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The Matching Interdiction Problem in some Special Classes of Graphs

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Abstract
This paper is based on two main concepts: the matching and interdiction. Consider the graph G that each edge has a positive weight. The purpose of the matching interdiction problem in the weighted graph G with the weight $v(G)$ is to delete a subset of vertices, denoted by R^*, such that the weight of the maximum matching in the resulted graph is minimized. We first restrict this problem to the case of deleting two vertices of the in graph. Then, we compute the quantity $(v(G)-v(G[V\setminus R^*]))/(v(G)-v(G[V\setminus R]))$ some special classes of graphs, where $v(G[V\setminus R^*])$ and $v(G[V\setminus R])$ are the weights of the maximum matching in the graphs $G[V\setminus R^*]$ and $G[V\setminus R]$, respectively. We show that the limit of this proportion in some sequences of graphs tends to its maximum value. We prove that the limit of this proportion can be decreased in the sequence of the wheel graphs. Also, if the weight of all edges in the complete graph is equal, the approximate and optimal solutions will be the same. At end, we generalize the results related to the path graph when the purpose is to delete B even vertices.

Keywords: Interdiction, matching, weighted graph.

1. Introduction

The interdiction problem is a classical problem in networks such that the researchers have studied this problem since the 1960s [1, 2]. In the network interdiction problems, there are two inconsistent forces, the follower and interdictor. The follower tries to optimize the objective function by considering the constraints of problem. The interdictor attempts to restrict the action of follower by destroying the vertices or arcs of the network. The topic of interdiction has been entered in majority of the network problems that some of them are expressed [3-6].

The maximum flow network interdiction problem is defined as follows. Consider the network $G=(V,E)$ such that each arc (i,j) has the capacity u_{ij} and the deletion cost r_{ij}. Suppose that

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R is positive integer. The purpose is to find the set $E' \subseteq E$ such that $\sum_{(i,j)\in E'} r_{ij} \le R$ and the maximum flow in $G - E'$ is minimized. McMasters and Mustin [3] presented an algorithm for this problem on the planar undirected networks. They use the dual of network in this algorithm. Wood [7] showed that the maximum flow network interdiction problem is NP-complete even when $r_{ij} = 1$ for each $(i,j) \in E$. Then, he presented an integer linear programming model for solving this problem. The presented model by Wood is a basic model that has been used in some other papers [8, 9]. Kennedy et al. [9] expressed the maximum flow network interdiction problem as follows. Let r_i be the deletion cost of vertex i. The purpose is to find the set $V' \subseteq V$ such that $\sum_{i\in V'} r_i \le R$ and the maximum flow in $G[V \setminus V']$ is minimized. They presented a model for this problem similar to the Wood's model. Zenklusen [10] considered the network flow interdiction problems on the planar networks with a single source and sink having no restrictions on the demand and supply.
Shen et al. [11] presented another interdiction problem in graphs. The purpose is to remove the subset $V' \subseteq V$ in the graph $G = (V, E)$ such that the disconnectivity of graph is maximized. They introduced three metrics for measuring the connectivity of graph.

In this paper, we consider the matching interdiction problem. This problem was started by Zenklusen in 2010 [6]. Consider the graph $G = (V, E)$ such that each arc (i, j) has the weight w_{ij} and each vertex i has the deletion cost r_i. Suppose that B is positive integer. The purpose is to find a set $R \subseteq V$ such that $\sum_{i\in R} r_i \le B$ and the weight of the maximum matching in $G[V \setminus R]$ is minimized. Zenklusen proved that for some classes of graphs and under some conditions, this problem is NP−complete. He also considered the proportion between differences of the optimal and approximate solutions of the matching interdiction problem from the weight of the maximum matching in G ,which is denoted by $e_{G|B}$. The quantity $e_{G|B}$ can be bounded when $r_i = 1$ for each $i \in V$.

The rest of this paper is organized as follows. In Section 2, some needed definitions and basic facts are introduced. In Section 3, the values of $e_{G|2}$ for some classes of graphs such as path, cycle, complete, wheel and complete bipartite are computed. The limit of this proportion in some sequences of graphs tends to its maximum value, i.e., two. We join the path graphs from a specific vertex and prove that $e_{G|2}$ will be attained its maximum value. The value of this limit can be decreased in the sequence of wheel graphs. Also, we show which this proportion in the complete graph with equal weights on the edges is equal to one. The limits of this proportion in the mentioned classes of graphs are considered into account. At end, we generalize the results related to the sequence of the path graphs when B is even.

2. Basic facts

Suppose that $G = (V, E)$ is an undirected graph with vertex and edge sets V and E, respectively. Let $|V| = n$ and $|E| = m$. Each edge $(i, j) \in E$ has a positive weight w_{ij} [12]. A walk in G is a sequence of vertices $(v_1, v_2, \ldots, v_r)$ such that $(v_i, v_{i+1}) \in E$ for each $1 \le i \le r-1$. The graph G is called connected if there is at least one walk between every pair of its vertices. We say that two vertices u and v are adjacent if $(u,v) \in E$. Also, the edge (u,v) is incident to vertices u and v. Let $V' \subseteq V$. $G[V \setminus V']$ is a subgraph of G with the vertex set

$V \setminus V'$ and edge set $\{(u,v) \in E \,|\, u,v \in V \setminus V'\}$. $M' \subseteq E$ is called a matching in G if no two edges in M' are incident to the same vertices. The weight of matching M' is defined as follows:
$w(M') = \sum_{(i,j)\in M'} w_{ij}$.
We refer to the matching $M = \{e_1,\dots,e_k\}$ as the maximum matching in G if $w(M) \ge w(M')$, for every matching M' in G. The weight of the maximum matching in the graph G is denoted by $v(G)$, i.e., $v(G) = w(M)$. Let $B \in N$ be even. Consider the set:
$W = \{v(G[V \setminus V']) \,|\, V' \subseteq V, |V'| = B\}$,
where $v(G[V \setminus V'])$ is the weight of the maximum matching in the graph $G[V \setminus V']$. Let $v(G[V \setminus R_B^*]) = \text{Min}\{w \,|\, w \in W\}$. R_B^* is called the optimal interdiction set of G. Each element of the set $\{V' \subseteq V \,||\, V'| = B\}$ is called an interdiction set. Let $e_i = (u_i, v_i)$, where $1 \le i \le k$, and $w_{u_1v_1} \ge \cdots \ge w_{u_kv_k}$. The set $R_B = \left\{u_1,\dots,u_{\frac{B}{2}}, v_1,\dots,v_{\frac{B}{2}}\right\}$ is called the approximate interdiction set of G. If $B \ge 2k$, the graph $G[V \setminus R_B]$ is a graph without edges. Therefore, we suppose that $B < 2k$. Now, consider the following proportion:

$$e_{G|B} = \frac{v(G) - v(G[V \setminus R_B^*])}{v(G) - v(G[V \setminus R_B])}. \quad (1)$$

Since $M \setminus \left\{e_1,\dots,e_{\frac{B}{2}}\right\}$ is the maximum matching in the graph $G[V \setminus R_B]$. Therefore,

$$v(G[V \setminus R_B]) = v(G) - \sum_{i=1}^{B/2} w_{u_iv_i}. \quad (2)$$

It has been showed that $e_{G|B} \le 2$ [8]. If $e_{G|B} = 1$, then we can conclude that $R_B^* = R_B$. Hence, it is better that the value of $e_{G|B}$ to be near one. In this paper, we denote the sets R_2^*, R_2 and quantity $e_{G|2}$ by R^*, R and e_G, respectively.

3. Main results

The aim of this section is to compute e_G, for some classes of graphs. We start by the path graph $P_n = (V,E)$. This graph has the vertex set $V = \{1,\dots,n\}$ and edge set $E = \{(i,i+1) \,|\, 1 \le i \le n-1\}$. We also assume that each edge $(i,i+1)$ has the weight $w_i \in N$. The graph P_n is depicted in Figure 1.

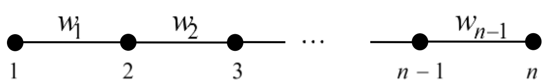

Figure 1: The path graph P_n

Theorem 1. Let $w_1 < \cdots < w_{n-1}$. Then,

$$e_{P_n} = \begin{cases} 1 & n = 2,3, \\ \dfrac{w_{n-3} + w_{n-1}}{w_{n-1}} & n \ge 4, \end{cases} \quad (3)$$

and $\lim_{n\to\infty} e_{P_n} = 2$. In particular, if $w_i = i$, for every $1 \le i \le n-1$, we obtain:

$$e_{P_n} = \begin{cases} 1 & n = 2,3, \\ \dfrac{2n-4}{n-1} & n \geq 4. \end{cases}$$

Proof. Let $n \geq 4$. There are two cases:

1. n is even. The maximum matching in P_n and its weight are as follows:

$M = \{(1,2),(3,4),\ldots,(n-1,n)\}$,

$$v(P_n) = \sum_{i=1}^{n/2} w_{2i-1}. \tag{4}$$

Therefore, $R = \{n-1, n\}$ and

$$v(P_n[V \setminus R]) = v(P_{n-2}) = \sum_{i=1}^{(n-2)/2} w_{2i-1}. \tag{5}$$

Claim 1. $R^* = \{n-3, n-1\}$.

Proof of Claim 1. Let $R_1 = \{n-3, n-1\}$. The maximum matching in $P_n[V \setminus R_1]$ and its weight are as follows:

$M_1 = \{(1,2),(3,4),\ldots,(n-5,n-4)\}$,

$$\begin{aligned} v(P_n[V \setminus R_1]) = v(P_{n-4}) &= \sum_{i=1}^{(n-4)/2} w_{2i-1} \\ &= v(P_n) - w_{n-3} - w_{n-1}. \end{aligned} \tag{6}$$

Let $R_2 = \{j,k\} \subseteq V$ such that $1 \leq j < k \leq n$ and $M' = \{(a,b) \in M \mid a \in R_2 \text{ or } b \in R_2\}$. Therefore,

$$\sum_{(a,b)\in M'} w_{ab} \leq w_{n-3} + w_{n-1}. \tag{7}$$

By Eqs. (6) and (7), we have:

$$\begin{aligned} v(P_n[V \setminus R_1]) &= v(P_n) - w_{n-3} - w_{n-1} \\ &\leq v(P_n) - \sum_{(a,b)\in M'} w_{ab} \\ &= w(M_2) \leq v(P_n[V \setminus R_2]), \end{aligned}$$

where $M_2 = M \setminus M'$ is a matching in $P_n[V \setminus R_2]$. Therefore, $R^* = R_1$ and the claim follows.

By Eqs. (4), (5) and (6), e_{P_n} is obtained as follows:

$$e_{P_n} = \frac{v(P_n) - v(P_{n-4})}{v(P_n) - v(P_{n-2})} = \frac{w_{n-3} + w_{n-1}}{w_{n-1}}.$$

2. n is odd. The maximum matching in P_n and its weight are as follows:

$M = \{(2,3),(4,5),\ldots,(n-1,n)\}$,

$$v(P_n) = \sum_{i=1}^{(n-1)/2} w_{2i}. \tag{8}$$

Therefore, $R = \{n-1, n\}$ and

$$v(P_n[V \setminus R]) = v(P_{n-2}) = \sum_{i=1}^{(n-3)/2} w_{2i}. \tag{9}$$

Similar to claim 1, we can show that $R^* = \{n-3, n-1\}$. Hence, we have:

$$v(P_n[V \setminus R^*] = v(P_{n-4}) = \sum_{i=1}^{(n-5)/2} w_{2i}. \tag{10}$$

By Eqs. (8), (9) and (10), e_{P_n} is obtained as follows:

$$e_{P_n} = \frac{v(P_n) - v(P_{n-4})}{v(P_n) - v(P_{n-2})} = \frac{w_{n-3} + w_{n-1}}{w_{n-1}}.$$

Therefore, for every $n \geq 4$, $e_{P_n} = \frac{w_{n-3} + w_{n-1}}{w_{n-1}}$. It is obvious that $e_{P_2} = e_{P_3} = 1$. ■

The cycle graph $C_n = (V, E)$ is a connected undirected graph such that $V = \{1, \ldots, n\}$ and $E = \{(i, i+1) \mid 1 \leq i \leq n-1\} \cup \{(n,1)\}$. Each edge $(i, i+1)$ has the weight $w_i \in N$ and the weight of edge $(n,1)$ is $w_n \in N$. The graph C_n is depicted in Figure 2.

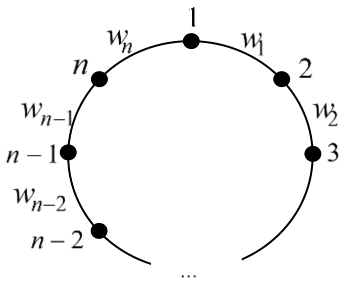

Figure 2: The cycle graph C_n

Theorem 2. Let $w_1 \leq w_2 < w_3 < \cdots < w_n$. Then,

$$e_{C_n} = \begin{cases} 1 & n = 2, \\ \frac{w_{n-2} + w_n}{w_n} & \text{if } n \geq 4 \text{ is even}, \\ \frac{w_{n-2} + w_n - 1}{w_n} & \text{if n} \geq 3 \text{ is odd and } w_1 = 1, \end{cases} \tag{11}$$

and $\lim_{n\to\infty} e_{C_{2n}} = \lim_{n\to\infty} e_{C_{2n+1}} = 2$. If $w_i = i$, for every $1 \leq i \leq n$, then for every $n \geq 2$, we have:

$$e_{C_n} = \begin{cases} \frac{2n-2}{n} & \text{if } n \text{ is even}, \\ \frac{2n-3}{n} & \text{if n is odd.} \end{cases}$$

Proof. It is obvious that $e_{C_2} = 1$. Let $n \geq 4$ be even. The maximum matching in C_n and $v(C_n)$ are as follows:

$M = \{(2,3), (4,5), \ldots, (n-2, n-1), (n,1)\}$,

$$v(C_n) = v(P_{n+1}) = \sum_{i=1}^{n/2} w_{2i}. \tag{12}$$

Therefore, $R = \{1, n\}$ and

$$v(C_n[V \setminus R]) = v(P_{n-1}) = \sum_{i=1}^{(n-2)/2} w_{2i}. \tag{13}$$

Similar to claim 1 in Theorem 1, we can show that $R^* = \{n-2, n\}$. Hence,

$$v(C_n[V \setminus R^*]) = v(P_{n-3}) = \sum_{i=1}^{(n-4)/2} w_{2i}. \tag{14}$$

e_{C_n} is obtained by Eqs. (12), (13) and (14):

$$e_{C_n} = \frac{v(P_{n+1}) - v(P_{n-3})}{v(P_{n+1}) - v(P_{n-1})} = \frac{w_{n-2} + w_n}{w_n}.$$

Now, let $n \geq 3$ be odd and $w_1 = 1$. The maximum matching in C_n and $v(C_n)$ are as follows:

$M = \{(3,4),(5,6),\ldots,(n-2,n-1),(n,1)\}$,

$$v(C_n) = v(P_{n+1}) - w_1 = \sum_{i=2}^{(n+1)/2} w_{2i-1}. \tag{15}$$

Therefore, $R = \{1, n\}$ and we have:

$$v(C_n[V \setminus R]) = v(P_{n-1}) - w_1 = \sum_{i=2}^{(n-1)/2} w_{2i-1}. \tag{16}$$

Claim 1. $R^* = \{n-2, n\}$.

Proof of Claim 1. Let $R_1 = \{n-2, n\}$. The maximum matching in $C_n[V \setminus R_1]$ and its weight are as follows:

$M_1 = \{(1,2),(3,4),\ldots,(n-4,n-3)\}$,

$$v(C_n[V \setminus R_1]) = v(P_{n-3}) = \sum_{i=1}^{(n-3)/2} w_{2i-1}. \tag{17}$$

Suppose that $R_2 = \{\{j,k\} \mid 1 \leq j < k \leq n, \{j,k\} \notin \{\{1,n-1\},\{1,n-2\},\{n-1,n\},\{n-2,n\}\}\}$ and $M' = \{(a,b) \in M \mid a \in R_2 \text{ or } b \in R_2\}$. Since $w_1 = 1$, we have:

$$\sum_{(a,b) \in M'} w_{ab} + w_1 \leq w_{n-2} + w_n.$$

Therefore,

$$\begin{aligned} v(C_n[V \setminus R_1]) &= \sum_{i=1}^{(n-3)/2} w_{2i-1} \\ &= v(C_n) - w_{n-2} - w_n + w_1 \\ &\leq v(C_n) - \sum_{(a,b) \in M'} w_{ab} \\ &= w(M_2) \\ &\leq v(C_n[V \setminus R_2]), \end{aligned} \tag{18}$$

where $M_2 = M \setminus M'$ is a matching in $C_n[V \setminus R_2]$. Now, we consider the following three interdiction sets:

1. $R_2 = \{1, n-1\}$. Then, we have:

$$\begin{aligned} v(C_n[V \setminus R_2]) &= \sum_{i=1}^{(n-3)/2} w_{2i} \\ &\geq \sum_{i=1}^{(n-3)/2} w_{2i-1} \\ &= v(C_n[V \setminus R_1]). \end{aligned}$$

2. $R_2 = \{1, n-2\}$. Then, we drive:

$$
\begin{aligned}
v(C_n[V \setminus R_2]) &= \sum_{i=2}^{(n-3)/2} w_{2i-1} + w_{n-1} \\
&\geq \sum_{i=1}^{(n-3)/2} w_{2i-1} \\
&= v(C_n[V \setminus R_1]).
\end{aligned}
$$

3. $R_2 = \{n-1, n\}$. Therefore,

$$
\begin{aligned}
v(C_n[V \setminus R_2]) &= \sum_{i=1}^{(n-3)/2} w_{2i} \\
&\geq \sum_{i=1}^{(n-3)/2} w_{2i-1} \\
&= v(C_n[V \setminus R_1]).
\end{aligned}
$$

According to Rel. (18) and the above three cases, we can conclude the following inequality holds for every set $R_2 = \{j,k\} \subseteq V$, where $1 \leq j < k \leq n$:

$$v(C_n[V \setminus R_1]) \leq v(C_n[V \setminus R_2]). \tag{19}$$

By (19), R_1 is the optimal interdiction set and the claim follows.
The value of e_{C_n} can be obtained by Eqs. (15), (16) and (17) as follows:

$$
\begin{aligned}
e_{C_n} &= \frac{v(P_{n+1}) - w_1 - v(P_{n-3})}{v(P_{n+1}) - w_1 - v(P_{n-1}) + w_1} \\
&= \frac{w_{n-2} + w_n - w_1}{w_n}.
\end{aligned}
$$

This completes the proof. ■

Consider the graph $A_n = (V,E)$, depicted in Fig. 3. In this graph, $V = \{1,\dots,n\}$ and $E = \{(i,i+1) \mid 1 \leq i \leq n-2\} \cup \{(2,n)\}$. Each edge $(i,i+1)$ has the weight $w_i \in N$ and the weight of edge $(2,n)$ is $w_{n-1} \in N$.

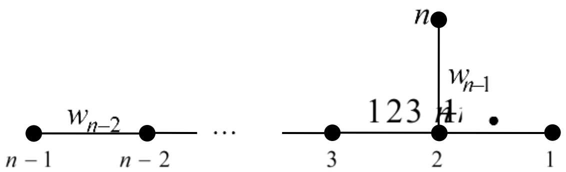

Figure 3: The graph A_n

Theorem 3. Let $w_1 < w_2 < \cdots < w_{n-1}$. Then, the value of e_{A_n} is given by:

$$
e_{A_n} = \begin{cases} 1 & n = 4, \\ \dfrac{w_{n-2} + w_{n-1}}{w_{n-1}} & n \geq 5. \end{cases} \tag{20}
$$

Furthermore, $\lim_{n\to\infty} e_{A_n} = 2$. In particular, if $w_i = i$, for every $1 \leq i \leq n-1$, we have:

$$
e_{A_n} = \begin{cases} 1 & n = 4, \\ \dfrac{2n-3}{n-1} & n \geq 5. \end{cases}
$$

Proof. Since $v(A_4[V \setminus R]) = v(A_4[V \setminus R^*] = 0$, we have $e_{A_4} = 1$.Let $n \geq 5$. There are two cases:

1. n is even. The maximum matching in A_n and its weight are as follows:
$M = \{(4,5),(6,7),\dots,(n-2,n-1),(2,n)\}$,

$$\begin{aligned} v(A_n) &= v(P_{n-1}) - w_2 + w_{n-1} \\ &= \sum_{i=2}^{(n-2)/2} w_{2i} + w_{n-1}. \end{aligned} \tag{21}$$

2. n is odd. The maximum matching in A_n and its weight are as follows:
$M = \{(3,4),(5,6),\dots,(n-2,n-1),(2,n)\}$,

$$\begin{aligned} v(A_n) &= v(P_{n-1}) - w_1 + w_{n-1} \\ &= \sum_{i=2}^{(n-1)/2} w_{2i-1} + w_{n-1}. \end{aligned} \tag{22}$$

In each two cases, $R = \{2,n\}$. Therefore,

$$v(A_n[V \setminus R]) = \begin{cases} v(P_{n-1}) - w_2 = \sum_{i=2}^{(n-2)/2} w_{2i} & \text{if } n \text{ is even,} \\ v(P_{n-1}) - w_1 = \sum_{i=2}^{(n-1)/2} w_{2i-1} & \text{if } n \text{ is odd.} \end{cases} \tag{23}$$

Similar to Claim 1 in Theorem 1, we can show that in the above two cases, $R^* = \{2, n-2\}$. Hence,

$$v(A_n[V \setminus R^*]) = \begin{cases} v(P_{n-3}) - w_2 = \sum_{i=2}^{(n-4)/2} w_{2i} & \text{if } n \text{ is even,} \\ v(P_{n-3}) - w_1 = \sum_{i=2}^{(n-3)/2} w_{2i-1} & \text{if } n \text{ is odd.} \end{cases} \tag{24}$$

We can derive e_{A_n}, for every $n \ge 5$, by Eqs. (21), (22), (23) and (24) as follows:

$$\begin{aligned} e_{A_n} &= \frac{v(P_{n-1}) - v(P_{n-3}) + w_{n-1}}{w_{n-1}} \\ &= \frac{w_{n-2} + w_{n-1}}{w_{n-1}}, \end{aligned}$$

whence the theorem. ■

Consider the graph $G_k = (V,E)$, depicted in Fig. 4. In this graph, $V = \{1,\dots,n\}$ and $E = \{(i,i+1) \mid 1 \le i \le n-1\} \cup \{(n,1)\} \cup \{(i,i+3) \mid i = 1,5,9,\dots,n-3\}$ such that $n = 4k$ and $m = 5k$. Each edge $(i,i+1)$ has the weight $w_i \in N$ and the weight of edge $(n,1)$ is $w_n \in N$. Also, we put the weight of edges $\{(i,i+3) \mid i = 1,5,9,\dots,n-3\}$ from w_{n+1} to w_{n+k}, respectively

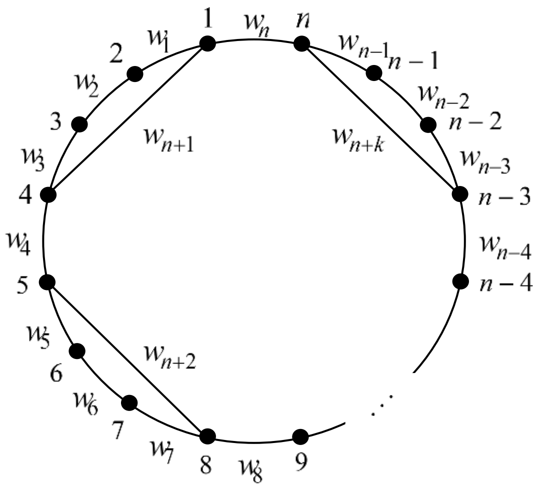

Figure 4: The graph G_k

Theorem 4. Let $w_1 < w_2 < \cdots < w_{n+k}$. Then, for each $k \geq 2$, we have:

$$e_{G_k} = \frac{w_{5k-1} + w_{5k}}{w_{5k}}. \tag{25}$$

Also, $\lim_{k\to\infty} e_{G_k} = 2$. If $w_i = i$ for every $1 \leq i \leq n+k$, then $e_{G_k} = \frac{10k-1}{5k}$.

Proof. Notice that the graph G_k is obtained by repeating the subgraphs depicted in Fig. 5. The maximum matching in these subgraphs is denoted by the red edges. This motivates our main proof.

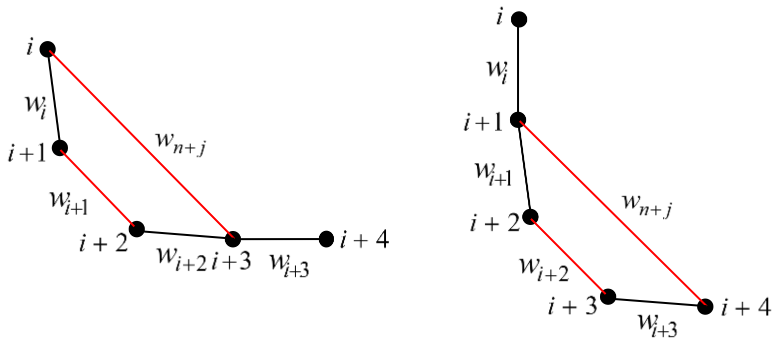

Figure. 5 The subgraphs of G_k

Let $k \geq 2$. Consider the following matchings in the graph G_k:

$$M_1 = \{(1,2),(3,4),\ldots,(n-1,n)\} \quad \rightarrow \quad w(M_1) = \sum_{i=1}^{n/2} w_{2i-1},$$

$$M_2 = \{(2,3),(4,5),\ldots,(n-2,n-1),(n,1)\} \quad \rightarrow \quad w(M_2) = \sum_{i=1}^{n/2} w_{2i},$$

$$M_3 = \{(2,3),(6,7),\ldots,(n-2,n-1),(1,4),(5,8),\ldots,(n-3,n)\} \quad \rightarrow \quad w(M_3) = \sum_{i=0}^{(n-4)/4} w_{4i+2} + \sum_{i=n+1}^{n+k} w_i.$$

The following relation holds between these matchings:
$w(M_3) > w(M_2) > w(M_1)$.
If we consider an arbitrary matching in G_k, then its weight will be less than $w(M_3)$. Therefore, M_3 is the maximum matching in G_k and

$$v(G_k) = \sum_{i=0}^{(n-4)/4} w_{4i+2} + \sum_{i=n+1}^{n+k} w_i. \quad (26)$$

The approximate interdiction set in G_k is $R = \{n-3, n\}$. By Eq. (2), we have:

$$v(G_k[V \setminus R]) = v(G_k) - w_{n+k}. \quad (27)$$

Similar to Claim 1 in Theorem 1, we can show that $R^* = \{n-4, n\}$. Hence,

$$v(G_k[V \setminus R^*]) = v(G_k) - w_{n+k-1} - w_{n+k}. \quad (28)$$

We derive e_{G_k} by Eqs. (26), (27) and (28):

$$e_{G_k} = \frac{w_{n+k-1} + w_{n+k}}{w_{n+k}} = \frac{w_{5k-1} + w_{5k}}{w_{5k}}.$$

The theorem follows. ■

The complete graph $K_n = (V, E)$ is a connected graph such that $V = \{1, \ldots, n\}$ and $E = \{(i, j) \mid i, j \in V, i < j\}$, see Fig.6.

Note 1. If $V' = \{i, j\} \subseteq V$, where $i \neq j$, then $K_n[V \setminus V']$ is isomorphic to the complete graph K_{n-2}.

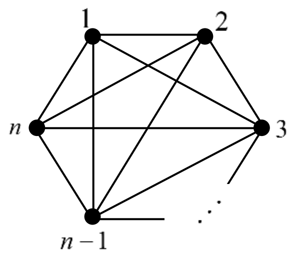

Figure 6: The complete graph K_n

Theorem 5. Suppose that the weight of edge $(i, i+1)$, where $1 \leq i \leq n-1$, in the graph K_n is $w_i \in N$ and the weight of edge $(n,1)$ is $w_n \in N$ such that $w_1 < w_2 < \cdots < w_n$. Let the weight of remaining edges in K_n be w_1. Then,

$$e_{K_n} = \begin{cases} 1 & n = 2, \\ \dfrac{w_{n-2} + w_n - w_1}{w_n} & n \geq 3, \end{cases} \quad (29)$$

and $\lim_{n\to\infty} e_{K_n} = 2$. In particular, if $w_i = i$, for every $1 \leq i \leq n$, we have the following relation:

$$e_{K_n} = \begin{cases} 1 & n = 2, \\ \dfrac{2n-3}{n} & n \geq 3. \end{cases}$$

Proof. It is obvious that $e_{K_2} = 1$. From Fig. 6, one can see that the maximum number of nonadjacent edges in K_n is equal to the maximum number of nonadjacent edges in K_{n-2} plus one. Suppose $n \geq 3$ and M_n is the maximum matching in K_n with given edge weights in the theorem. Then, we have the following relations:

$|M_n| = |M_{n-2}| + 1,$

$w(M_n) = w(M_{n-2}) + w_n.$

Therefore, the weight of the maximum matching in K_n is equal to:

$$v(K_n) = v(C_n). \tag{30}$$

Hence, $R = \{1, n\}$ and

$$v(K_n[V \setminus R]) = v(K_{n-2}) = v(C_{n-2}). \tag{31}$$

By Note 1, we can conclude that $R^* = \{n-2, n\}$. Then,

$$v(K_n[V \setminus R^*]) = \begin{cases} v(P_{n-3}) + w_1 & \text{if } n \text{ is even,} \\ v(P_{n-3}) & \text{if } n \text{ is odd.} \end{cases} \tag{32}$$

First, we use Eqs. (30), (31) and (32) for obtaining e_{K_n}. Then, by substituting Eqs. (12) and (15) into it, the following relation is obtained:

$$e_{K_n} = \frac{v(P_{n+1}) - v(P_{n-3}) - w_1}{v(P_{n+1}) - v(P_{n-1})}$$
$$= \frac{w_{n-2} + w_n - w_1}{w_n},$$

proving the result. ■

The complete bipartite graph $K_{m,n} = (V, E)$ is a connected graph such that the set V is partitioned into two subsets $V_m = \{v_1, \ldots, v_m\}$ and $U_n = \{u_1, \ldots, u_n\}$. Every edge has one end in V_m and one end in U_n. In other words, $E = \{(v_i, u_j) \,|\, 1 \leq i \leq m, 1 \leq j \leq n\}$. The weight of each edge (v_i, u_j) is $w_i \in N$. This graph is depicted in Fig. 7.

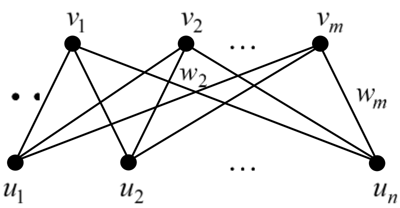

Figure 7: The complete bipartite graph $K_{m,n}$

Theorem 6. Suppose $m, n \in N$, $m \leq n$ and $w_1 < w_2 < \cdots < w_m$. Then, we have:

$$e_{K_{m,n}} = \begin{cases} \dfrac{w_m + w_{m-1}}{w_m} & m \geq 2, \\ 1 & m = 1, n \geq 1. \end{cases} \tag{33}$$

Furthermore, $\lim_{m \to \infty} e_{K_{m,n}} = 2$. If $w_i = i$ for each $1 \leq i \leq m$, then:

$$e_{K_{m,n}} = \frac{2m-1}{m}, \qquad \forall m \geq 1.$$

Proof. Assume that $m \le n$ and $m \ge 2$. By the structure of graph $K_{m,n}$, the maximum number of nonadjacent edges in $K_{m,n}$ is equal to m. Hence, the maximum matching and its weight are as follows:
$M = \{(v_m, u_n), (v_{m-1}, u_{n-1}), \ldots, (v_1, u_{n-m+1})\}$,

$$\nu(K_{m,n}) = \sum_{i=1}^{m} w_i. \quad (34)$$

Therefore, $R = \{v_m, u_n\}$ and we have:

$$\nu(K_{m,n}[V \setminus R]) = \sum_{i=1}^{m-1} w_i. \quad (35)$$

Now, we determine the optimal interdiction set. There are four cases:

1. $R_1^* = \{v_{m-1}, v_m\} \subseteq V_m$. In this case, we have:

$$\nu(K_{m,n}[V \setminus R_1^*]) = \sum_{i=1}^{m-2} w_i. \quad (36)$$

2. $R_2^* = \{v_i, v_j\} \subseteq V_m$, where $i < j$ and $R_2^* \ne R_1^*$. Hence,
$\nu(K_{m,n}[V \setminus R_2^*]) = \sum_{i=1}^{m} w_i - w_i - w_j$.
Since $w_i + w_j < w_{m-1} + w_m$, the following relation holds:
$\nu(K_{m,n}[V \setminus R_1^*]) < \nu(K_{m,n}[V \setminus R_2^*])$.

3. $R_3^* = \{u_i, u_j\} \subseteq U_n$, where $i < j$. There are two cases:

I. $m \le n-2$. In this case, $\nu(K_{m,n}[V \setminus R_3^*]) = \sum_{i=1}^{m} w_i$.

II. $m > n-2$. This inequality holds when $n = m$ or $n = m+1$. If $n = m$, then $\nu(K_{m,n}[V \setminus R_3^*]) = \sum_{i=3}^{m} w_i$. If $n = m+1$, we have $\nu(K_{m,n}[V \setminus R_3^*]) = \sum_{i=2}^{m} w_i$.

By cases I and II, $\nu(K_{m,n}[V \setminus R_2^*]) < \nu(K_{m,n}[V \setminus R_3^*])$.

4. $R_4^* = \{v_i, u_j\}$, where $1 \le i \le m$ and $1 \le j \le n$. Therefore,
$\nu(K_{m,n}[V \setminus R_4^*]) = \sum_{k=1}^{m} w_k - w_i$.
It is now obvious that $\nu(K_{m,n}[V \setminus R_1^*]) < \nu(K_{m,n}[V \setminus R_4^*])$.

By the above cases, we can conclude that $R^* = R_1^*$. By substituting Eq. (34), (35) and (36) into (1), $e_{K_{m,n}}$ is obtained as follows:

$e_{K_{m,n}} = \frac{w_m + w_{m-1}}{w_m}$.

Let $m = 1$ and $n \ge 1$. In this case, $R^* = R = \{v_1, u_n\}$ and
$\nu(K_{1,n}[V \setminus R]) = \nu(K_{1,n}[V \setminus R^*]) = 0$.
Therefore, $e_{K_{1,n}} = 1$ and the theorem is proved. ■

The wheel graph $W_n = (V, E)$ is a connected graph such that $V = \{1, \ldots, n\}$ and $E = \{(i, i+1) \mid 1 \le i \le n-2\} \cup \{(n-1, 1)\} \cup \{(n, i) \mid 1 \le i \le n-1\}$. The weights of edges in W_n are defined as follows:

1. The weight of edge $(i, i+1)$, for every $1 \le i \le n-2$, is denoted by w_i.
2. The weight of edge $(n, 1)$ is w_{n-1}.

3. The weight of edge (n,i), for every $1 \le i \le n-1$, is denoted by w_{2n-i}.
The graph W_n is depicted in Figure 8.

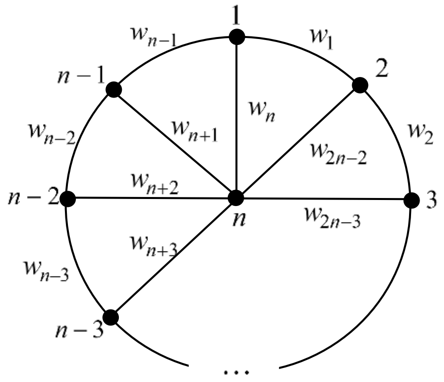

Figure 8: The wheel graph W_n

Theorem 7. Let $n \ge 3$ and $w_1 \le w_2 < w_3 < \cdots < w_{2n-2}$. Then, we have:

1. If n is even and $w_1 = 1$, then $e_{W_n} = \dfrac{w_{2n-2} + w_{n-1} - 1}{w_{2n-2}}$.
2. If n is odd, $w_1 = 1$ and $w_2 = 2$, then $e_{W_n} = \dfrac{w_{2n-2} + w_{n-1} - 2}{w_{2n-2}}$.

In particular, let $w_i = i$, for every $1 \le i \le 2n-2$. Then,

$$e_{W_n} = \begin{cases} \dfrac{3n-4}{2n-2} & \text{if } n \text{ is even,} \\ \dfrac{3n-5}{2n-2} & \text{if } n \text{ is odd,} \end{cases}$$

and $\lim\limits_{n\to\infty} e_{W_{2n}} = \lim\limits_{n\to\infty} e_{W_{2n+1}} = \dfrac{3}{2}$.

Proof. Let n be even. The maximum number of nonadjacent edges in W_n is equal to $\frac{n}{2}$. Therefore, we have to select $\frac{n}{2}$ edges with the greatest weight for the maximum matching in W_n. The maximum matching in C_{n-1}, $M_{C_{n-1}}$, is as follows:
$M_{C_{n-1}} = \{(3,4),(5,6),\ldots,(n-3,n-2),(n-1,1)\}$.
Since an edge adjacent to vertex n can be in the maximum matching of graph W_n, M is containing $M_{C_{n-1}}$ and the edge $(2,n)$. Therefore,
$M = \{(3,4),(5,6),\ldots,(n-3,n-2),(n-1,1),(2,n)\}$,
and

$$\begin{aligned} v(W_n) &= v(C_{n-1}) + w_{2n-2} \\ &= \sum_{i=2}^{n/2} w_{2i-1} + w_{2n-2}. \end{aligned} \tag{37}$$

Since $R = \{2,n\}$, we have:

$$v(W_n[V \setminus R]) = v(C_{n-1}). \tag{38}$$

Claim 1. $R^* = \{n-1, n\}$.

Proof of Claim 1. Consider the interdiction set $R_1 = \{n-1, n\}$. As $W_n[V \setminus R_1]$ is the path graph P_{n-2}. Hence,

$$v(W_n[V \setminus R_1]) = v(P_{n-2}) = \sum_{i=1}^{(n-2)/2} w_{2i-1}. \tag{39}$$

We consider the following two sets:

$R_2 = \{\{j,k\} \subseteq V \mid 1 \le j < k \le n, \{j,k\} \notin \{\{1,n\}, \{1,2\}, \{2,n-1\}, \{n-1,n\}\}\}$,

$M' = \{(a,b) \in M \mid a \in R_2 \text{ or } b \in R_2\}$.

Therefore,

$$\begin{aligned} v(W_n[V \setminus R_1]) - w_1 &= \sum_{i=2}^{(n-2)/2} w_{2i-1} \\ &= v(W_n) - w_{n-1} - w_{2n-2} \\ &< v(W_n) - \sum_{(a,b)\in M'} w_{ab} \\ &= w(M_2) \\ &\le v(W_n[V \setminus R_2]), \end{aligned} \tag{40}$$

where $M_2 = M \setminus M'$ is a matching in $W_n[V \setminus R_2]$. Since $w_1 = 1$, Rel. (40) yields:

$$v(W_n[V \setminus R_1]) \le v(W_n[V \setminus R_2]). \tag{41}$$

Now, we consider the following three interdictions set:

1. $R_2 = \{1, n\}$. Therefore,

$$\begin{aligned} v(W_n[V \setminus R_2]) = v(P_{n-1}) &= \sum_{i=1}^{(n-2)/2} w_{2i} \\ &\ge \sum_{i=1}^{(n-2)/2} w_{2i-1} = v(W_n[V \setminus R_1]). \end{aligned}$$

2. $R_2 = \{1, 2\}$. We have:

$$\begin{aligned} v(W_n[V \setminus R_2]) &= \sum_{i=2}^{(n-2)/2} w_{2i} + w_{2n-3} \\ &\ge \sum_{i=1}^{(n-2)/2} w_{2i-1} = v(W_n[V \setminus R_1]). \end{aligned}$$

3. $R_2 = \{2, n-1\}$. In this case, the value of $v(W_n[V \setminus R_2])$ is equal to $\sum_{i=2}^{(n-2)/2} w_{2i-1} + w_n$ or $\sum_{i=3}^{(n-2)/2} w_{2i-1} + w_{2n-3}$. In each case, the following relation holds:

$v(W_n[V \setminus R_2]) \ge v(W_n[V \setminus R_1])$.

According to (41) and the above three cases, for each set $R_2 = \{j,k\} \subseteq V$, where $1 \le j < k \le n$, we have:

$v(W_n[V \setminus R_2]) \ge v(W_n[V \setminus R_1])$.

Then, it is concluded that R_1 is the optimal interdiction set and Claim 1 is proved.

We proceed by substituting Eqs. (37), (38), and (39) into (1):

$$e_{W_n} = \frac{\sum_{i=2}^{n/2} w_{2i-1} + w_{2n-2} - \sum_{i=1}^{(n-2)/2} w_{2i-1}}{w_{2n-2}}$$
$$= \frac{w_{2n-2} + w_{n-1} - w_1}{w_{2n-2}}$$
$$= \frac{w_{2n-2} + w_{n-1} - 1}{w_{2n-2}}.$$

Let n be odd. The maximum number of nonadjacent edges in W_n is equal to $\frac{n-1}{2}$. Therefore, the maximum matching in W_n and $v(W_n)$ are as follows:
$M = \{(4,5),(6,7),\ldots,(n-1,1),(2,n)\}$,

$$v(W_n) = v(C_{n-1}) - w_2 + w_{2n-2}$$
$$= \sum_{i=2}^{(n-1)/2} w_{2i} + w_{2n-2}. \quad (42)$$

Also, $R = \{2, n\}$ and

$$v(W_n[V \setminus R]) = v(C_{n-1}) - w_2. \quad (43)$$

Similar to claim 1, we can show that $R^* = \{n-1, n\}$. Hence,

$$v(W_n[V \setminus R^*]) = \sum_{i=1}^{(n-3)/2} w_{2i}. \quad (44)$$

From Eqs. (42), (43) and (44), we can drive the following equality:

$$e_{W_n} = \frac{\sum_{i=2}^{(n-1)/2} w_{2i} + w_{2n-2} - \sum_{i=1}^{(n-3)/2} w_{2i}}{w_{2n-2}}$$
$$= \frac{w_{2n-2} + w_{n-1} - w_2}{w_{2n-2}}$$
$$= \frac{w_{2n-2} + w_{n-1} - 2}{w_{2n-2}}.$$

The proof of the theorem is complete. ■

Theorem8. Let the weight of each edge in the complete graph K_n be $w > 0$. Then, $e_{K_n} = 1$, for every $n \geq 2$.

Proof. By Note 1 and since the weight of all edges is equal to w, we have:
$v(K_n[V \setminus R]) = v(K_{n-2})$,
$v(K_n[V \setminus R^*]) = v(K_{n-2})$.
By substituting the above relations into Eq. (1), we obtain e_{K_n} as follows:

$$e_{K_n} = \frac{v(K_n) - v(K_{n-2})}{v(K_n) - v(K_{n-2})} = 1. \quad ■$$

Let $m \geq 2$. Consider the graph $S_n = (V, E)$, depicted in Fig. 9. This graph is obtained by joining the vertices 1 in m path graph P_n.

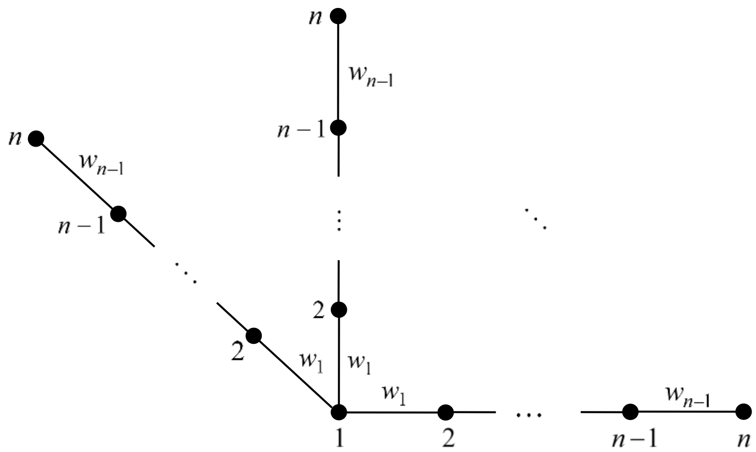

Figure 9: The graph S_n

Theorem 9. Let $w_1 < w_2 < \cdots < w_{n-1}$. Then, we have:

$$e_{S_n} = \begin{cases} 1 & n = 2, \\ 2 & n \geq 3. \end{cases} \tag{45}$$

Proof. Obviously, $e_{S_2} = 1$. Let $n \geq 3$. According to the maximum matching of the graph P_n in Theorem 1, $v(S_n)$ is obtained as follows:

$$v(S_n) = \begin{cases} mv(P_n) & \text{if } n \text{ is odd,} \\ mv(P_n) - (m-1)w_1 & \text{if } n \text{ is even.} \end{cases} \tag{46}$$

Therefore, $R = \{n-1, n\}$ such that $(n-1, n) \in E$. We immediately derive:

$$v(S_n[V \setminus R]) = \begin{cases} (m-1)v(P_n) + v(P_{n-2}) & \text{if } n \text{ is odd,} \\ (m-1)v(P_n) + v(P_{n-2}) - (m-1)w_1 & \text{if } n \text{ is even.} \end{cases} \tag{47}$$

Similar to Claim 1 in Theorem 1, we can show that the set R^* in the graph S_n is containing two vertices $n-1$ on two path graphs P_n. Hence,

$$v(S_n[V \setminus R^*]) = \begin{cases} 2v(P_{n-2}) + (m-2)v(P_n) & \text{if } n \text{ is odd,} \\ (m-2)v(P_n) + 2v(P_{n-2}) - (m-1)w_1 & \text{if } n \text{ is even.} \end{cases} \tag{48}$$

Using Eqs. (46), (47) and (48), e_{S_n} is obtained as follows:

$$e_{S_n} = \frac{2v(P_n) - 2v(P_{n-2})}{v(P_n) - v(P_{n-2})} = 2,$$

as desired. ■

Now, we want to generalize the result of Theorem 1. In other words, we remove B even vertices of the graph P_n and obtain $e_{P_n|B}$.

Theorem 10. Let $w_1 < \cdots < w_{n-1}$. Then,

$$e_{P_n|B} = \begin{cases} \dfrac{v(P_n) - v(P_{n-2B})}{v(P_n) - v(P_{n-B})} & \text{if } B < [n/2], \\ \dfrac{v(P_n)}{v(P_n) - v(P_{n-B})} & \text{if } [n/2] \leq B < n-1, \\ 1 & \text{if } B \geq n-1, \end{cases} \tag{49}$$

where $[x]$ is the greatest integer less than or equal to x.

Proof. By a similar argument as Theorem 1, we have:

$$v(P_n)=\begin{cases}\sum_{i=1}^{(n-1)/2} w_{2i} & \text{if } n \text{ is odd,}\\ \sum_{i=1}^{n/2} w_{2i-1} & \text{if } n \text{ is even.}\end{cases} \tag{50}$$

Notice that the following recurrence relation holds for $v(P_n)$:

$$\begin{cases}v(P_0)=0, v(P_1)=0,\\ v(P_n)=v(P_{n-2})+w_{n-1}, & \forall n\geq 2.\end{cases}$$

Now, there are two cases:

1. n is odd. Since $|M|=\frac{n-1}{2}$, then $v(P_n[V\setminus R_B])=0$ for $B\geq n-1$. Let $V_1=\{n-1,n-3,\ldots,4,2\}$.

The first B vertices of the set V_1 are in R_B^*. Since $|V_1|=\frac{n-1}{2}$, $v(P_n[V\setminus R_B^*])=0$ for $B\geq\frac{n-1}{2}$.

There are three subcases as follows:

I. $B<\frac{n-1}{2}$. In this case,

$$v(P_n[V\setminus R_B])=v(P_{n-B})=\sum_{i=1}^{(n-B-1)/2} w_{2i},$$

$$v(P_n[V\setminus R_B^*])=v(P_{n-2B})=\sum_{i=1}^{(n-2B-1)/2} w_{2i}.$$

Therefore, $e_{P_n|B}$ is obtained as follows:

$$e_{P_n|B}=\frac{v(P_n)-v(P_{n-2B})}{v(P_n)-v(P_{n-B})}=\frac{\sum_{i=(n-2B+1)/2}^{(n-1)/2} w_{2i}}{\sum_{i=(n-B+1)/2}^{(n-1)/2} w_{2i}}.$$

II. $\frac{n-1}{2}\leq B<n-1$. Therefore,

$$v(P_n[V\setminus R_B])=v(P_{n-B})=\sum_{i=1}^{(n-B-1)/2} w_{2i},$$

$$v(P_n[V\setminus R_B^*])=0.$$

By substituting the above relations into Eq. (1), we have:

$$e_{P_n|B}=\frac{v(P_n)}{v(P_n)-v(P_{n-B})}=\frac{\sum_{i=1}^{(n-1)/2} w_{2i}}{\sum_{i=(n-B+1)/2}^{(n-1)/2} w_{2i}}.$$

III. $B\geq n-1$. Obviously, $v(P_n[V\setminus R_B^*])=0$ and $v(P_n[V\setminus R_B])=0$. Hence, $e_{P_n|B}=1$.

2. n is even. Since $|M|=\frac{n}{2}$, then $v(P_n[V\setminus R_B])=0$ for $B\geq n$. Let $V_2=\{n-1,n-3,\ldots,3,1\}$.

The first B vertices of the set V_2 are in R_B^*. Since $|V_2|=\frac{n}{2}$, $v(P_n[V\setminus R_B^*])=0$ for $B\geq\frac{n}{2}$. There are three cases:

I. $B<\frac{n}{2}$. Now, we have:

$$v(P_n[V \setminus R_B]) = v(P_{n-B}) = \sum_{i=1}^{(n-B)/2} w_{2i-1},$$

$$v(P_n[V \setminus R_B^*]) = v(P_{n-2B}) = \sum_{i=1}^{(n-2B)/2} w_{2i-1}.$$

Therefore, $e_{P_n|B}$ is obtained as follows:

$$e_{P_n|B} = \frac{v(P_n) - v(P_{n-2B})}{v(P_n) - v(P_{n-B})} = \frac{\sum_{i=(n-2B+2)/2}^{n/2} w_{2i-1}}{\sum_{i=(n-B+2)/2}^{n/2} w_{2i-1}}.$$

II. $\frac{n}{2} \le B < n$. Then,

$$v(P_n[V \setminus R_B]) = v(P_{n-B}) = \sum_{i=1}^{(n-B)/2} w_{2i-1},$$

$$v(P_n[V \setminus R_B^*]) = 0.$$

By substituting the above relations into Eq. (1), we drive $e_{P_n|B}$ as follows:

$$e_{P_n|B} = \frac{v(P_n)}{v(P_n) - v(P_{n-B})} = \frac{\sum_{i=1}^{n/2} w_{2i-1}}{\sum_{i=(n-B+2)/2}^{n/2} w_{2i-1}}.$$

III. $B \ge n$. Obviously, $v(P_n[V \setminus R_B^*]) = 0$ and $v(P_n[V \setminus R_B]) = 0$. Therefore, $e_{P_n|B} = 1$.

This completes the proof. ■

Corollary 1. Consider the path graph P_n such that $w_i = i$,for every $1 \le i \le n-1$. Then, $e_{P_n|B}$ can be computed by the following formula:

$$e_{P_n|B} = \begin{cases} \frac{4n-4B}{2n-B} & \text{if } B < [n/2], \\ \frac{n^2-1}{2Bn-B^2} & \text{if } n \text{ is odd and } \frac{n-1}{2} \le B < n-1, \\ \frac{n^2}{2Bn-B^2} & \text{if } n \text{ is even and } \frac{n}{2} \le B < n, \\ 1 & \text{if } B \ge n-1. \end{cases}$$

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