for applying eq.(23) in eq.(21)

$D^{\frac{3}{4}}(c_1e^{3t} + c_2e^{-3t} + c_3e^{2t} + c_4e^{-3t}) + \left(\frac{t^2e^t}{5}\right)(c_1e^{3t} + c_2e^{-3t} + c_3e^{2t} + c_4e^{-2t}) -$

$e^t\int_0^t s\,(c_1e^{3s} + c_2e^{-3s} + c_3e^{2s} + c_4e^{-2s})ds = f(t)$ …(25)

where $f(t) = \frac{6t^{9/4}}{\Gamma(\frac{13}{4})}$

As in example (2) Gauss elimination method may be used to solve system (20) for eq.(25) and eq.(24) to find

$c_1 = 0.043$, $c_2 = 0.777$, $c_3 = 0.026$, $c_4 = -0.838$,

so the approximate solution y(t) is:

$y(t) = 0.043e^{3t} + 0.777e^{-3t} + 0.026e^{2t} - 0.838e^{-2t}$

Table-3, presents a comparison between the exact and numerical solution:

Table 3-Numerical results of example (3):

x	y(x)	s(x)	$\lvert y(x) - s(x)\rvert$
0	0	8×10^{-3}	8×10^{-3}
0.1	1×10^{-3}	-0.021	0.022
0.2	8×10^{-3}	-0.018	0.026
0.3	0.027	9.138×10^{-3}	0.018

0.4	0.064	0.058	5.881×10^{-3}
0.5	0.125	0.128	3.477×10^{-3}
0.6	0.216	0.222	6.494×10^{-3}
0.7	0.343	0.345	2.081×10^{-3}
0.8	0.512	0.504	7.926×10^{-3}
0.9	0.729	0.711	0.018
1	1	0.981	0.019

Figure-3, give the exact solution ex(t) and its approximation s(t) for the considered example.

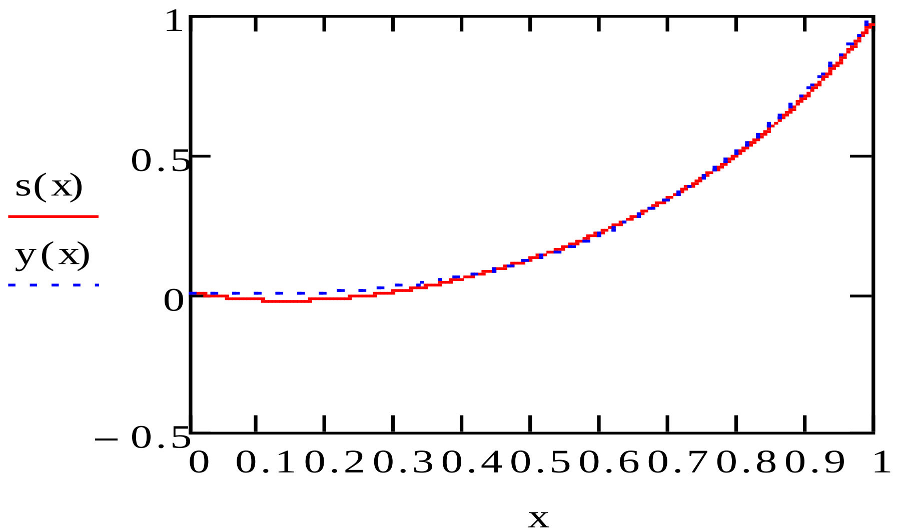

Figure 3-Exact and approximate solutions of example (3).

5. Conclusions

In this paper, the application of generalized spline functions investigated to obtain approximate solution of fractional integro-differential equations. Three test examples are considered with different operators of order *2n*. As a comparison with the exact solution , Tables- (1, 2, 3), Figures-(1, 2, 3) showed the result.

References

1. Ahlberge, J.H., Nilson, E.N. and walsh, J.L. **1967**. "*The theory of splines and their applications*", Academic press, New york
2. Rodrigues, R.C. and Torres, M. **2006.** "Generalized splines in R^n and optimal control", *Rend. Sem. Mat. Univ. Pol. Torino*, **64**(1).
3. Oladejo S.o. and olurode K.A. **2008.** "The Application of cubic spline collection to the solution of integral equations ", *Applied sciences Research* , **4**(6): 748-753.
4. Osama, H., Mohammed Fadhel, S., Fadhel and Akram M. Al-abood. **2007**. "G-spline interpolation for approximating the solution of fractional differential equation using linear multi-step methods", *Journal of Al-Nahrain University – Science*, **10**(2): 118-123.
5. Fadhel, S., Fadhel ,Suha N.Al-Rawi and Nabaa N. Hassan, **2010.** "Generalized Spline Approximation Method for Solving Ordinary and partial Differential Equations", *Eng.&Tech .journal*.
6. Mariya kamenova Ishtera. **2005**."Properties and applications of the Caputo fractional operator",Thesis Department of Mathmetics Unversity Karlsruhe(TH).
7. D.sh. Mohammed,"Numerical solution of Fractional Integro-Differential Equations by Least squares method and shifted chebyshev polynomial", *Mathematical Problems in Engineering*, **2014**: 1-5
8. Gulsu, M., Ozturk, Y. Anapal, A. **2013.** "Numerical approach for solving fractional fredholm integro-differential eqqution", *International Journal of computer Mathematics*, **90**(7): 1413-1434.
9. Rawashdeh, E.A. **2006**. "Numerical solution of fractional integro-differential equations by collocation method",*Apppl.math.comput*, **176**: 1-6.