DOI: 10.24996/ijs.2022.63.3.28

ISSN: 0067-2904

Numerical Solution of Linear Fractional Differential Equation with Delay Through Finite Difference Method

Auras K. Hameed[1*], Muna M. Mustafa[2]
[1]Department of Computer Science, College of Science for Women, University of Baghdad,
[2]Department of Mathematics, College of Science for Women, University of Baghdad,

Received: 14/12/2020 Accepted: 13/6/2021

Abstract
This article addresses a new numerical method to find a numerical solution of the linear delay differential equation of fractional order α; $1 < \alpha < 2$, the fractional derivatives described in the Caputo sense. The new approach is to approximating second and third derivatives. A backward finite difference method is used. Besides, the composite Trapezoidal rule is used in the Caputo definition to match the integral term. The accuracy and convergence of the prescribed technique are explained. The results are shown through numerical examples.

Keywords: Caputo derivative, Delay differential equation, fractional order, finite difference, Trapezoidal rule.

الحل العددي للمعادلة التفاضلية الكسرية الخطية مع التأخير بطريقة الفروق المنتهية

اوراس خالد حميد[1]*، منى منصور مصطفى[2]
[1]قسم علوم الحاسوب، كلية العلوم للبنات، جامعة بغداد، بغداد، العراق
[2]قسم علوم الرياضيات، كلية العلوم للبنات، جامعة بغداد، بغداد، العراق

الخلاصة
تتناول هذه المقالة طريقة عددية جديدة لإيجاد حل عددي للمعادلة التفاضلية للتأخير الخطي ذات الترتيب الكسري α ، $1 < \alpha < 2$; المشتقات الكسرية الموصوفة بالمعنى Caputo. الأسلوب الجديد لتقريب المشتقات الثانية والثالثة يستخدم طريقة الفروق المنتهية التراجعي. إلى جانب ذلك ، يتم استخدام قاعدة شبه منحرف المركبة في تعريف Caputo لمطابقة المصطلح المتكامل.تم تشرح دقة وتقارب التقنية الموصوفة والنتائج تم توضيحها من خلال الأمثلة العددية.

.

1. Introduction
Fractional differential equations are applied in Engineering, science, finance, applied mathematics, bioengineering, and others [1]. However, many researchers investigated a solution for delay fractional differential equations . In [2], AL-Saltani (2007) used a linear multistep method to find a solution for delay fractional differential equations. Khader and Hendy (2012) used the Legendre seudospactral method to approximate the solution of

*Email: aurask_comp@csw.uobaghdad.edu.iq

fractional-order delay differential equations [1]. Kaslik, and Sivasundaram (2012) presented several analytical and numerical approaches for the stability analysis of linear fractional-order delay differential equations [3]. Morgado et al. (2013) interested in linear fractional differential equations with delay and they summarised existence and uniqueness theory based on the method of steps. They also discussed the solution's dependence on the equation [4]. Moghaddam and Mostaghim (2013) used a finite difference method to solve nonlinear fractional delay differential equations with $0 < \alpha < 1$ [5]. The Hermite wavelet method is used by Saeed and Rehman in 2014 to find a solution for fractional delay differential equations [6]. Iqbal et al. (2015) introduced the Laguerre wavelet method and combined the steps method to solve linear and nonlinear delay differential equations of fractional-order [7]. A new predictor-corrector process expanded by Daftardar-Gejji et al. [8] to solve fractional delay differential equations and to perform related error analysis. Saeed et al. (2015) used the Chebyshev wavelet method for solving the fractional delay differential equations and integro-differential equations [9]. Xu and Lin (2016) modified the traditional Runga-Kutta method to derive the numerical solutions of fractional delay differential equations [10]. Moghaddam et al. [11] presented a numerical technique based on the Adams-Bashforth-Moulton scheme to solve Variable order fractional delay differential equations. In 2016 Vivek et al. [12] described the application of the improved predictor-corrector method to solve delay differential equations of the fractional-order. Mohammed and Wadi (2016) generalized the Hat function operational matrices, then combined with a process depends on steps method to solve linear and nonlinear delay differential equations of fractional order [13]. Muthukumar and Priya (2017) used shifted Jacobi polynomial to find the numerical solution of linear fractional delay differential equation with $0 < \alpha < 1$ [14]. In[15], Moghaddam, and Mostaghim found numerical solutions and boundary conditions of nonlinear fractional differential equations with delay using the finite difference method. In 2017 Sumudu transform decomposition method has been introduced by Eltayeb and Abdeldaim to solve the linear and nonlinear fractional delay differential equations [16]. In [17], Li and Wang presented a concept of delayed Mittag-Leffler type matrix function then they studied the finite time stability of fractional delay differential equation. Valizadeh et al. (2019) proposed a new method called perturbed decomposition natural transform method to solve the fractional delay pantograph differential equation [18]. Raslan et al. (2019) used spectral Tau method for solving general fractional-order differential equations with a linear functional argument [19]. Malmir (2019) construct a new fractional integration operational matrix of Chebyshev wavelets for fractional differential delay systems [20]. Yang et al. (2019) find a solutions to a linear fractional delay differential equation of Hadamard type by introducing the Mittag-Leffler delay matrix functions with logarithmic functions [21].

In this paper, we propose a new method for finding a numerical solution based on backward finite difference formula for the linear fractional differential equation with a delay of the form:

$$D^{\alpha} y(t) = A_1(t) y(t) + A_0(t) y(t-\tau) + f(t) , \quad (1)$$

with

$$y(t) = \emptyset(t) \; for \; t \in [-\tau, a] \; , \quad (2)$$

where $t \in [a, b]$ and $1 < \alpha < 2$. The functions $A_0 , A_1 , f \; and \; \emptyset$ are continuous on $[a, b]$, $\tau > 0$, $y(t)$ the unknown function.

The structure of this paper is organized as follows: Section 2 includes basic concepts. Section 3 provides the basic idea of the proposed approach, and section 4 describes the solution's algorithm. Section 5 contains numerical examples. Finally, section 6 gives conclusions and recommendations.

2. Preliminaries

This section presents notations, definitions of fractional derivative and the backward finite differences method to approximate the second and third derivative. The description of the composite Trapezoidal rule is also mentioned.

Definition 1 The Caputo fractional derivative operator D^{α} of order α is defined in the following form [19]:

$$D^{\alpha}f(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^{(m)}(z)}{(t-z)^{\alpha-m+1}}dz \ ,\alpha>0 \tag{3}$$

where $m-1\le\alpha\le m\ ,m\in\mathbb{N},t>0$.

Definition 2 The backward four points difference formula of order two to approximate the second derivative is [21]:

$$f''(x_i)=\frac{2f(x_i)-5f(x_{i-1})+4f(x_{i-2})-f(x_{i-3})}{h^2} \quad . \tag{4}$$

Definition 3 The backward five points difference formula of order two to approximate the third derivative is [21]:

$$f'''(x_i)=\frac{5f(x_i)-18f(x_{i-1})+24f(x_{i-2})-14f(x_{i-3})+3f(x_{i-4})}{2h^3} \quad , \tag{5}$$

where h represents the distance between points.

Definition 4 Suppose that the interval $[a,\ b]$ is subdivided into s equally spaced intervals $[x_k\ ,x_{k+1}]$ of width $h=\frac{b-a}{s}$, by using the equally spaced nodes $x_k=a+kh\ ,\ k=0,1,\dots,s$. The composite Trapezoidal rule for s subintervals is (21):

$$\int_a^b f(x)dx=\frac{h}{2}\sum_{k=0}^{s-1}(f(x_k)+f(x_{k+1})) \ . \tag{6}$$

3. Method of Solution

This section introduces a numerical solution of fractional delay differential equation of order $1<\alpha<2$ using the finite differences method. First of all, an approximate to the fractional derivative D^{α} is proposed for $t=t_j=a+jh, j=1,2,\dots,l$ and $h=\frac{b-a}{l}$:

According to definition.1 and the value of $1<\alpha<2$, we obtain:

$$D^{\alpha}f(t)=\frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{f''(z)}{(t-z)^{\alpha-1}}dz \quad . \tag{7}$$

The following results obtained by applying integration by parts formula for the integral in eq.7:

$$D^{\alpha}f(t)=\frac{1}{(2-\alpha)\Gamma(2-\alpha)}\left[f''(0)t^{2-\alpha}-\int_0^t f^{(3)}(z)(t-z)^{2-\alpha}dz\right] \ , \tag{8}$$

using the composite Trapezoidal rule in eq.6 for the integral in eq.8, to get:

$$\int_0^{t_j} f^{(3)}(z)(t-z)^{2-\alpha}dz=\frac{h}{2}\left[f^{(3)}(0)(t-0)^{2-\alpha}+2\sum_{m=1}^{j-1}f^{(3)}(t_m)(t-t_m)^{2-\alpha}+f^{(3)}(t_j)\left(t-t_j\right)^{2-\alpha}\right] \ .$$

Substituting eq.8 in eq.7, for $t=t_j\ ,j=0,1,\dots,l$, yields:

$$D^{\alpha}f(t)|_{t=t_j}=\frac{1}{(2-\alpha)\Gamma(2-\alpha)}\left[f''(0)t_j^{2-\alpha}-\frac{h}{2}\left[f^{(3)}(0)t_j^{2-\alpha}+2\sum_{m=1}^{j-1}f^{(3)}(t_m)\left(t_j-t_m\right)^{2-\alpha}\right]\right].j=1,2,\dots,l$$

Then, for the 2nd and 3rd derivatives that appear previous equation, the backward difference formula (eq.4 and eq.5) are used to obtain:

$$D^{\alpha}f(t)|_{t=t_j} = \frac{1}{(2-\alpha)\Gamma(2-\alpha)}\left[\frac{2f(0)-5f(a-h)+4f(a-2h)-f(a-3h)}{h^2}t_j^{2-\alpha} - \frac{h}{2}\left[\frac{5f(0)-18f(a-h)+24f(a-2h)-14f(a-3h)+3f(a-4h)}{2h^3}t_j^{2-\alpha} + 2\sum_{m=1}^{j-1}\frac{5f(x_m)-18f(x_{m-1})+24f(x_{m-2})-14f(x_{m-3})+3f(x_{m-4})}{2h^3}\left(t_j-t_m\right)^{2-\alpha}\right]\right] \quad . \quad j=1,2,..,l \qquad (9)$$

Now, a numerical solution of delay differential equation with fractional order is considered by substituting $t = t_j$, $j = 1,2,\dots l$ in eq.1, then we get:

$$D^{\alpha}y(t)|_{t=t_j} = A_1(t_j)y(t_j) + A_0(t_j)y(t_j-\tau) + f(t_j) \ , j = 1,2,\dots,l$$

using eq.9 for approximating $D^{\alpha}f(t)|_{t=t_j}$, to obtain the following system of linear equations:

$$CY + D = Q \quad , \qquad (10)$$

where C is a matrix of $l \times l$ dimension, Y, D, and Q are vectors of length l, which have the following form:

$$C_{ij} = \begin{cases} 0 & ,if\ i<j \\ A_1(t_i) & ,if\ i=j \\ \frac{5(x_i-x_j)^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} & ,if\ i-j=1 \\ \frac{5(x_i-x_j)^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} - \frac{18(x_i-x_{j+1})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} & ,if\ i-j=2 \\ \frac{5(x_i-x_j)^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} - \frac{18(x_i-x_{j+1})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} + \frac{24(x_i-x_{j+2})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} & ,if\ i-j=3 \\ \frac{5(x_i-x_j)^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} - \frac{18(x_i-x_{j+1})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} + \frac{24(x_i-x_{j+2})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} - \frac{14(x_i-x_{j+3})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} & ,if\ i-j=4 \\ \frac{5(x_i-x_j)^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} - \frac{18(x_i-x_{j+1})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} + \frac{24(x_i-x_{j+2})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} - \frac{14(x_i-x_{j+3})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} + \frac{3(x_i-x_{j+4})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} & ,otherwise \end{cases} \qquad (11)$$

$, Y = [y_1 \quad y_2 \quad \cdots \quad y_l]^T, d_j = A_0(t_j)y(t_j-\tau) \ , j = 1,2,\dots,l$

and

$$q_j = \begin{cases} f_j - \frac{x_j^{2-\alpha}}{(2-\alpha)\Gamma(2-\alpha)} * \frac{(2\emptyset_0-5\emptyset_{-1}+4\emptyset_{-2}-\emptyset_{-3})}{h^2} & , if\ j=1 \\ f_j - \frac{x_j^{2-\alpha} * z_1}{(2-\alpha)\Gamma(2-\alpha)} - \frac{x_j^{2-\alpha} * z_2}{(2-\alpha)\Gamma(2-\alpha)} + \frac{18(x_j-x_{j-1})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)}\emptyset_0 - \frac{24(x_j-x_{j-1})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)}\emptyset_{-1} + \frac{14(x_j-x_{j-1})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)}\emptyset_{-2} - \frac{24(x_j-x_{j-1})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)}\emptyset_{-3} & , if\ j=2 \\ f_j - \frac{x_j^{2-\alpha} * z_1}{(2-\alpha)\Gamma(2-\alpha)} - \frac{x_j^{2-\alpha} * z_2}{(2-\alpha)\Gamma(2-\alpha)} + \begin{pmatrix} \frac{18(x_j-x_{j-2})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \\ - \frac{24(x_j-x_{j-1})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \end{pmatrix}\emptyset_0 + \begin{pmatrix} \frac{-24(x_j-x_{j-2})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \\ + \frac{14(x_j-x_{j-1})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \end{pmatrix}\emptyset_{-1} + \begin{pmatrix} \frac{14(x_j-x_{j-2})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \\ - \frac{3(x_j-x_{j-1})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \end{pmatrix}\emptyset_{-2} - \frac{3(x_j-x_{j-2})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)}\emptyset_{-3} & , if\ j=3 \\ f_j - \frac{x_j^{2-\alpha} * z_1}{(2-\alpha)\Gamma(2-\alpha)} - \frac{x_j^{2-\alpha} * z_2}{(2-\alpha)\Gamma(2-\alpha)} + \begin{pmatrix} \frac{18(x_j-x_{j-3})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \\ - \frac{24(x_j-x_{j-2})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \\ + \frac{14(x_j-x_{j-2})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \end{pmatrix}\emptyset_0 + \begin{pmatrix} \frac{-24(x_j-x_{j-3})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \\ + \frac{14(x_j-x_{j-2})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \\ - \frac{3(x_j-x_{j-2})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \end{pmatrix}\emptyset_{-1} + \begin{pmatrix} \frac{14(x_j-x_{j-3})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \\ - \frac{3(x_j-x_{j-2})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \end{pmatrix} - \emptyset_{-2} - \frac{3(x_j-x_{j-2})^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)}\emptyset_{-3} & , if\ j=4 \\ f_j - \frac{x_j^{2-\alpha} * z_1}{(2-\alpha)\Gamma(2-\alpha)} - \frac{x_j^{2-\alpha} * z_2}{(2-\alpha)\Gamma(2-\alpha)} + \begin{pmatrix} \frac{18(x_j-x_1)^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \\ - \frac{24(x_j-x_2)^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \\ + \frac{14(x_j-x_3)^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \\ - \frac{3(x_j-x_4)^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \end{pmatrix}\emptyset_0 + \begin{pmatrix} \frac{-24(x_j-x_1)^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \\ + \frac{14(x_j-x_2)^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \\ - \frac{3(x_j-x_3)^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \end{pmatrix}\emptyset_{-1} + \begin{pmatrix} \frac{14(x_j-x_1)^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \\ - \frac{3(x_j-x_2)^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)} \end{pmatrix} - \emptyset_{-2} - \frac{3(x_j-x_1)^{2-\alpha}}{2h^2(2-\alpha)\Gamma(2-\alpha)}\emptyset_{-3} & , otherwise \end{cases} \quad (12)$$

where $z_1 = \frac{(2\emptyset_0-5\emptyset_{-1}+4\emptyset_{-2}-\emptyset_{-3})}{h^2}$ and $z_2 = \frac{(5\emptyset_0-18\emptyset_{-1}+24\emptyset_{-2}-14\emptyset_{-3}+3\emptyset_{-4})}{(2h)^3}$.

4. Algorithm of the Proposed Method:

The following steps to evaluate numerical solutions of linear fractional differential equation with a delay using the finite difference method :

Step 1: assume that $h = \frac{b-a}{n}$, $n \epsilon \mathbb{N}$, u $\quad y(t) = \emptyset(t)\ for\ t \in [-\tau, a]$.

Step 2: put $x_i = a + ih$, with $x_0 = a$ and $x_n = b$, $i = 0,1,\cdots,n$.

Step 3: Calculate the linear system $CY + D = Q$ using eq.(11) and (12).

Step 4: Solve the system of step3 using the Gaussian elimination method with partial pivoting.

5. Numerical Examples:

The following examples are constructed to illustrate the effectiveness of the proposed technique .

Test Example 1: Consider the linear fractional differential equation with a delay of the form:

$$D^{\alpha}y(t) = 2y(t) + ty(t-1) + \cos t - 2\sin(t-1) - t\sin t\ , \quad (13)$$

with

$$y(t) = t - \frac{t^3}{6}\ for\ t \in [-1,0]\ , \quad (14)$$

where the exact solution $y(t) = sin\, t \quad for\ \alpha = 1$. Figure 1 shows a comparison between the results of using the proposed method with different values of $\alpha = 1, \frac{5}{4}, \frac{3}{2}, and\ \frac{7}{4}$ with the exact solution to the problem.

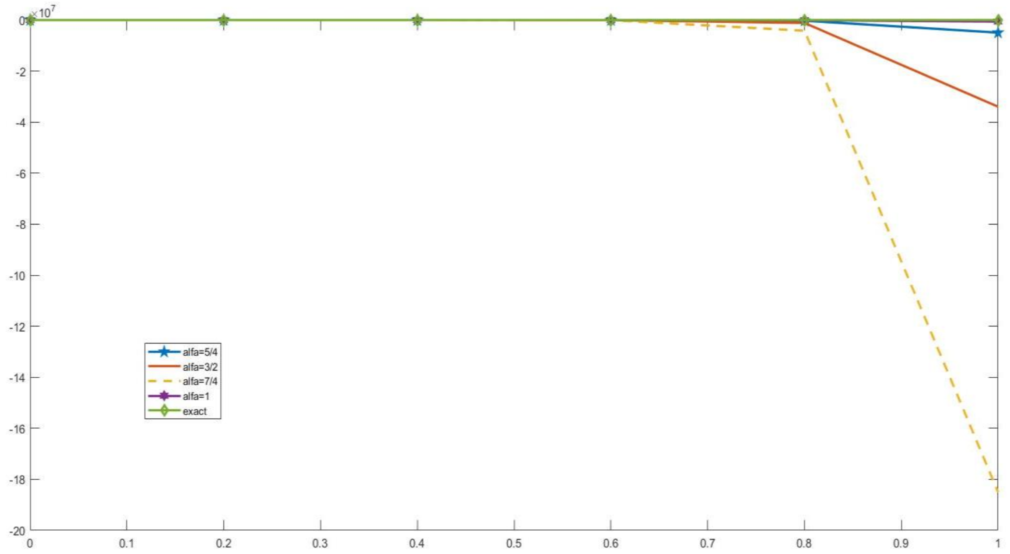

Figure 1- Comparison between the exact and numerical solution with $\alpha = 1, \frac{5}{4}, \frac{3}{2}, and\ \frac{7}{4}$.

The previous Figure shows that the numerical solution converges to the exact solution when α approaching one where it is the exact solution.

Test Example 2: Consider the linear fractional differential equation with a delay of the form:

$$D^{\alpha}y(t) = t^2 y(t-2) + e^t - t^2 e^{t-2} - te^t \ . \tag{15}$$

with

$$y(t) = 1 + t \ \ for\ t \in [-2,0] \ \ , \tag{16}$$

where the exact solution $y(t) = e^t \ \ for\ \alpha = 1$. Figure 2 shows a comparison between the results of using the proposed method with different values of $\alpha = 1, \frac{5}{4}, \frac{3}{2}, and\ \frac{7}{4}$ with the exact solution to the problem.

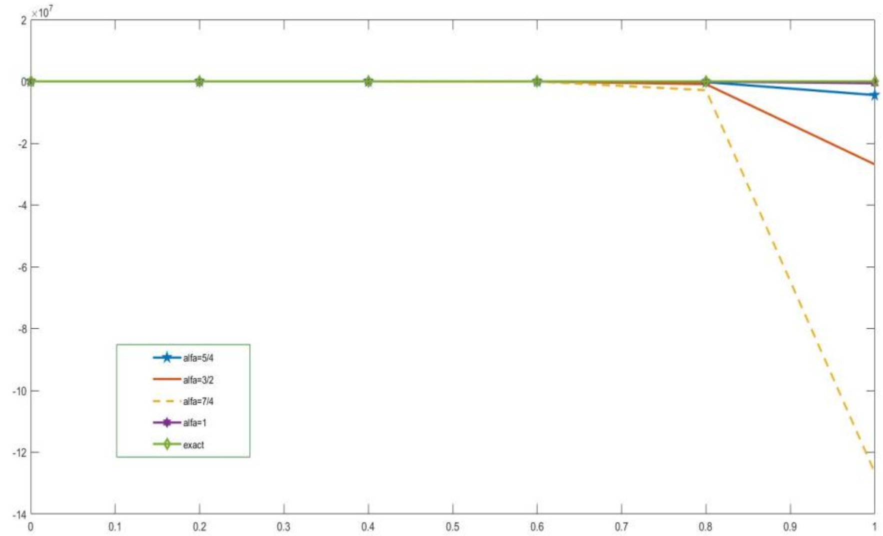

Figure 2- Comparison between the exact and numerical solution with $\alpha = 1, \frac{5}{4}, \frac{3}{2}, and\ \frac{7}{4}$.

The previous Figure shows that the numerical solution converges to the exact solution when α approaching one where it is the exact solution.

Test Example 3: Consider the linear fractional differential equation with a delay of the form:

$$D^{\alpha}y(t) = y(t) + (t-2)y(t-3) - \cos t - \sin(t) - (t-2)\cos(t-3) \ , \quad (17)$$

with

$$y(t) = 1 + \frac{t^2}{2} \ for\ t \in [-3,0] \ , \quad (18)$$

where the exact solution $y(t) = \cos t \quad for\ \alpha = 1$. Figure 3 shows a comparison between the results of using the proposed method with different values of $\alpha = 1, \frac{5}{4}, \frac{3}{2}, and\ \frac{7}{4}$ with the exact solution to the problem.

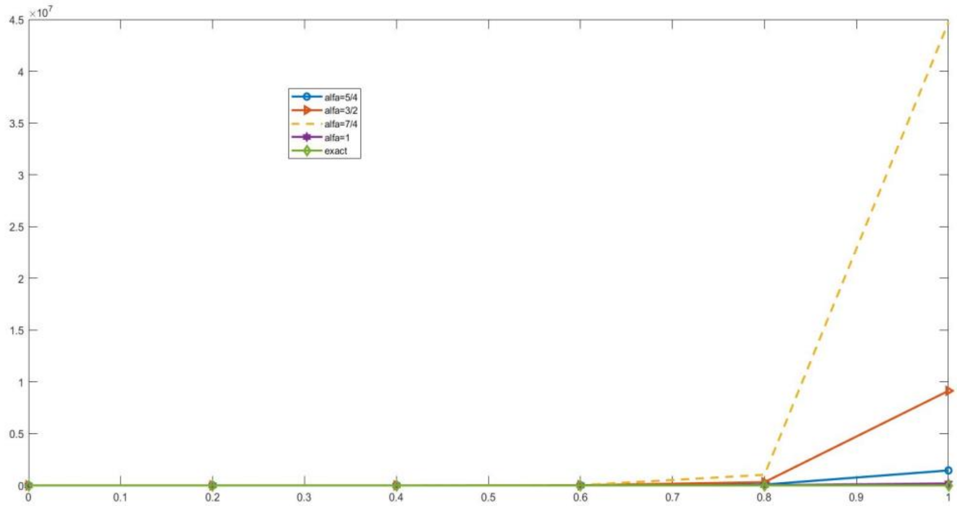

Figure 3- Comparison between the exact and numerical solution with $\alpha = 1, \frac{5}{4}, \frac{3}{2}, and\ \frac{7}{4}$.

The previous Figure shows that the numerical solution converges to the exact solution when α approaching one where it is the exact solution.

6. Conclusions

In this paper, a new approach is based on the backward finite difference method to approximate linear fractional differential equations with a delay of order α, $1 < \alpha < 2$ in the Caputo sense have successfully applied. The efficiency and accuracy of the achieved numerical solutions compared with exact solutions are considered. A suitable method for the fractional differential equation with a delay of order α; $1 < \alpha < 2$ in the Caputo sense is solved. Finally, for future work, we suggest using another form based on the forward difference method or central finite difference method. We also recommended using other approaches for finding an integral part in the definition of Caputo like Simpson's rule.

REFERENCES

[1] Khader, M.M., Hendy, A.S. **2012**. The approximate and exact solutions of the fractional-order delay differential equations using Legendre seudospectral method. International Journal of Pure and Applied Mathematics.;74(3):287–97.

[2] AL-Saltani, B.K. **2007.** Solution of delay fractional differential equations by using linear multistep method. Journal of Kerbala University. 5(4):217–22.

[3] Kaslik, E., Sivasundaram, S. **2012.** Analytical and numerical methods for the stability analysis of linear fractional delay differential equations. 236, Journal of Computational and Applied Mathematics. p. 4027–41.

[4] Moghaddam, B.P, Mostaghim, Z.S. **2013.** A numerical method based on finite difference for solving fractional delay differential equations. Journal of Taibah University for Science. 7(3):120–7.

[5] Saeed, U., Rehman, M.U. **2014.** Hermite wavelet method for fractional delay differential equations. Journal of Difference Equations. 2014.

[6] Iqbal M.A, Saeed U., Mohyud-Din S.T. **2015.** Modified Laguerre wavelets method for delay differential equations of fractional-order. Egyptian Journal of basic and applied sciences.2(1):50–4.

[7] Daftardar-Gejji V., Sukale Y., Bhalekar S. **2015.** Solving fractional delay differential equations: A new approach. 18, Fractional Calculus and Applied Analysis: 400–18.

[8] Saeed U., Rehman M.U., Iqbal M.A. **2015**. Modified Chebyshev wavelet methods for fractional delay-type equations. 264, Applied Mathematics and Computation: 431–42.

[9] Xu M-Q., Lin Y-Z. **2016**. Simplified reproducing kernel method for fractional differential equations with delay. Applied Mathematics Letters.52:156–61.

[10] Parsa Moghaddam B., Yaghoobi S., Tenreiro Machado J.A. **2016.** An extended predictor--corrector algorithm for variable-order fractional delay differential equations. Journal of Computational and Nonlinear Dynamics. 11(6).

[11] Mohammed O.H., Wadi Q. **2016.** A modified method for solving delay differential equations of fractional order. IOSR journal of mathematics (IOSR-JM). 12(3):15–21.

[12] Muthukumar P., Ganesh Priya B. **2017.** Numerical solution of fractional delay differential equation by shifted Jacobi polynomials. International Journal of Computer Mathematics. 94(3):471–92.

[13] Moghaddam B.P., Mostaghim Z.S. **2017.** Modified finite difference method for solving fractional delay differential equations. 35, Boletim da Sociedade Paranaense de Matematica: 49–58.

[14] Eltayeb H., Abdeldaim E. **2017.** Sumudu Decomposition Method for Solving Fractiona Delay Differential Equations. Research in Applied Mathematics. 1(1):1–13.

[15] Li M., Wang J. **2017.** Finite time stability of fractional delay differential equations. Applied Mathematics Letters. 64:170–6.

[16] Valizadeh M., Mahmoudi Y., Dastmalchi Saei F. **2019.** Application of Natural Transform Method to Fractional Pantograph Delay Differential Equations. Journal of Mathematics.2019.

[17] Raslan K.R., Ali K.K., Mohamed E.M. **2019.** Spectral Tau method for solving general fractional order differential equations with linear functional argument. Journal of the Egyptian Mathematical Society.27(1):33.

[18] Malmir I. **2019.** A new fractional integration operational matrix of Chebyshev wavelets in fractional delay systems. Fractal and Fractional. 2019;3(3):46.

[19] Yang P., Wang J., Zhou Y. **2019.** Representation of solution for a linear fractional delay differential equation of Hadamard type. Advances in Difference Equations. 2019(1):300.

[20] Subramaniam AGV. **1997.** Numerical Methods for Engineers and Scientists An Introduction with Applications using MATLAB® [Internet]. 69–71, Journal of Wind Engineering and Industrial Aerodynamics. 131–132 p. Available from http://linkinghub.elsevier.com/retrieve/pii/S0167610597002274

[21] Young T., Mohlenkamp M.J. **2011.** Introduction to Numerical Methods and Matlab Programming for Engineers.