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Gompertz Fréchet stress-strength Reliability Estimation

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Abstract

In this paper, the reliability of the stress-strength model is derived for probability P(Y<X) of a component having its strength X exposed to one independent stress Y, when X and Y are following Gompertz Fréchet distribution with unknown shape parameters θ,λ and known parameters α,β,γ. Different methods were used to estimate reliability R and Gompertz Fréchet distribution parameters, which are maximum likelihood, least square, weighted least square, regression, and ranked set sampling. Also, a comparison of these estimators was made by a simulation study based on mean square error (MSE) criteria. The comparison confirms that the performance of the maximum likelihood estimator is better than that of the other estimators.

Keywords: Reliability, Stress- Strength, Gompertz Fréchet Distribution, Maximum Likelihood Estimator, Least square Estimator, Weighted Least square Estimator, Regression Estimator, Ranked set sampling estimator.

تقدير معولية الإجهاد– المتانة لجومبرتز فريشت

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قسم الرياضيات، كلية التربية، الجامعة المستنصرية، بغداد، العراق

الخلاصة

في هذا البحث ، تم اشتقاق نموذج لمعولية الإجهاد–المتانة للاحتمال P (Y <X) لمكونة تتعرض متانتها X لإجهاد واحد مستقل Y، عندما يكون X و Y تتبع توزيع جومبرتز فريشت مع معلمات شكل غير معروفة λ,θ ومعلمات معروفة α,β,γ. استخدمت طرق مختلفة لتقدير المعوليه R ومعلمات توزيع جومبرتز فريشت والتي هي طريقة الامكان الاعظم ، طريقة المربعات الصغرى ، طريقة المربعات الصغرى الموزونة ، طريقة الانحدار وطريقة العينة المحددة التصنيف ، والمقارنة بين هذه التقديرات من خلال دراسة المحاكاة بناءً على معيار متوسط الخطأ التربيعي (MSE). تؤكد المقارنة أن أداء مقدر الامكان الاعظم يعمل بشكل أفضل من غيره.

1-Introduction

The reliability of stress-strength was used by Church and Harris in 1970. It is defined as $R = P(Y < X)$, where X represents the strength random variable and Y represents the stress random variable [1]. The word stress-strength refers to a part of a system that has a random strength component X subjected to a random stress Y to determine reliability [2]. If the stress applied exceeds the strength, the component fails, whereas the component works whenever Y is less than X [3].

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Some attempts have been made to define modern types of distributions, extend renewed families, and at the same time, provide higher resilience in forming data in practice. Many families employing more than one parameters to generate modern distributions have been suggested in the statistical literature. Some renowned generators include the exponentiated half-logistic family created by Alizadeh *et al.* (2014), Lomax-G by Ortega *et al*. (2014), Kumaraswamy odd logistic-G by Emadi *et al*. (2015), Kumareswamy Marshall-Olkin by Tahis *et al*.(2015), odd generalized exponential –G by Tahir *et al*. (2015), type I half –logistic family by Cordeior *et al*. (2016), and Gompertz-Frechet by Alizadeh *et al*. (2016). Some special models of the Go-G family include Gompertz-Weibul, Gompertz-Gamma, Gompertz-beta, Gompertz-log logistic, and Gompertz-Fréchet. The cumulative distribution function CDF of the Gompertz-G family is defined as [4]:

$$F(x) = \int_0^{-\log[1-G(x;\epsilon)}\theta e^{\gamma t} e^{-\frac{\theta}{\gamma}(e^{\gamma t}-1)}dt = 1 - e^{\frac{\theta}{\gamma}\{1-[1-G(x;\epsilon)]^{-\gamma}\}} \quad \dots (1)$$

where $G(x,\epsilon)$ is the baseline CDF, which depends on a parameter vector ϵ, and $\gamma > 0$ and $\theta > 0$ are two shape parameters. Also, the probability density function pdf of the Gompertz-G family is defined by:

$$f(x;\theta,\gamma,\epsilon) = \theta g(x;\epsilon)\,[1 - G(x;\epsilon)]^{-\gamma-1}\, e^{\frac{\theta}{\gamma}\{1-[1-G(x;\epsilon)]^{-\gamma}\}} \quad \dots (2)$$

The CDF and pdf of Fréchet distribution are [5]:

$$G(x,\alpha,\beta)=\exp\left[-(\tfrac{\alpha}{x})^{\beta}\right] \quad \dots (3)$$

$$\text{and } g(x,\alpha,\beta) = \beta\,\alpha^{\beta}\,x^{-\beta-1}\exp\left[-(\tfrac{\alpha}{x})^{\beta}\right] \quad \dots (4)$$

respectively , where $\alpha > 0$ is the scale parameter and $\beta > 0$ is the shape parameter.
The CDF of the Gompertz Fréchet (GF) distribution is obtained by substituting equation (3) in equation (1), given by [5]:

$$F(x,\theta,\alpha,\beta,\gamma) = 1 - \exp\left[\frac{\theta}{\gamma}\left(1 - \left\{1 - \exp\left[-(\tfrac{\alpha}{x})^{\beta}\right]\right\}^{-\gamma}\right)\right] \quad \dots (5)$$

And pdf can be derived from equation (5) as:

$$f(x,\theta,\alpha,\beta,\gamma) = \theta\beta\alpha^{\beta}x^{-\beta-1}\exp\left[-(\frac{\alpha}{x})^{\beta}\right]\left\{1 - \exp\left[-(\frac{\alpha}{x})^{\beta}\right]\right\}^{-\gamma-1} * \exp\left[\frac{\theta}{\gamma}\left(1 - \left\{1 - \exp\left[-(\tfrac{\alpha}{x})^{\beta}\right]\right\}^{-\gamma}\right)\right] \quad \dots (6)$$

where $\theta > 0$, $\beta > 0$, and $\gamma > 0$ are the shape parameters and $\alpha > 0$ is the scale parameter.
The main aim of this paper is to obtain a mathematical formula of Reliability R of probability P(Y<X), based on Gompertz Fréchet distribution, as presented in section 2. In order to find the estimators of the shape parameters (θ,λ) for the two random variables, five different estimation methods (maximum likelihood, least square method, weighted least square method, regression method, and ranked set sampling method) are used and then the reliability parameters is estimated in section 3. A simulation study is conducted in section 4 to compare the performance of the five different estimators of the reliability, based on six experiments of shape parameter values and at different sample sizes of (15) for small, (30) for medium, and (90) for large sample sizes. A comparison is made by using the MSE approach and the conclusions are discussed in section 5.

2- The Reliability expression

Let X~GF$(\theta,\alpha,\beta,\gamma)$ be a strength random variable and Y~ GF$(\lambda,\alpha,\beta,\gamma)$ be a stress random variable .

Let $u_x = 1 - \exp\left[-(\frac{\alpha}{x})^{\beta}\right]$

and we can write $\exp\left[-(\frac{\alpha}{x})^{\beta}\right] = 1 - u_x$.

Then F(x) and f(x) can be written as:

$\mathrm{F}(x,\theta,\alpha,\beta,\gamma) = 1 - \exp\left[\frac{\theta}{\gamma}\left(1 - u_x^{-\gamma}\right)\right]$

$\mathrm{f}(x,\theta,\alpha,\beta,\gamma) = \theta\beta\alpha^{\beta}x^{-\beta-1}\left(1-u_x\right)u_x^{-\gamma-1}\exp\left[\frac{\theta}{\gamma}\left(1-u_x^{-\gamma}\right)\right]$ and

$\mathrm{G}(y,\lambda,\alpha,\beta,\gamma) = 1 - \exp\left[\frac{\lambda}{\gamma}\left(1 - u_y^{-\gamma}\right)\right]$

$\mathrm{g}(y,\lambda,\alpha,\beta,\gamma) = \lambda\beta\alpha^{\beta}y^{-\beta-1}\left(1-u_y\right)u_y^{-\gamma-1}\exp\left[\frac{\lambda}{\gamma}\left(1-u_y^{-\gamma}\right)\right].$

The reliability is given by [6]:

$\mathrm{R} = \mathrm{P}(\mathrm{Y} < \mathrm{X})$

$\mathrm{R} = \int_{x=0}^{\infty}\mathrm{G}_y(x)\mathrm{f}(x)dx$

$$\mathrm{R} = \int_{x=0}^{\infty}\left(1 - \exp\left[\frac{\lambda}{\gamma}\left(1-u_x^{-\gamma}\right)\right]\right)\theta\beta\alpha^{\beta}x^{-\beta-1}\left[1-u_x\right]u_x^{-\gamma-1} * \exp\left[\frac{\theta}{\gamma}\left(1-u_x^{-\gamma}\right)\right]dx$$

$\mathrm{R} = 1 - \theta\int_{x=0}^{\infty}\beta\alpha^{\beta}x^{-\beta-1}\left[1-u_x\right]u_x^{-\gamma-1}\exp\left[\frac{\lambda+\theta}{\gamma}\left(1-u_x^{-\gamma}\right)\right]dx$

Since $\int_{x=0}^{\infty}\mathrm{f}(x)dx = 1$,

then $\int_{x=0}^{\infty}\beta\alpha^{\beta}x^{-\beta-1}\left[1-u_x\right]u_x^{-\gamma-1}\exp\left[\frac{\lambda+\theta}{\gamma}\left(1-u_x^{-\gamma}\right)\right] = \frac{1}{\lambda+\theta}$ … (7)

Therefore, $\mathrm{R} = 1 - \frac{\theta}{\lambda+\theta}$

$\mathrm{R} = \frac{\lambda}{\lambda+\theta}$ … (8)

Figure (1) shows the change in the reliability curve with the impact of the strength parameter values θ in three different cases of the parameter values of λ, where the reliability in this case is decreasing.

Figure (2) shows the change in the reliability curve with the impact of the stress parameter values λ in three different cases of the parameter values of θ, where the reliability in this case is increasing, but it reaches a certain point and begins to decrease.

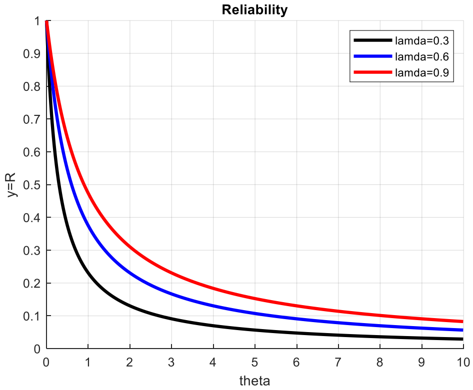

Figure 1-The Reliability curve aginst θ

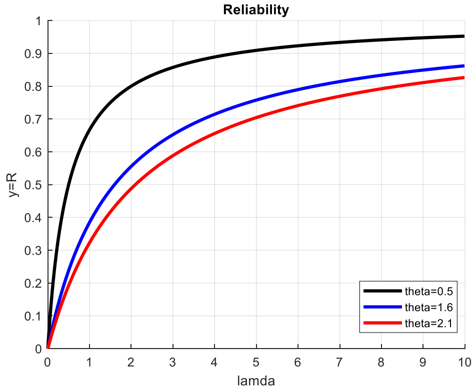

Figure2- The Reliability curve aginst λ

3- Estimation method

In this subsection, the shape parameters θ and λ of the GF distribution and the reliability are estimated using five methods of estimation: maximum likelihood, least square, weighted least square, regression, and rank set sampling.

3-1- Maximum likelihood estimation (MLE) method

The maximum likelihood estimation (MLE) method is one of the most important and

common parameter estimation methods. In 1922, R. A. Fisher introduced the method and then first presented the numerical procedure in 1912 [7]. Let $x_1, x_2,\ldots, x_n$ be a strength random sample of size n from GF($\theta,\alpha,\beta,\gamma$) where θ is an unknown parameter and α,β,γ are known.
Then, the likelihood function is given by [5]:

$$L = \prod_{i=1}^{n}[f(x_1, x_2, \ldots, x_n; \theta,\alpha,\beta,\gamma)]$$

$$L = \prod_{i=1}^{n}\left[\begin{array}{c}\theta\beta\alpha^{\beta}x_i^{-\beta-1}\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]\left\{1-\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]\right\}^{-\gamma-1} \\ *\exp\left[\frac{\theta}{\gamma}\left(1-\left\{1-\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]\right\}^{-\gamma}\right)\right]\end{array}\right]$$

$$L=\theta^n\beta^n\alpha^{n\beta}\prod_{i=1}^{n}x_i^{-\beta-1}\exp\left(\sum_{i=1}^{n}\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]\right)\prod_{i=1}^{n}\left\{1-\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]\right\}^{-\gamma-1}\exp\left(\sum_{i=1}^{n}\left[\frac{\theta}{\gamma}\left(1-\left\{1-\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]\right\}^{-\gamma}\right)\right]\right)$$

$$\ln L = n\ln\theta + n\ln\beta + n\beta\ln\alpha - (\beta+1)\sum_{i=1}^{n}\ln x_i + \sum_{i=1}^{n}\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right] - (\gamma+1)\sum_{i=1}^{n}\ln\left\{1-\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]\right\} + \frac{\theta}{\gamma}\sum_{i=1}^{n}\left(1-\left\{1-\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]\right\}^{-\gamma}\right)$$

$$\frac{\partial \ln L}{\partial\theta} = \frac{n}{\hat{\theta}} + \frac{1}{\gamma}\sum_{i=1}^{n}\left(1-\left\{1-\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]\right\}^{-\gamma}\right) = 0$$

$$\hat{\theta}_{MLE} = \frac{-n\gamma}{\sum_{i=1}^{n}\left(1-\left\{1-\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]\right\}^{-\gamma}\right)} \quad \ldots (9)$$

In the same way, let $y_1, y_2 \ldots\ldots y_n$ be a stress random sample of size m from GF($\lambda,\alpha,\beta,\gamma$) where λ is an unknown parameter and α,β,γ are known.
Then, $\hat{\lambda}_{MLE}$ is given by:

$$\hat{\lambda}_{MLE} = \frac{-m\gamma}{\sum_{j=1}^{m}\left(1-\left\{1-\exp\left[-\left(\frac{\alpha}{y_j}\right)^{\beta}\right]\right\}^{-\gamma}\right)} \quad \ldots (10)$$

Then, by substituting equation (9) and (10) in equation (8), we get:

$$\hat{R}_{MLE} = \frac{\hat{\lambda}_{MLE}}{\hat{\lambda}_{MLE}+\hat{\theta}_{MLE}} \quad \ldots (11)$$

3-2- Least Square Estimation Method (LS)

As early as (1794), the German mathematician Carl Friedrich Gauss investigated the least square method, which he published in 1809. This method of estimation is very common for model fitting, especially in linear regression and non-linear regression. The (LS) method is generated by minimizing the number of squares between the value and the estimated value of the error [8].The combination of the parametric and the non-parametric distribution functions is the least square form. The minimization is achieved by using the following equation [6]:

$$S = \sum_{i=1}^{n}(F(x_i) - E(F(x_i)))^2 \quad \ldots (12)$$

Suppose that $x_1, x_2, \ldots\ldots x_n$ is a random sample that has GF($\theta,\alpha,\beta,\gamma$) distribution with sample size n.
The procedure attempts to minimize the following function with respect to $\theta,\alpha,\beta,\gamma$:

$$S(\theta,\alpha,\beta,\gamma) = \sum_{i=1}^{n}\left(\left(1-\exp\left[\frac{\theta}{\gamma}\left(1-\left\{1-\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]\right\}^{-\gamma}\right)\right]\right)-P_i\right)^2 \quad \ldots (13)$$

where $E(F(x_i)) = P_i$ and P_i is the plotting position, where $P_i = \frac{i}{n+1}$, i=1,2,…,n.

To obtain the formula of F(x_i) by equation (5):

$$F(x_i) = 1 - \exp\left[\frac{\theta}{\gamma}\left(1 - \left\{1 - \exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]\right\}^{-\gamma}\right)\right]$$

$$1 - F(x_i) = \exp\left[\frac{\theta}{\gamma}\left(1 - \left\{1 - \exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]\right\}^{-\gamma}\right)\right]$$

$$\text{Ln}(1 - F(x_i)) = \left[\frac{\theta}{\gamma}\left(1 - \left\{1 - \exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]\right\}^{-\gamma}\right)\right] \quad \ldots (14)$$

where $x_{(i)}$ is the i:th order statistics of the random sample of the size n from the GF, hence for the GF. To obtain the LS estimation $\hat{\theta}_{LS}$ of the parameter θ, the following equation is applied (14):

$$S(\theta,\alpha,\beta,\gamma) = \sum_{i=1}^{n}\left(\left[\frac{\theta}{\gamma}\left(1 - \left\{1 - \exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]\right\}^{-\gamma}\right)\right] - q_i\right)^2 \quad \ldots (15)$$

where $q_i = \text{Ln}\big(1 - F(x_i)\big) = \text{Ln}(1 - P_i)$.

By taking the derivative to equation (15) with respect to the parameter θ and equating the result to zero, we obtain:

$$\frac{dS(\theta,\alpha,\beta,\gamma)}{d\theta} = \sum_{i=1}^{n} 2\left(\left[\frac{\theta}{\gamma}\left(1 - \left\{1 - \exp\left[-\left(\frac{\alpha}{x_{(i)}}\right)^{\beta}\right]\right\}^{-\gamma}\right)\right] - q_i\right) * \frac{1}{\gamma}\left(1 - \left\{1 - \exp\left[-\left(\frac{\alpha}{x_{(i)}}\right)^{\beta}\right]\right\}^{-\gamma}\right)$$

$$\frac{\hat{\theta}}{\gamma^2}\sum_{i=1}^{n}\left(1 - \left\{1 - \exp\left[-\left(\frac{\alpha}{x_{(i)}}\right)^{\beta}\right]\right\}^{-\gamma}\right)^2 - q_i\frac{1}{\gamma}\sum_{i=1}^{n}\left(1 - \left\{1 - \exp\left[-\left(\frac{\alpha}{x_{(i)}}\right)^{\beta}\right]\right\}^{-\gamma}\right) = 0$$

$$\hat{\theta}_{LS} = \frac{\gamma\sum_{i=1}^{n} q_i\left(1 - \left\{1 - \exp\left[-\left(\frac{\alpha}{x_{(i)}}\right)^{\beta}\right]\right\}^{-\gamma}\right)}{\sum_{i=1}^{n}\left(1 - \left\{1 - \exp\left[-\left(\frac{\alpha}{x_{(i)}}\right)^{\beta}\right]\right\}^{-\gamma}\right)^2} \quad \ldots (16)$$

In the same way above, we can estimate λ as bellow :

$$\hat{\lambda}_{LS} = \frac{\gamma\sum_{j=1}^{m} q_j\left(1 - \left\{1 - \exp\left[-\left(\frac{\alpha}{y_{(j)}}\right)^{\beta}\right]\right\}^{-\gamma}\right)}{\sum_{j=1}^{m}\left(1 - \left\{1 - \exp\left[-\left(\frac{\alpha}{y_{(j)}}\right)^{\beta}\right]\right\}^{-\gamma}\right)^2} \quad \ldots (17)$$

Then, by substituting (16) and (17) in (8), we get :

$$\hat{R}_{LS} = \frac{\hat{\lambda}_{LS}}{\hat{\lambda}_{LS} + \hat{\theta}_{LS}} \quad \ldots (18)$$

3-3- Weighted Least Squares (WLS) Estimation Method

The weighted least squares method extends the least squares method to the case where there are different variations in the sample data. In other words, some of the samples have more errors or less effects than others. This approach represents the action of the model's random errors, and can be used for parameter functions that are either linear or nonlinear. It works by integrating the fitting criteria with additional non-negative weights or constants associated with all data points. The size of the weight indicates the accuracy of the details found in the relevant observations [9].The weighted last-square method can be used to minimize the following equation [6]:

$$\mathrm{Q} = \sum_{i=1}^{n} W_i\left(F\left(x_{(i)}\right) - E\left(F(x_{(i)})\right)\right)^2 \qquad \ldots (19)$$

where $W_i = \frac{1}{var[F(x_{(i)}]} = \frac{(n+1)^2(n+2)}{i(n-i+1)}$, i=1,2,…,n.

Let $x_1, x_2, \ldots, x_n$ be a strength random sample with size n from GF$(\theta, \alpha, \beta, \gamma)$ distribution. The procedure attempts to minimize the following function with respect to $\theta, \alpha, \beta,$ and γ:

$$\mathrm{Q}(\theta, \alpha, \beta, \gamma) = \sum_{i=1}^{n} W_i \left(\left(1 - \exp\left[\frac{\theta}{\gamma} \left(1 - \left\{ 1 - \exp\left[-\left(\frac{\alpha}{x_i} \right)^{\beta} \right] \right\}^{-\gamma} \right) \right] \right) - P_i \right)^2 \qquad \ldots (20)$$

As steps in equations (13) and (15), we get:

$$\mathrm{Q}(\theta, \alpha, \beta, \gamma) = \sum_{i=1}^{n} W_i \left(\left[\frac{\theta}{\gamma} \left(1 - \left\{ 1 - \exp\left[-\left(\frac{\alpha}{x_{(i)}} \right)^{\beta} \right] \right\}^{-\gamma} \right) \right] - q_i \right)^2 \qquad \ldots (21)$$

By taking the derivative equation (21) with respect to the parameter θ and equating the result to zero, we get $\hat{\theta}_{WLS}$ as :

$$\hat{\theta}_{WLS} = \frac{\gamma \sum_{i=1}^{n} W_i q_i \left(1 - \left\{ 1 - \exp\left[-\left(\frac{\alpha}{x_{(i)}} \right)^{\beta} \right] \right\}^{-\gamma} \right)}{\sum_{i=1}^{n} W_i \left(1 - \left\{ 1 - \exp\left[-\left(\frac{\alpha}{x_{(i)}} \right)^{\beta} \right] \right\}^{-\gamma} \right)^2} \qquad \ldots (22)$$

In the same way, we can estimate λ as bellow:

$$\hat{\lambda}_{WLS} = \frac{\gamma \sum_{j=1}^{m} W_i q_j \left(1 - \left\{ 1 - \exp\left[-\left(\frac{\alpha}{y_{(j)}} \right)^{\beta} \right] \right\}^{-\gamma} \right)}{\sum_{j=1}^{m} W_i \left(1 - \left\{ 1 - \exp\left[-\left(\frac{\alpha}{y_{(j)}} \right)^{\beta} \right] \right\}^{-\gamma} \right)^2} \qquad \ldots (23)$$

Then by substituting (22) and (23) in (8), we get :

$$\hat{R}_{WLS} = \frac{\hat{\lambda}_{WLS}}{\hat{\lambda}_{WLS} + \hat{\theta}_{WLS}} \qquad \ldots (24)$$

3-4- Regression estimation method (Rg)

Regression is one of the essential procedures that uses additional knowledge to create good efficiency estimators. Regression is the basic method of analyzing functional relations between variables conceptually.The relationship is expressed in the form of an equation or model connecting the response variable (Y) and one (X) or more explanatory variables. The simple true relations can be approximated by the standard regression equation [10]:

$$Z_i = \mathrm{a} + \mathrm{b}\, u_i + e_i \qquad \ldots (25)$$

where Z_i is dependent variable, u_i is independent variable, e_i is the independent error random variable, and a, b are called regression coefficients where a is the intercept and b is the slope.

Let $x_1, x_2, \ldots, x_n$ be a random strength sample of size (*n*) from GF$(\theta, \alpha, \beta, \gamma)$, then the GF estimators of the unknown parameter θ can be obtained by taking the natural logarithm to equation (5), as follows:

$$\ln(1 - \mathrm{F}(x_i)) = \frac{\theta}{\gamma} \left(1 - \left\{ 1 - \exp\left[-\left(\frac{\alpha}{x_i} \right)^{\beta} \right] \right\}^{-\gamma} \right) \qquad \ldots (26)$$

By putting the plotting position p_i instead of $\mathrm{F}(x_i)$ in equation (26), we get:

$$\ln(1 - p_i) = \frac{\theta}{\gamma} \left(1 - \left\{ 1 - \exp\left[-\left(\frac{\alpha}{x_i} \right)^{\beta} \right] \right\}^{-\gamma} \right) \qquad \ldots (27)$$

By comparison between equation (27) and equation (25), we get:

$$Z_i = \ln(1 - p_i)\ , \mathrm{a}=0\ , \mathrm{b}=\theta\ , u_i = \frac{1}{\gamma} \left(1 - \left\{ 1 - \exp\left[-\left(\frac{\alpha}{x_i} \right)^{\beta} \right] \right\}^{-\gamma} \right) \qquad \ldots (28)$$

where b can be estimated by the minimizing summation of the squared error with respect to b. Then we get [11]:

$$\hat{b} = \frac{n\sum_{i=1}^{n} Z_i u_i - \sum_{i=1}^{n} Z_i \sum_{i=1}^{n} u_i}{n\sum_{i=1}^{n}(u_i)^2 - \left(\sum_{i=1}^{n} u_i\right)^2} \quad \ldots (29)$$

By substituting (28) in (29), the GF estimator for the unknown parameter θ, say $\hat{\theta}_{Rg}$, is:

$$\hat{\theta}_{Rg} = \frac{\frac{n}{\gamma}\sum_{i=1}^{n} \text{Ln}(1-p_i)\left(1-\left\{1-\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]\right\}^{-\gamma}\right) - \frac{1}{\gamma}\sum_{i=1}^{n} \text{Ln}(1-p_i)\sum_{i=1}^{n}\left(1-\left\{1-\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]\right\}^{-\gamma}\right)}{\frac{n}{\gamma^2}\sum_{i=1}^{n}\left(\left(1-\left\{1-\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]\right\}^{-\gamma}\right)\right)^2 - \left(\frac{1}{\gamma}\sum_{i=1}^{n}\left(1-\left\{1-\exp\left[-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]\right\}^{-\gamma}\right)\right)^2} \quad \ldots (30)$$

In the same way, we can estimate λ as bellow:

$$\hat{\lambda}_{Rg} = \frac{\frac{m}{\gamma}\sum_{j=1}^{m} \text{Ln}(1-p_j)\left(1-\left\{1-\exp\left[-\left(\frac{\alpha}{y_j}\right)^{\beta}\right]\right\}^{-\gamma}\right) - \frac{1}{\gamma}\sum_{j=1}^{m} \text{Ln}(1-p_j)\sum_{j=1}^{m}\left(1-\left\{1-\exp\left[-\left(\frac{\alpha}{y_j}\right)^{\beta}\right]\right\}^{-\gamma}\right)}{\frac{m}{\gamma^2}\sum_{j=1}^{m}\left(\left(1-\left\{1-\exp\left[-\left(\frac{\alpha}{y_j}\right)^{\beta}\right]\right\}^{-\gamma}\right)\right)^2 - \left(\frac{1}{\gamma}\sum_{j=1}^{m}\left(1-\left\{1-\exp\left[-\left(\frac{\alpha}{y_j}\right)^{\beta}\right]\right\}^{-\gamma}\right)\right)^2} \quad \ldots (31)$$

Then, by substituting (30) and (31) in (8), we get:

$$\hat{R}_{Rg} = \frac{\hat{\lambda}_{Rg}}{\hat{\lambda}_{Rg} + \hat{\theta}_{Rg}} \quad \ldots (32)$$

3-5- Ranked set sample method (RSS)

Ranked set sampling was introduced and applied to the issue of McIntyre's estimate of mean pasture yields (1952). This function was used to increase the efficiency of the sample mean as a population mean estimator in circumstances where it was difficult or costly to calculate the characteristic of interest. But when ranked, it becomes cheaper. Recent interest in classified set sampling for environmental applications has been expressed. This can entail costly testing to quantify the variable of interest for sample units. However, it can be achieved more cheaply by rating small sets of samples with respect to the characteristic [12]. Let $(x_1, x_2, \ldots, x_n)$ be a random sample from GF. Assume that $(x_{(1)}, x_{(2)}, \ldots, x_{(n)})$ are the order statistics obtained by ordering the sample in an increasing order.

The pdf of $x_{(i)}$ is [13]:

$$\text{f}(x_{(i)}) = \frac{n!}{(i-1)!(n-i)!}\left[F(x_{(i)})\right]^{i-1}\left[1-F(x_{(i)})\right]^{n-i}\text{f}(x_{(i)}) \quad \ldots (33)$$

Then, by applying equations (5) and (6) in (33), we get :

$$\text{f}(x_{(i)}) = \frac{n!}{(i-1)!(n-i)!}\left[1-\exp\left[\frac{\theta}{\gamma}\left(1-u_{x_{(i)}}^{-\gamma}\right)\right]\right]^{i-1}\left[\exp\left[\frac{\theta}{\gamma}\left(1-u_{x_{(i)}}^{-\gamma}\right)\right]\right]^{n-i}$$
$$\theta\beta\alpha^{\beta}x_{(i)}^{-B-1}\left(1-u_{x_{(i)}}\right)u_{x_{(i)}}^{-\gamma-1}\exp\left[\frac{\theta}{\gamma}\left(1-u_{x_{(i)}}^{-\gamma}\right)\right]$$

where $u_x = 1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]$

Suppose that Q= $\frac{n!}{(i-1)!(n-i)!}$, then we get:

$$\text{f}(x_{(i)}) = \text{Q}\,\theta\beta\alpha^{\beta}x_{(i)}^{-B-1}\left[1-\exp\left[\frac{\theta}{\gamma}\left(1-u_{x_{(i)}}^{-\gamma}\right)\right]\right]^{i-1}\left(1-u_{x_{(i)}}\right)u_{x_{(i)}}^{-\gamma-1}$$
$$*\left[\exp\left[\frac{\theta}{\gamma}\left(1-u_{x_{(i)}}^{-\gamma}\right)\right]\right]^{n-i+1}$$

The likelihood function of the order sample $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ is :

$$\text{L}(x_{(1)}, x_{(2)}, \ldots, x_{(n)};\ \theta, \alpha, \beta, \gamma) = Q^n\theta^n\beta^n\alpha^{n\beta}\prod_{i=1}^{n} x_{(i)}^{-(B+1)}\prod_{i=1}^{n}\left(1-u_{x_{(i)}}\right)$$

$$* \prod_{i=1}^{n} \left[1 - \exp\left[\frac{\theta}{\gamma}\left(1 - u_{x_{(i)}}^{-\gamma}\right)\right]\right]^{i-1} * \prod_{i=1}^{n} u_{x_{(i)}}^{-\gamma-1}$$

$$* \prod_{i=1}^{n} \left[\exp\left[\frac{\theta}{\gamma}\left(1 - u_{x_{(i)}}^{-\gamma}\right)\right]\right]^{n-i+1} \quad \ldots \quad (34)$$

Then, the natural logarithm function for equation (34) can be written as :

$$\ln L = n \ln Q + n \ln \theta + n \ln \beta + n\beta \ln \alpha - (\beta + 1) \sum_{i=1}^{n} \ln x_{(i)} + \sum_{i=1}^{n} (i-1) \ln\left[1 - \exp\left[\frac{\theta}{\gamma}\left(1 - u_{x_{(i)}}^{-\gamma}\right)\right]\right] + \sum_{i=1}^{n} \ln\left(1 - u_{x_{(i)}}\right) - (\gamma + 1) \sum_{i=1}^{n} \ln u_{x_{(i)}} + \frac{\theta}{\gamma} \sum_{i=1}^{n} (n-i+1)\left(1 - u_{x_{(i)}}^{-\gamma}\right) \quad \ldots (35)$$

To minimize equation (35) , we must calculate the great endings by taking the partial derivative with respect to the unknown parameter θ. Then we get :

$$\frac{d \ln L}{d\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} (i-1) \frac{\frac{-1}{\gamma}\left(1 - u_{x_{(i)}}^{-\gamma}\right)\exp\left[\frac{\theta}{\gamma}\left(1 - u_{x_{(i)}}^{-\gamma}\right)\right]}{\left[1 - \exp\left[\frac{\theta}{\gamma}\left(1 - u_{x_{(i)}}^{-\gamma}\right)\right]\right]} + \frac{1}{\gamma} \sum_{i=1}^{n} (n-i+1)\left(1 - u_{x_{(i)}}^{-\gamma}\right)$$

By equating the partial derivative to zero, the right –hand side will be:

$$\frac{n}{\hat{\theta}} = \frac{1}{\gamma} \sum_{i=1}^{n} \frac{(i-1)\left(1 - u_{x_{(i)}}^{-\gamma}\right)\exp\left[\frac{\hat{\theta}}{\gamma}\left(1 - u_{x_{(i)}}^{-\gamma}\right)\right]}{\left[1 - \exp\left[\frac{\hat{\theta}}{\gamma}\left(1 - u_{x_{(i)}}^{-\gamma}\right)\right]\right]} - \frac{1}{\gamma} \sum_{i=1}^{n} (n-i+1)\left(1 - u_{x_{(i)}}^{-\gamma}\right)$$

$$\hat{\theta}_{RSS} = \frac{n\gamma}{\sum_{i=1}^{n} \frac{(i-1)\left(1 - u_{x_{(i)}}^{-\gamma}\right)\exp\left[\frac{\theta_0}{\gamma}\left(1 - u_{x_{(i)}}^{-\gamma}\right)\right]}{\left[1 - \exp\left[\frac{\theta_0}{\gamma}\left(1 - u_{x_{(i)}}^{-\gamma}\right)\right]\right]} - \sum_{i=1}^{n} (n-i+1)\left(1 - u_{x_{(i)}}^{-\gamma}\right)} \quad \ldots (36)$$

In the same way ,can be estimation λ as bellow:

$$\hat{\lambda}_{RSS} = \frac{m\gamma}{\sum_{j=1}^{m} \frac{(j-1)\left(1 - u_{y_{(j)}}^{-\gamma}\right)\exp\left[\frac{\lambda_0}{\gamma}\left(1 - u_{y_{(j)}}^{-\gamma}\right)\right]}{\left[1 - \exp\left[\frac{\lambda_0}{\gamma}\left(1 - u_{y_{(j)}}^{-\gamma}\right)\right]\right]} - \sum_{j=1}^{m} (m-j+1)\left(1 - u_{y_{(j)}}^{-\gamma}\right)} \quad \ldots (37)$$

Then by substituting (36) and (37) in (8), we get:

$$\hat{R}_{RSS} = \frac{\hat{\lambda}_{RSS}}{\hat{\lambda}_{RSS} + \hat{\theta}_{RSS}} \quad \ldots (38)$$

4- Simulation study

In this section, a simulation study is used to determine the best reliability estimate with unknown Gompertz Fréchet distribution parameters and to evaluate the selected five different estimates, where regression estimators were used as the initial value. The MSE for various sample sizes (15,30,90) and (α= 0.3, γ= 0.5, β= 0.7 and α= 0.2, γ= 0.2, β= 0.9) are assessed for six different experiments, each with the parameters α, γ and β.

A simulation study is conducted using MATLAB 2020 for the six different experiments to compare the performance of reliability estimators using the following steps:

Step1: Generating the random values of the random variables by the inverse function, according to the following formula:

$$x = \alpha\left[-\ln\left(1 - \left\{1 - \frac{\gamma}{\theta}\ln(1 - F(x))\right\}^{\frac{-1}{\gamma}}\right)\right]^{\frac{-1}{\beta}}$$

Step2: Finding the MLE for reliability, using equation (11), LS using equation (18), WLS using equation (24), Rg, using equation (32), and RSS using equation (38).

Step3: Finding the mean by the equation: Mean $= \frac{\sum_{i=1}^{N} \hat{R}_i}{N}$.

Step4: The estimation methods are compared using the mean square error criteria: MSE $= \frac{1}{N}\sum_{i=1}^{N}(\hat{R}_i - R)^2$, where N is 500 in each experiment.
The results are recorded from experiments 1 to 6 in the following tables. The comparison of the performance of these estimators based on MSE values was noted as follows:

1- In Table 1 and 2

- For experiments (1) and (2) in the case of sample size (15,15) and (30,30), experiments (3) and (6) in sample size (30,30) and experiment (5) with sample size of (15,15) the best value of MSE was that obtained by MLE followed by LS, RSS, WLS and Rg .
- For experiments (1),(4),(5) and (6) in the case of sample size (90,90), the best value of MSE was that obtained by MLE followed by LS, RSS, Rg and WLS.
- For experiments (1),(2),(3),(5) and (6) in the case of sample size (15,30) and (30,15) and also in experiments (4) in size (15,30),(30,15) and (30,90), the best value of MSE was that obtained by MLE followed by LS, WLS ,Rg and RSS.
- For experiments (1) and (6) in the case of sample size (30,90), the best value of MSE was that obtained by MLE followed by LS, WLS, Rg and RSS.
- For experiments (3),(4) and (6) in the case of sample size (15,15) and in experiment (3) in sample size (30,90) the best value of MSE was that obtained by MLE followed by LS, WLS, RSS and Rg .
- For experiments (2) and (5) in the case of sample size (30,90) the best value of MSE was that obtained by MLE followed by LS, Rg, WLS and RSS.

2- In Table 1

- For experiments (2) and (3) in the case of sample size (90,90) the best value of MSE was that obtained by MLE followed by LS, RSS, Rg and WLS.

3- In Table 2

- For experiments (4),(5) in the case of sample size (30,30), the best value of MSE was that obtained by MLE followed by RSS, LS, WLS, and Rg.

Table 1-Real reliability values and their estimators performance for experiments 1, 2, 3 .

Exp1: R=0.2857 when λ=1.2 ,θ=3 for α=0.3 ,γ=0.5 ,β=0.7							
(n,m)		**MLE**	**LS**	**WLS**	**Rg**	**RSS**	**Best**
(15,15)	Mean MSE	0.2905 0.0053	0.2916 0.0063	0.2932 0.0076	0.2960 0.0092	02944 0.0073	MLE
(30,30)	Mean MSE	0.2874 0.0030	0.2886 0.0037	0.2897 0.0051	0.2908 0.0054	0.2887 0.0039	MLE
(90,90)	Mean MSE	0.2867 0.0008	0.2864 0.0010	0.2862 0.0023	0.2866 0.0016	0.2874 0.0012	MLE
(15,30)	Mean MSE	0.2887 0.0042	0.2977 0.0054	0.2955 0.0066	0.2995 0.0078	0.1806 0.0139	MLE
(30,15)	Mean MSE	0.2945 0.0046	0.2854 0.0049	0.2889 0.0061	0.2860 0.0070	0.4443 0.0341	MLE
(30,90)	Mean MSE	0.2891 0.0019	0.2974 0.0025	0.2943 0.0037	0.2987 0.0037	0.1261 0.0262	MLE
Exp2: R=0.4000 when λ=2 ,θ=3 for α=0.3 ,γ=0.5 ,β=0.7							
(n,m)		**MLE**	**LS**	**WLS**	**Rg**	**RSS**	**Best**
(15,15)	Mean MSE	0.4044 0.0069	0.4035 0.0076	0.4036 0.0091	0.4037 0.0110	0.4080 0.0099	MLE
(30,30)	Mean MSE	0.4049 0.0037	0.4073 0.0048	0.4090 0.0071	0.4102 0.0074	0.4025 0.0050	MLE
(90,90)	Mean MSE	0.4024 0.0013	0.4017 0.0017	0.4012 0.0035	0.4012 0.0026	0.4032 0.0017	MLE
(15,30)	Mean MSE	0.4032 0.0054	0.4147 0.0069	0.4125 0.0088	0.4173 0.0100	0.2640 0.0234	MLE
(30,15)	Mean	0.4083	0.3977	0.4014	0.3984	0.5654	

	MSE	0.0049	0.0059	0.0077	0.0091	0.0344	MLE
(30,90)	Mean MSE	0.4044 0.0069	0.4035 0.0076	0.4036 0.0091	0.4037 0.0110	0.4080 0.0099	MLE
Exp3: R= 0.6667 when λ=1.2 ,θ=0.6 for α=0.3 ,γ=0.5 ,β=0.7							
(n,m)		**MLE**	**LS**	**WLS**	**Rg**	**RSS**	**Best**
(15,15)	Mean MSE	0.6620 0.0064	0.6605 0.0075	0.6590 0.0090	0.6571 0.0107	0.6608 0.0084	MLE
(30,30)	Mean MSE	0.6660 0.0031	0.6640 0.0039	0.6613 0.0057	0.6620 0.0060	0.6670 0.0041	MLE
(90,90)	Mean MSE	0.6615 0.0011	0.6614 0.0015	0.6600 0.0030	0.6607 0.0024	0.6611 0.0015	MLE
(15,30)	Mean MSE	0.6575 0.0056	0.6676 0.0062	0.6642 0.0078	0.6682 0.0085	0.5014 0.0364	MLE
(30,15)	Mean MSE	0.6669 0.0044	0.6545 0.0056	0.6561 0.0070	0.6511 0.0084	0.7917 0.0193	MLE
(30,90)	Mean MSE	0.6627 0.0021	0.6750 0.0024	0.6717 0.0038	0.6774 0.0037	0.4044 0.0723	MLE

Table 2-Real reliability values and their estimators performance for experiments 4, 5, 6 .

Exp4: R=0.2857 when λ=1.2 ,θ=3 for α=0.2 ,γ=0.2 ,β=0.9							
(n,m)		**MLE**	**LS**	**WLS**	**Rg**	**RSS**	**Best**
(15,15)	Mean MSE	0.2910 0.0057	0.2904 0.0065	0.2907 0.0077	0.2918 0.0091	0.2951 0.0078	MLE
(30,30)	Mean MSE	0.2868 0.0027	0.2877 0.0036	0.2881 0.0051	0.2898 0.0054	0.2864 0.0034	MLE
(90,90)	Mean MSE	0.2862 0.0010	0.2869 0.0012	0.2890 0.0025	0.2881 0.0019	0.2855 0.0013	MLE
(15,30)	Mean MSE	0.2846 0.0039	0.2943 0.0054	0.2919 0.0071	0.2966 0.0083	0.1747 0.0149	MLE
(30,15)	Mean MSE	0.2910 0.0041	0.2837 0.0048	0.2883 0.0063	0.2861 0.0072	0.4369 0.0307	MLE
(30,90)	Mean MSE	0.2839 0.0018	0.2945 0.0023	0.2913 0.0036	0.2975 0.0037	0.1216 0.0277	MLE
Exp5: R= 0.6667 when λ=1.2 ,θ=0.6 for α=0.2 ,γ=0.2 ,β=0.9							
(n,m)		**MLE**	**LS**	**WLS**	**Rg**	**RSS**	**Best**
(15,15)	Mean MSE	0.6653 0.0068	0.6606 0.0083	0.6564 0.0099	0.6543 0.0118	0.6688 0.0088	MLE
(30,30)	Mean MSE	0.6672 0.0036	0.6658 0.0047	0.6653 0.0066	0.6642 0.0071	0.6679 0.0044	MLE
(90,90)	Mean MSE	0.6638 0.0011	0.6639 0.0014	0.6629 0.0029	0.6637 0.0022	0.6634 0.0015	MLE
(15,30)	Mean MSE	0.6562 0.0058	0.6673 0.0065	0.6637 0.0080	0.6668 0.0095	0.5009 0.0367	MLE
(30,15)	Mean MSE	0.6611 0.0049	0.6501 0.0062	0.6514 0.0078	0.6466 0.0092	0.7860 0.0180	MLE
(30,90)	Mean MSE	0.6645 0.0022	0.6751 0.0027	0.6703 0.0044	0.6767 0.0041	0.4079 0.0705	MLE
Exp6: R= 0.4000when λ=2 ,θ=3 for α=0.2 ,γ=0.2 ,β=0.9							
(n,m)		**MLE**	**LS**	**WLS**	**Rg**	**RSS**	**Best**
(15,15)	Mean MSE	0.4105 0.0071	0.4106 0.0082	0.4118 0.0101	0.4121 0.0120	0.4122 0.0106	MLE
(30,30)	Mean MSE	0.4008 0.0035	0.4004 0.0042	0.4004 0.0059	0.4009 0.0063	0.4014 0.0049	MLE
(90,90)	Mean MSE	0.4012 0.0012	0.4017 0.0015	0.4023 0.0029	0.4022 0.0023	0.4010 0.0016	MLE
(15,30)	Mean MSE	0.4001 0.0057	0.4112 0.0076	0.4077 0.0096	0.4135 0.0113	0.2615 0.0244	MLE
(30,15)	Mean	0.4101	0.3989	0.4024	0.3970	0.5701	

	MSE	0.0057	0.0066	0.0085	0.0096	0.0367	MLE
(30,90)	Mean	0.3987	0.4121	0.4097	0.4166	0.1866	
	MSE	0.0024	0.0031	0.0047	0.0047	0.0469	MLE

5- Conclusions

Reliability of stress-strength p (y<x), where the strength x and stress y follow the Gompertz Fréchet distribution with different parameters, was estimated using five methods of estimation (maximum likelihood, least square, weighted least square, regression and ranked set sampling). MSE was used to compare the performance of estimators, as the value of MSE decreases by increasing the size of the sample. Through the results shown in the six tables, the maximum likelihood method revealed the best estimations for all experiments .

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