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Boundary Optimal Control for Triple Nonlinear Hyperbolic Boundary Value Problem with State Constraints

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Abstract
The paper is concerned with the state and proof of the solvability theorem of unique state vector solution (SVS) of triple nonlinear hyperbolic boundary value problem (TNLHBVP), via utilizing the Galerkin method (GAM) with the Aubin theorem (AUTH), when the boundary control vector (BCV) is known. Solvability theorem of a boundary optimal control vector (BOCV) with equality and inequality state vector constraints (EINESVC) is proved. We studied the solvability theorem of a unique solution for the adjoint triple boundary value problem (ATHBVP) associated with TNLHBVP. The directional derivation (DRD) of the Hamiltonian (DRDH) is deduced. Finally, the necessary theorem (necessary conditions "NCOs") and the sufficient theorem (sufficient conditions" SCOs"), together denoted as NSCOs, for the optimality (OP) of the state constrained problem (SCP) are stated and proved.

Key words: Boundary optimal control vector, necessary condition, sufficient condition, directional derivative.

سيطرة حدودية مثلى لمسالة القيم الحدودية الزائدية غير الخطية الثلاثية مع قيود الحالة

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الخلاصة
يهتم هذا البحث نص وبرهان مبرهنة قابلية الحل الوحيد لمتجه الحالة لمسالة القيم الحدودية الزائدية غير الخطية الثلاثية باستخدام طريقة كالركن مع مبرهنة ابين عندما يكون متجه السيطرة الحدودية معلوما" , تم برهان مبرهنة قابلية الحل لسيطرة امثلية حدودية مع قيود التساوي والتباين . تمت دراسة قابلية الحل لمسالة القيم الحدودية المصاحبة لمسالة القيم الحدودية الزائدية غير الخطية الثلاثية .تم ايجاد الاشتقاق الاتجاهي لهمالتونيان. اخيرا تم كتابة نص وبرهان مبرهنتي الشروط الضرورية والكافية للمسالة .

1. Introduction

The problems of optimal control (OCPs) have a major significant and vital role in numerous fields, such as biology [1], electric power [2], robotics [3], economic [4], and many other different fields. This significance has motivated many investigators to be concerned with studding the OCPs for mathematical modules dominated by the three types of nonlinear PDEs; elliptic [5], hyperbolic [6] and parabolic [7], whilst many others [8-10] are concerned with studying the boundary OCPs (BOCPs).

In the latest years, numerous investigations were conducted about the BOCP dominated by the couple nonlinear BVPs (CNBVPs) of these three types, respectively, as indicated in [11-13]. Furthermore, many other investigations were performed about the BOCPs dominated by the nonlinear

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triple PDEs (TNBVPs) of elliptic and parabolic types [14-15]. All these investigations took our attention to think about generalizing the work in [12] for the BOCP dominated by CNBVPs into BOCP dominated by NTBVPs of a hyperbolic type (NTHBVPs). This includes the investigation of the solvability theorem for the SVS, the solvability theorem of a BOCV with the EINESVC, the derivation for the DRDH, and the demonstration theorems for both the NCOs and the SCOs of optimality.

This work starts with investigating the solvability theorem of the SVS of the NTHBVPS using the GAM when the BCV is given. Next, the solvability theorem of a BOCV dominated by the considered NTHBVPS with the EINESVC is demonstrated. The solvability theorem of the SVS of the Triple adjoint BVPs (ATHBVP) associated with the NTHBVPS is demonstrated. The DRDH is derived and, finally, the theorems of both the NCOs and SCOs of optimality of the SCP are demonstrated.

2. Description of the problem: Let $Q = \Omega \times I$, with Ω is open and bounded in $\mathbb{R}^3$, with "Lipschitz boundary" $\Gamma = \partial\Omega$, $I = [0,T]$, (with $T < \infty$) and $\Sigma = \Gamma \times I$. Then the NTHBVPS are given by:

$$y_{1tt} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(\alpha_{ij} \frac{\partial y_1}{\partial x_j})) + \beta_1 y_1 - \beta_4 y_2 - \beta_5 y_3 = h_1(y_1), \text{ in } Q \quad (1)$$

$$y_{2tt} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(\beta_{ij} \frac{\partial y_2}{\partial x_j}) + \beta_2 y_2 + \beta_4 y_1 + \beta_6 y_3 = h_2(y_2) , \text{ in } Q \quad (2)$$

$$y_{3tt} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(\gamma_{ij} \frac{\partial y_2}{\partial x_j}) + \beta_3 y_3 - \beta_6 y_2 + \beta_5 y_1 = h_3(y_3) , \text{ in } Q \quad (3)$$

$$\frac{\partial y_1}{\partial v_\alpha} = u_1(x,t), \text{ on } \Sigma \quad (4)$$

$$y_1(x,0) = y_1^0(x) , \quad \text{and } y_{1t}(x,0) = y_1^1(x) , \text{ on } \Omega \quad (5)$$

$$\frac{\partial y_2}{\partial v_\beta} = u_2(x,t) , \text{ on } \Sigma \quad (6)$$

$$y_2(x,0) = y_2^0(x) , \quad \text{and } y_{2t}(x,0) = y_2^1(x) , \text{ on } \Omega \quad (7)$$

$$\frac{\partial y_3}{\partial v_\gamma} = u_3(x,t) , \text{ on } \Sigma \quad (8)$$

$$y_3(x,0) = y_3^0(x) , \quad \text{and } y_{3t}(x,0) = y_3^1(x) , \text{ on } \Omega \quad (9)$$

where $\vec{y} = (y_1, y_2, y_3) \in (H^1(Q))^3 = \boldsymbol{H^1}(\mathbf{Q})$ is the SVS, $\vec{u} = (u_1, u_2, u_3) \in (L^2(\Sigma))^3 = \boldsymbol{L^2}(\boldsymbol{\Sigma})$ is the BCV, $(h_1, h_2, h_3) \in (L^2(Q))^3 = \boldsymbol{L^2}(\mathbf{Q})$ is a given "vector" function with $h_i(y_i) = h_i(x,t,y_i)$, $\alpha_{ij} = \alpha_{ij}(x,t)$,$\beta_{ij} = \beta_{ij}(x,t), \beta = \beta(x,t)$, $\gamma_{ij} = \gamma_{ij}(x,t)$, $\beta_i = \beta_i(x,t) \in C^\infty(Q)$, $\forall 1 \le i \le 6$, and each of v_α, v_β , v_γ is a normal unit vector to Σ.

The admissible set of the BCV is

$\vec{W}_A = \{\vec{u} \in \vec{U}_c = \boldsymbol{L^2}(\boldsymbol{\Sigma}) | \vec{u} \in \vec{U} \text{ a.e. in } \Sigma, J_1(\vec{u}) = 0 , J_2(\vec{u}) \le 0 \}$, $\vec{U} \subset \mathbb{R}^3$.

The objective function (OBF) (where $l = 0$) and the EINESVC (where $l = 1,2$) are

$$J_l(\vec{u}) = \sum_{i=1}^{3}[\int_Q p_{l1}(y_i)dxdt + \int_\Sigma q_{l1}(u_i)d\sigma], \quad (10)$$

where $\vec{y} = (y_1, y_2, y_3)$ is the SVS of (1-9), which corresponds to the BCV $\vec{u}$, $p_{li}(y_i) = p_{li}(x,t,y_i)$, and $q_{li}(u_i) = q_{li}(x,t,u_i)$, for $l = 0,1,2$ and $i = 1,2,3$, are given .

The BOCV is to find $\vec{u} \in \vec{W}_A$ such that $J_0(\vec{\vec{u}}) = \min_{\vec{u} \in \vec{W}_A} J_0(\vec{u})$.

Let $\vec{V} = V \times V \times V = \{\vec{v}: \vec{v} \in (H^1(\Omega))^3 = \boldsymbol{H^1}(\Omega)\}$, $\vec{v} = (v_1, v_2, v_3)$. We symbolize by (v_1,v_2)_Ω and ‖v‖_0 the inner product (IP) and the norm (NR) in L^2 (Ω), by (u,u)_Γ and ‖u‖_Γ IP and the NR in L^2 (Σ), by (v_1,v_2)_1 and ‖v‖_1, the IP and the NR in H^1 (Ω), by (v ⃗,v ⃗)_Ω and ‖v ⃗ ‖_0 the IP and the NR in L^2 (Ω), by (v ⃗,v ⃗)_Γ and ‖v‖_Γ the IP and the NR in i L^2 (Σ), by (v ⃗,v ⃗)_1=∑_(i=1)^3▒(v_i,v_i)_1 and ‖v ⃗ ‖_1^2=∑_(i=1)^3▒‖v_i ‖_1^2 the IP and the NR in V ⃗, and finally V ⃗^* is the dual of V ⃗.

The weak form (WKF) of problem (1-9) when $\vec{y} \in \boldsymbol{H^1}(\mathbf{Q})$ is given almost everywhere (a.e.) on I ($\forall v_1, v_{2,}, v_3 \in V, y_1(.,t), y_2(.,t), y_3(.,t) \in V$) by

$$\langle y_{1tt}, v_1 \rangle + \alpha_1(t, y_1, v_1) + (\beta_1 y_1, v_1)_\Omega - (\beta_4 y_2, v_1)_\Omega - (\beta_5 y_3, v_1)_\Omega = (h_1, v_1)_\Omega + (u_1, v_1)_\Gamma , \quad (11a)$$

$$(y_1^0, v_1)_\Omega = (y_1(0), v_1)_\Omega \quad \text{and} \quad (y_1^1, v_1)_\Omega = (y_{1t}(0), v_1)_\Omega , \quad (11b)$$

$$\langle y_{2tt}, v_2 \rangle + \alpha_2(t, y_2, v_2) + (\beta_2 y_2, v_2)_\Omega + (\beta_4 y_1, v_2)_\Omega + (\beta_6 y_3, v_2)_\Omega = (h_2, v_2)_\Omega + (u_2, v_2)_\Gamma \quad (12a)$$

$$(y_2^0, v_2)_\Omega = (y_2(0), v_2)_\Omega , \text{ and } (y_2^1, v_2)_\Omega = (y_{2t}(0), v_2)_\Omega \quad (12b)$$

$$\langle y_{3tt}, v_3 \rangle + \alpha_3(t, y_3, v_3) + (\beta_3 y_3, v_3)_\Omega - (\beta_6 y_3, v_3)_\Omega + (\beta_5 y_1, v_3)_\Omega = (h_3, v_3)_\Omega + (u_3, v_3)_\Gamma, \quad (13a)$$

$(y_3^0, v_3)_\Omega = (y_3(0), v_3)_\Omega$ and $(y_3^1, v_3)_\Omega = (y_{3t}(0), v_3)_\Omega$ (13b)
where $\alpha_1(t, y_1, v_1) = \int_\Omega \sum_{i,j=1}^n \alpha_{ij} \frac{\partial y_1}{\partial x_i}\frac{\partial v_1}{\partial x_j} dx$, $\alpha_2(t, y_2, v_2) = \int_\Omega \sum_{i,j=1}^n \beta_{ij} \frac{\partial y_2}{\partial x_i}\frac{\partial v_2}{\partial x_j} dx$ and

$\alpha_3(t, y_3, v_3) = \int_\Omega \sum_{i,j=1}^n \gamma_{ij} \frac{\partial y_2}{\partial x_i}\frac{\partial v_2}{\partial x_j} dx$.

Assumptions : *"Assum." (A)*

(i) h_i on $Q \times \mathbb{R}$ is of a Carathéodory type "CTHDT" , and
$|h_i(x, t, y_i)| \le \psi_i(x, t) + c_i|y_i|$, where $y_i \in \mathbb{R}$, $c_i > 0$ and $\psi_i(x, t) \in L^2(Q, \mathbb{R})$, for each $i = 1,2,3$

(ii) h_i, have a Lipschitz property (LIP) with respect to (w.r.t.) y_i, for each $i = 1,2,3$, i.e.
$|h_i(x, t, y_i) - h_i(x, t, \bar{y}_i)| \le L_i|y_i - \bar{y}_i|$, where$(x, t) \in Q$, $y_i, \bar{y}_i \in \mathbb{R}$ and $L_i > 0$.

(iii) $s(t, \vec{y}, \vec{v}) = \alpha_1(t, y_1, v_1) + (\beta_1 y_1, v_1)_\Omega + \alpha_2(t, y_2, v_2) + (\beta_2 y_2, v_2)_\Omega + \alpha_3(t, y_3, v_3) + (\beta_3 y_3, v_3)_\Omega$

$t(t, \vec{y}, \vec{v}) = s(t, \vec{y}, \vec{v}) - (\beta_4 y_2, v_1)_\Omega - (\beta_5 y_3, v_1)_\Omega + (\beta_4 y_1, v_2)_\Omega + (\beta_6 y_3, v_2)_\Omega - (\beta_6 y_3, v_3)_\Omega + (\beta_5 y_1, v_3)_\Omega$,

and $|s(t, \vec{y}, \vec{v})| \le a\|\vec{y}\|_1\|\vec{v}\|_1$, $s(t, \vec{y}, \vec{y}) \ge \bar{a}\|\vec{y}\|_1^2$, $|s_t(t, \vec{y}, \vec{v})| \le b\|\vec{v}\|_1$, $s_t(t, \vec{y}, \vec{y}) \ge \bar{b}\|\vec{y}\|_1^2$,
where $a, \bar{a}$, b, $\bar{b}$ are positive real constants.

Theorem 2.1 (The AUTH theorem)[16]: Assume that X_0, X , and X_1 are Banach spaces with $X_0 \subset X \subset X_1$, where the injections being continuous , X_i is reflexive for $i = 0,1$, and the injection of X_0 into X is compact. Let > 0 be a fixed finite number and let α_0, $\alpha_1$be two finite numbers such that $\alpha_i > 1$, $i = 0,1$. We consider the following "Banach space" $Y = \left\{v \in L^{\alpha_0}(0, T; X_0), \acute{v} = \frac{dv}{dt} \in L^{\alpha_1}(0, T; X_1)\right\}$ with the norm $\|v\|_Y = \left\{\|v\|^2_{L^{\alpha_0}(0,T;X_0)} + \|\acute{v}\|^2_{L^{\alpha_1}(0,T;X_1)}\right\}^{\frac{1}{2}}$, $\forall v \in Y$.
Then, the injection $\subset L^{\alpha_0}(0, T; X_0)$ is continuous and compact from Y into $L^{\alpha_0}(0, T; X_0)$.

Lemma 2.1[17]: Let $V, H, \acute{V}$ be three Hilbert spaces, where $\acute{V}$ is the dual of V. If a function u belongs to $L^2(0, T; V)$ and its derivative $\acute{u}$ belongs to $L^2\left(0, T; \acute{V}\right)$, then u is almost everywhere equal to a function continuous from $[0, T]$ into H and the following equality holds in the scalar distribution sense on $(0, T)$: $\frac{d}{dt}\|u\|^2 = 2\langle \acute{u}, u\rangle$.

Proposition 2.1[12]**:** Suppose that Ω is a measurable subset of $\mathbb{R}^d$ $(d = 2,3)$. Let $l: \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ is of a "Carathéodory type" that satisfies $\|L(x, y)\| \le \theta(x) + \phi(x)\|y\|^a$, for each $(x, y) \in \Omega \times \mathbb{R}^n$, where $y \in L^b(\Omega \times \mathbb{R}^n)$, $\theta(x) \in L^1(\Omega \times \mathbb{R})$, $\phi \in L^{\frac{b}{b-a}}(\Omega \times \mathbb{R})$, and $a \in [0, b]$,$a \in \mathbb{N}$, if $b \in [1, \infty)$ and $\phi \equiv 0$, if $b = \infty$. Then, the functional $L(y) = \int_\Omega l\left(x, y(x)\right)dx$ is continuous.

Theorem 2.2 [16]: Assume that Ω is a measure space with finite measure. Let (h_n) be a sequence of measurable functions on Ω, then $h_n(x) \to h(x)$ a.e. on Ω (with $|h(x)| < \infty$ a.e.).

Theorem 2.3 (The TKL Theorem) **[16]:** Let X be a vector space, Z a vector space with norm, U a nonempty convex subset of X, and K (with $K^\circ \ne \emptyset$) a convex and positive cone in Z. Let the functional $G_0: U \to \mathbb{R}$, $G_1: U \to \mathbb{R}^m$ $m \ge 0$, and $G_2: U \to Z$ be $(m + 1)$ −locally continuous and have $(m + 1)$ −derivatives at u where $m \ne 0$, and let them be K-linear at the point u where $m = 0$, the set of constraints is $W = \{u \in U | G_1(u) = 0, G_2(u) \in -K\}$. If $G_0(u)$ has a minimum at u in$W$, then there exists $\lambda_0 \in \mathbb{R}$, $\lambda_1 \in \mathbb{R}^m$, $\lambda_2 \in \mathbb{Z}^*$, with $\lambda_0 \ge 0$, $\lambda_2 \ge 0$, $\sum_{l=0}^2 |\lambda_l| = 1$, such that u satisfies $\forall w \in W$ in the following**:**
$\lambda_0 DG_0(u, w - u) + \lambda_1^T DG_1(u, w - u) + \langle \lambda_2, DG_2(u, w - u)\rangle \ge 0$, and $\langle \lambda_2, G_2(u)\rangle = 0$.

Main Results

3. Solvability of the SVS: In this section, we will test the existence of a unique vector solution for the WKF(11−13) when the BCV is given.

Theorem 3.1: With assums. (A), for any given BCV $\vec{u} \in \boldsymbol{L^2(Q)}$, the WKF(11−13) has a unique solution $\vec{y} = (y_1, y_2, y_3)$ with $\vec{y} \in \boldsymbol{L^2(I, V)} = (L^2(I, V))^3$ and $\vec{y}_t = (y_{1t}, y_{2t}, y_{3t}) \in \boldsymbol{L^2(I, V^*)}$.

Proof: Let $\vec{V}_n = V_n \times V_n \times V_n \subset \vec{V}$ (for each n) be the set of piecewise affine function on Ω. Let $\left\{\vec{V}_n\right\}_{n=1}^\infty$ be a sequence of subspaces of $\vec{V}$, such that $\forall\, \vec{v} = (v_1, v_2, v_3) \in \vec{V}$, there exists a sequence $\{\vec{v}_n\}$ with $\vec{v}_n = (v_{1n}, v_{2n}, v_{3n}) \in \vec{V}_n$,$\forall n$, and $\vec{v}_n \longrightarrow \vec{v}$ strongly in $\vec{V} \Rightarrow \vec{v}_n \longrightarrow \vec{v}$ strongly in $\left(L^2(\Omega)\right)^2$. Let $\left\{\vec{v}_j = \left(v_{1j}, v_{2j}, v_{3j}\right): j = 1,2, \dots, M(n)\right\}$ be a finite basis of $\vec{V}_n$ (where $\vec{v}_j$ is piecewise affine function on Ω) and let $\vec{y}_n = (y_{1n}, y_{2n}, y_{3n})$ be the Galerkin approximate solution (GAS) to the exact solution $\vec{y} = (y_1, y_2, y_3)$ s.t.

$$y_{in} = \sum_{j=1}^{n} c_{ij}(t)\, v_{ij}(x) \tag{14}$$

where $c_{ij}(t)$ is an unknown function of t, $\forall\, i = 1,2,3, j = 1,2,\dots,n$.

The WKF ((11)-(13)) is approximated w.r.t. x by using the GAM, replacing y_int=z_in , ∀i=1,2,3 in the obtained equations, they become ($\forall\, v_i \in V_n$) :

$$\langle z_{1nt}, v_1\rangle + \alpha_1(t, y_{1n}, v_1) + (\beta_1 y_{1n} - \beta_4 y_{2n} - \beta_5 y_{3n}, v_1)_\Omega = (h_1, v_1)_\Omega + (u_1, v_1)_\Gamma, \tag{15.a}$$

$$(y_{1n}^0, v_1) = (y_1^0, v_1) \quad \text{and } (y_{1n}^1, v_1) = (y_1^1, v_1) \;, \tag{15.b}$$

$$\langle z_{2nt}, v_2\rangle + \alpha_2(t, y_{2n}, v_2) + (\beta_2 y_{2n} + \beta_4 y_{1n} + \beta_6 y_{3n}, v_2)_\Omega = (h_2, v_2)_\Omega + (u_2, v_2)_\Gamma, \tag{16.a}$$

$$(y_{2n}^0, v_2) = (y_2^0, v_2) \quad \text{and } (y_{2n}^1, v_2) = (y_2^1, v_2), \tag{16.b}$$

$$\langle z_{3nt}, v_3\rangle + +\alpha_3(t, y_{3n}, v_3) + (\beta_3 y_{3n} - \beta_6 y_{2n} + \beta_5 y_{1n}, v_3)_\Omega = (h_3, v_3)_\Omega + (u_3, v_3)_\Gamma, \tag{17.a}$$

$$(y_{3n}^0, v_3) = (y_3^0, v_3) \quad \text{and } (y_{3n}^1, v_3) = (y_3^1, v_3) \tag{17.b}$$

where $y_{in}^0 = y_{in}^0(x) = y_{in}(x,0) \in V_n$ (respectively $z_{in}^0 = y_{in}^1 = y_{in}^1(x) = y_{int}(x,0) \in L^2(\Omega)$) is the projection of y_i^0 onto V(the projection of $y_i^1 = y_{it}$ onto $L^2(\Omega)$), $\forall i = 1,2,3$, i.e.

$$y_{in}^0 \longrightarrow y_i^0 \text{ strongly in } V \;, \text{ with } \|\vec{y}_n^0\|_{\mathbf{1}} \le b_0 \text{ and } \|\vec{y}_n^0\|_{\mathbf{0}} \le b_0 \tag{18}$$

$$y_{in}^1 \longrightarrow y_i^1 \text{ strongly in } L^2(\Omega) \text{ and } \|\vec{y}_n^1\|_{\mathbf{0}} \le b_1 \tag{19}$$

By replacing (14) with $i = 1,2,3$ in (15−17), respectively, and then setting $v_i = v_{il}$, $\forall\, l = 1,2,\dots,n$, then the obtained equations are equivalent to the following nonlinear system (NLS) of 1^{st} order ODEs with ICs (which has a unique solution), i.e.

$$A_1 C_1'(t) + B_1 C_1(t) - E C_2(t) - F C_3(t) = b_1 \tag{20.a}$$

$$A_1 C_1(0) = b_1^0 \text{ and } A_1 \bar{C}_1(0) = b_1^1 \tag{20.b}$$

$$A_2 C_2'(t) + B_2 C_2(t) + G C_3(t) + H C_1(t) = b_2 \tag{21.a}$$

$$A_2 C_2(0) = b_2^0 \text{ and } A_2 \bar{C}_2(0) = b_2^1 \tag{21.b}$$

$$A_3 C_3'(t) + B_3 C_3(t) + R C_1(t) - W C_2(t) = b_3 \tag{22.a}$$

$$A_3 C_3(0) = b_3^0 \quad \text{and} \quad A_3 \bar{C}_3(0) = b_3^1 \tag{22.b}$$

where $A_i = (a_{ilj})_{n\times n}$, $a_{ilj} = (v_{ij}, v_{il})_\Omega$, $B_i = (b_{ilj})_{n\times n}$, $b_{ilj}=[\alpha_l(t, v_{ij}, v_{il}) + (\beta_l(t) v_{ij}, v_{il})_\Omega]$, $E = (e_{lj})_{n\times n}$, $e_{lj} = (\beta_4 v_{2j}, v_{1l})_\Omega$, $F = (f_{lj})_{n\times n}$, $f_{lj} = (\beta_5 v_{3j}, v_{1l})_\Omega$, G = $(g_{lj})_{n\times n}$, $g_{lj} = (\beta_4 v_{3j}, v_{2l})_\Omega$, H = $(h_{lj})_{n\times n}$, $h_{lj} = (\beta_6 v_{1j}, v_{2l})_\Omega$, R= $(r_{lj})_{n\times n}$, $r_{lj} = (\beta_6 v_{1j}, v_{3l})_\Omega$, W = $(w_{lj})_{n\times n}$, $w_{lj} = (\beta_5 v_{2j}, v_{3l})$, $b_{il}^0 = (y_i^0, v_{il})_\Omega$, $b_i^0 = (b_{il}^0)$, $b_i= (b_{il})_{n\times 1}$, $b_{il} =(h_i, v_{il})_\Omega + (u_i, v_i)_\Gamma$, $C_i'(t) = (c_{ij}'(t))_{n\times 1}$, $C_i(t) = (c_{ij}(t))_{n\times 1}$, $\bar{C}_i(0) = (\bar{c}_{ij}(0))_{n\times 1}$ $C_i(0) = (c_{ij}(0))_{n\times 1}$, $\forall\, l$ = 1,2,3,…,n , i= 1,2,3.

Then there is a sequence of unique solutions $\{\vec{y}_n\}$ for the following approximation problems corresponding to the sequence$\{\vec{V}_n\}$, , i.e. for each $\vec{v}_n = (v_{1n}, v_{2n}, v_{3n}) \subset \vec{V}_n$, and $n = 1,2,\dots$

$$\langle y_{1ntt}, v_{1n}\rangle + \alpha_1(t, y_{1n}, v_{1n}) + (\beta_1 y_{1n} - \beta_4 y_{2n} - \beta_5 y_{3n}, v_{1n})_\Omega = (h_1(y_{1n}), v_{1n})_\Omega + (u_1, v_{1n})_\Gamma \tag{23a}$$

$$(y_{1n}^0, v_{1n})_\Omega = (y_1^0, v_{1n})_\Omega \quad \text{and} \quad (y_{1n}^1, v_{1n})_\Omega = (y_1^1, v_{1n})_\Omega \;, \tag{23b}$$

$$\langle y_{2ntt}, v_{2n}\rangle + \alpha_2(t, y_{2n}, v_{2n}) + (\beta_4 y_{1n} + \beta_2 y_{2n} + \beta_6 y_{3n}, v_{2n})_\Omega = (h_1(y_{1n}), v_{2n})_\Omega + (u_2, v_{2n})_\Gamma, \tag{24a}$$

$$(y_{2n}^0, v_{2n})_\Omega = (y_2^0, v_{2n})_\Omega \quad \text{and} \quad (y_{2n}^1, v_{2n})_\Omega = (y_2^1, v_{2n})_\Omega \;, \tag{24b}$$

$$\langle y_{3ntt}, v_{3n}\rangle + \alpha_3(t, y_{3n}, v_{3n}) + (\beta_5 y_{1n} + \beta_3 y_{3n} - \beta_6 y_{2n}, v_{3n})_\Omega = (h_1(y_{1n}), v_3)_\Omega + (u_3, v_{3n})_\Gamma, \tag{25a}$$

$$(y_{3n}^0, v_{3n})_\Omega = (y_3^0, v_{3n})_\Omega \quad \text{and} \quad (y_3^1,, v_{3n})_\Omega = (y_3^1, v_{3n})_\Omega \tag{25b}$$

Adding the obtained three equations after replacing $v_{in} = y_{int}$, for $i = 1,2,3$ in (23a,24a,25a), respectively, then applying Lemma 2.1 for the 1st term of the LHS, yield

$$\frac{d}{dt}[\|\vec{y}_{nt}(t)\|_0^2 + s(t, \vec{y}_n, \vec{y}_n)] - s_t(t, \vec{y}_n, \vec{y}_n) =$$
$$2((\beta_4 y_{2n} + \beta_5 y_{3n}, y_{1nt})_\Omega - (\beta_4 y_{1n} + \beta_6 y_{3n}, y_{2nt})_\Omega + (\beta_6 y_{2n} - \beta_5 y_{1n}, y_{3nt})_\Omega + (h_1(y_{1n}), y_{1nt})_\Omega$$
$$+(u_1, y_{1nt})_\Gamma + (h_2(y_{2n}), y_{2nt})_\Omega + (u_2, y_{2nt})_\Gamma + (h_3(y_{3n}), y_{3nt})_\Omega + (u_3, y_{3nt})_\Gamma) \tag{26}$$

Now, assum. (A-iii) can be applied for the 2nd term in the LHS of (26) after taking the absolute value for its both sides, then it becomes

$$\frac{d}{dt}[\|\vec{y}_{nt}\|_0^2 + \bar{a}\|\vec{y}_n\|_1^2] \le b\|\vec{y}_n\|_1^2 + 2(|(\beta_4 y_{2n}, y_{1nt})_\Omega| + |(\beta_5 y_{3n}, y_{1nt})_\Omega| + |(u_1, y_{1nt})_\Gamma| +$$
$$|(u_2, y_{2nt})_\Gamma|$$

$$|(h_1(y_{1n}), y_{1nt})_\Omega| + |(\beta_4 y_{1n}, y_{2nt})_\Omega| + |(\beta_6 y_{3n}, y_{2nt})_\Omega| + |(h_2(y_{2n}), y_{2nt})_\Omega|$$

$+|(h_3(y_{3n}),y_{3nt})_\Omega| + |(\beta_6 y_{2n},y_{3nt})_\Omega| + |(\beta_5 y_{1n},y_{3nt})_\Omega| + |(u_3,y_{3nt})_\Gamma|)$ (27)

Integrating both sides (IBS)of (27) on $[0,t]$, applying $\|y_{in}\|_0 \le \|y_{in}\|_1 \le \|\vec{y}_n\|_1$, $\|y_{int}\|_0 \le \|\vec{y}_{nt}\|_0$, $\|u_i\|_\Gamma \le \|\vec{u}\|_\Gamma$ and the trace theorem (TTH), and applying assum. (A-i) for the RHS of the resulting equation, give

$$\int_0^t \frac{d}{dt}[\|\vec{y}_{nt}(t)\|_0^2 + \bar{a}\|\vec{y}_n\|_1^2]dt \le c_9\int_0^t(\|\vec{y}_{nt}\|_0^2 + \|\vec{y}_n\|_1^2)\,dt + \sum_{i=1}^3\int_0^T(\|\psi_i\|_Q^2 + \|u_i\|_\Sigma^2)dt$$
$$\le c_{10} + c_9\int_0^t(\|\vec{y}_{nt}\|_0^2 + \bar{a}\|\vec{y}_n\|_1^2)\,dt \quad (28)$$

where $|\beta_i| \le c_i$, for $i = 4,5,6$, $\beta = 2\max(\beta_4,\beta_5,\beta_6)$, $c = \max(c_1,c_2,c_3)$. $c_7 = 2 + \beta + c$, with $\|\psi_i\|_Q^2 \le \bar{b}_i$, $\|u_i\|_\Sigma^2 \le \breve{b}_i$, for each $i = 1,2,3$, $c_{10} = \sum_{i=1}^3(\bar{b}_i + \breve{b}_i)$, $c_9 = \max(c_7,\frac{c_8}{\bar{a}})$, $c_8 = b + \beta + c$.

Since $\|\vec{y}_n^0\|_{\mathbf{1}} \le b_1$ and $\|\vec{y}_n^1\|_{\mathbf{0}} \le b_0$, with $c_0 = b_0 + b_1 + c_{10}$, the inequality (28) becomes

$$\|\vec{y}_{nt}(t)\|_0^2 + \bar{a}\|\vec{y}_n(t)\|_1^2 \le c_0 + c_9\int_0^t(\|\vec{y}_{nt}\|_0^2 + \bar{a}\|\vec{y}_n\|_1^2)\,dt$$

Applying the Belman-Gronwall inequality(BGI) gives

$$\|\vec{y}_{nt}(t)\|_0^2 + \bar{a}\|\vec{y}_n(t)\|_1^2 \le c_0 e^{c_9} = b^2(c) \Rightarrow \|\vec{y}_{nt}(t)\|_0^2 \le b^2(c) \text{ and } \|\vec{y}_n(t)\|_1^2 \le b^2(c), \forall\, t \in [0,T]$$

Easily, one can obtain that $\|\vec{y}_{nt}(t)\|_{\boldsymbol{Q}} \le b_1(c)$ and $\|\vec{y}_n(t)\|_{\boldsymbol{L^2(I,V)}} \le b(c)$.

Then, the Alauglu's theorem "ALGTH" can be utilized here, which leads to that there is a subsequence of $\{\vec{y}_n\}_{n\in N}$, let we say again "for simplicity" $\{\vec{y}_n\}_{n\in N}$ s.t $\vec{y}_{nt} \longrightarrow \vec{y}$ weakly in $\boldsymbol{L^2(Q)}$ and $\vec{y}_n \longrightarrow \vec{y}$ weakly in $\boldsymbol{L^2(I,V)}$, and since

$$\boldsymbol{L^2(I,V)} \subset \boldsymbol{L^2(Q)} \cong \boldsymbol{L^2(Q)}^* \subset \boldsymbol{L^2(R,V^*)} \quad (29)$$

hence, Theorem 2.1 can be utilized to get that $\vec{y}_n \longrightarrow \vec{y}$ strongly in $\boldsymbol{L^2(Q)}$.

Now, multiplying both sides "MBS" of (23a), (24a), (25a) by $\varphi_i(t) \in C^2[0,T]$, $\forall i = 1,2,3$, respectively, s.t. $\varphi_i(T) = \acute{\varphi}_i(T) = 0$, $\varphi_i(0) \ne 0$, $\acute{\varphi}_i(0) \ne 0$, integrating on $[0,T]$, and finally integrating by parts twice (IBP) the 1st term in the LHS of each one of the obtained three equations, yield

$$-\int_0^T \frac{d}{dt}(y_{1n},v_{1n})\acute{\varphi}_1(t)dt + \int_0^T[\,\alpha_1(t,y_{1n},v_{1n}) + (\beta_1 y_{1n} - \beta_4 y_{2n} - \beta_5 y_{3n},v_{1n})_\Omega\,]\varphi_1(t)dt$$
$$= \int_0^T[(h_1(y_{1n})_1,v_{1n})_\Omega + (u_1,v_{1n})_\Gamma]\varphi_1(t)dt + (y_{1n}^1,v_{1n})\varphi_1(0), \quad (30)$$
$$\int_0^T(y_{1n},v_{1n})\acute{\acute{\varphi}}_1(t)dt + \int_0^T[\,\alpha_1(t,y_{1n},v_{1n}) + (\beta_1 y_{1n} - \beta_4 y_{2n} - \beta_5 y_{3n},v_{1n})_\Omega\,]\varphi_1(t)dt$$
$$= \int_0^T[(h_1(y_{1n}),v_{1n})_\Omega + (u_1,v_{1n})_\Gamma]\varphi_1(t)dt + (y_{1n}^1,v_{1n})\varphi_1(0) + (y_{1n}^0,v_{1n})\acute{\varphi}_1(0) \quad (31)$$
$$-\int_0^T \frac{d}{dt}(y_{2n},v_{2n})\acute{\varphi}_2(t)dt + \int_0^T[\,\alpha_2(t,y_{2n},v_{2n}) + (\beta_4 y_{1n} + \beta_2 y_{2n} + \beta_6 y_{3n},v_{2n})_\Omega]\varphi_2(t)dt$$
$$= \int_0^T[(h_2(y_{2n}),v_{2n})_\Omega + (u_2,v_{2n})_\Gamma]\varphi_2(t)dt + (y_{2n}^1,v_{2n})\varphi_2(0), \quad (32)$$
$$\int_0^T(y_{2n},v_{2n})\,\acute{\acute{\varphi}}_2(t)dt + \int_0^T[\,\alpha_2(t,y_{2n},v_{2n}) + (\beta_4 y_{1n} + \beta_2 y_{2n} + \beta_6 y_{3n},v_{2n})_\Omega]\varphi_2(t)dt$$
$$= \int_0^T[(h_2(y_{2n}),v_{2n})_\Omega + (u_2,v_{2n})_\Gamma]\varphi_2(t)dt + (y_{2n}^1,v_{2n})\varphi_2(0) + (y_{2n}^0,v_{2n})\acute{\varphi}_2(0) \quad (33)$$
$$-\int_0^T \frac{d}{dt}(y_{3n},v_{3n})\acute{\varphi}_3(t)dt + \int_0^T[\,\alpha_3(t,y_{3n},v_{3n}) + (\beta_5 y_{1n} + \beta_3 y_{3n} - \beta_6 y_{2n},v_{3n})_\Omega]\varphi_3(t)dt$$
$$= \int_0^T[(h_3(y_{3n}),v_3)_\Omega + (u_3,v_{3n})_\Gamma]\varphi_3(t)dt + (y_{3n}^1,v_{3n})\varphi_3(0), \quad (34)$$
$$\int_0^T(y_{3n},v_{3n})\acute{\acute{\varphi}}_1(t)dt + \int_0^T[\,\alpha_3(t,y_{3n},v_{3n}) + (\beta_5 y_{1n} + \beta_3 y_{3n} - \beta_6 y_{2n},v_{3n})_\Omega]\varphi_3(t)dt$$
$$= \int_0^T[(h_3(y_{3n}),v_{3n})_\Omega + (u_3,v_{3n})_\Gamma]\varphi_3(t)dt + (y_{3n}^1,v_{3n})\varphi_3(0) + (y_{3n}^0,v_{3n})\acute{\varphi}_3(0) \quad (35)$$

First, since

$$\left.\begin{array}{l} v_{in} \longrightarrow v_i \;\text{ strongly in } V \;\Rightarrow \\[4em] v_{in} \longrightarrow v_i \;\text{ strongly in } L^2(\Omega) \Rightarrow \end{array}\right\{ \begin{array}{l} \left.\begin{cases} v_{in}\varphi_i(t) \longrightarrow v_i\varphi_i(\mathrm{t}) \\ v_{in}\acute{\varphi}_i(\mathrm{t}) \longrightarrow v_i\acute{\varphi}_i(\mathrm{t}) \end{cases}\right\} \text{strongly in } L^2(I,V) \\ \quad v_{in}\varphi_i(0) \to v_i\varphi_i(0) \quad \text{strongly in } L^2(\Omega) \\ \left.\begin{cases} v_{in}\acute{\varphi}_i(\mathrm{t}) \longrightarrow \acute{\varphi}_i(\mathrm{t}) \\ v_{in}\acute{\acute{\varphi}}_i(t) \longrightarrow v_i\acute{\acute{\varphi}}_i(t) \end{cases}\right\} \text{strongly in } L^2(Q) \\ \quad v_{in}\acute{\varphi}_i(0) \longrightarrow \acute{\varphi}_i(0) \quad \text{strongly in } L^2(\Omega) \end{array} \text{, for each } i = 1,2,3,$$

Second, $y_{int} \longrightarrow y_{it}$ weakly in $L^2(Q)$ and $y_{in} \longrightarrow y_i$ weakly in $L^2(I,V)$ and strongly in $L^2(Q)$.

Third, since $\eta_{in} = v_{in}\varphi_i \longrightarrow v_i\varphi_i = \eta_i$ strongly in $L^2(Q)$ and η_{in} is measurable w.r.t. (x,t), so using assumption (A-i), applying proposition 2.1, the integral $\int_Q h_i(x,t,y_{in})\eta_{in}dxdt$ is continuous w.r.t. (y_{in},η_{in}), then

$\int_0^T (h_i(y_{in}), u_{in})\zeta_i(t)dt \to \int_0^T (h_i(y_i), u_i)\zeta_i(t)dt$, $\forall i = 1,2,3$.
On the other hand, since $y_{in} \longrightarrow y_i$ in $L^2(\Sigma)$ from the TTH, then $\forall i = 1,2,3$
$\int_0^T (u_i, v_{in})_\Gamma \varphi_i(t)dt \to \int_0^T (u_i, v_i)_\Gamma \varphi_i(t)dt$
From these convergences, (18) and (19), we can passage the limits in (30-35) to get

$$-\int_0^T \frac{d}{dt}(y_1, v_1)\acute{\varphi}_1(t)dt + \int_0^T [\,\alpha_1(t, y_1, v_1) + (\beta_1 y_1 - \beta_4 y_2 - \beta_5 y_3, v_1)_\Omega\,]\varphi_1(t)dt$$
$$= \int_0^T [(h_1(y_1), v_1)_\Omega + (u_1, v_1)_\Gamma]\varphi_1(t)dt + (y_1^1, v_1)\varphi_1(0) \quad (34)$$
$$\int_0^T (y_1, v_1)\acute{\varphi}_1(t)dt + \int_0^T [\,\alpha_1(t, y_1, v_1) + (\beta_1 y_1 - \beta_4 y_2 - \beta_5 y_3, v_1)_\Omega\,]\varphi_1(t)dt$$
$$= \int_0^T [(h_1(y_1), v_1)_\Omega + (u_1, v_1)_\Gamma]\varphi_1(t)dt + (y_1^1, v_1)\varphi_1(0) + (y_1^0, v_1)\acute{\varphi}_1(0) \quad (35)$$
$$-\int_0^T \frac{d}{dt}(y_2, v_2)\acute{\varphi}_2(t)dt + \int_0^T [\,\alpha_2(t, y_2, v_2) + (\beta_4 y_1 + \beta_2 y_2 + \beta_6 y_3, v_2)_\Omega]\varphi_2(t)dt$$
$$= \int_0^T [(h_2(y_2), v_2)_\Omega + (u_2, v_2)_\Gamma]\varphi_2(t)dt + (y_2^1, v_2)\varphi_2(0), \quad (36)$$
$$\int_0^T (y_2, v_2)\,\acute{\varphi}_2(t)dt + \int_0^T [\,\alpha_2(t, y_2, v_2) + (\beta_4 y_1 + \beta_2 y_2 + \beta_6 y_3, v_2)_\Omega]\varphi_2(t)dt$$
$$= \int_0^T [(h_2(y_2), v_2)_\Omega + (u_2, v_2)_\Gamma]\varphi_2(t)dt + (y_2^1, v_2)\varphi_2(0) + (y_2^0, v_2)\acute{\varphi}_2(0) \quad (37)$$
$$-\int_0^T \frac{d}{dt}(y_3, v_3)\acute{\varphi}_3(t)dt + \int_0^T [\,\alpha_3(t, y_3, v_3) + (\beta_5 y_1 + \beta_3 y_3 - \beta_6 y_2, v_3)_\Omega]\varphi_3(t)dt$$
$$= \int_0^T [(h_3(y_3), v_3)_\Omega + (u_3, v_3)_\Gamma]\varphi_3(t)dt + (y_3^1, v_3)\varphi_3(0) \quad (38)$$
$$\int_0^T (y_3, v_3)\acute{\varphi}_1(t)dt + \int_0^T [\,\alpha_3(t, y_3, v_3) + (\beta_5 y_1 + \beta_3 y_3 - \beta_6 y_2, v_3)_\Omega]\varphi_3(t)dt$$
$$= \int_0^T [(h_3(y_3), v_3)_\Omega + (u_3, v_3)_\Gamma]\varphi_3(t)dt + (y_3^1, v_3)\varphi_3(0) + (y_3^0, v_3)\acute{\varphi}_3(0) \quad (39)$$

Case1: We choose $\varphi_i \in C^2[0,T]$, s.t. $\varphi_i(0) = \acute{\varphi}_i(0) = \varphi_i(T) = \acute{\varphi}_i(T) = 0$, $\forall i = 1,2,3$. in (35), (37), (39), IBP twice the 1^{st} terms in the LHS of each one of these three equation, to obtain

$$\int_0^T < y_{1tt}, v_1 > \varphi_1(t)dt + \int_0^T [\,\alpha_1(t, y_1, v_1) + (\beta_1 y_1 - \beta_4 y_2 - \beta_5 y_3, v_1)_\Omega\,]\varphi_1(t)dt$$
$$= \int_0^T [(h_1(y_1), v_1)_\Omega + (u_1, v_1)_\Gamma]\varphi_1(t)dt \quad (40)$$
$$\int_0^T < y_{2tt}, v_2 > \varphi_2(t)dt + \int_0^T [\,\alpha_2(t, y_2, v_2) + (\beta_2 y_2 + \beta_4 y_1 + \beta_6 y_3, v_2)_\Omega]\varphi_2(t)dt$$
$$= \int_0^T [(h_2(y_2), v_2)_\Omega + +(u_2, v_2)_\Gamma]\varphi_2(t)dt \quad (41)$$
$$\int_0^T < y_{3tt}, v_3 > \varphi_3(t)dt + \int_0^T [\,\alpha_3(t, y_3, v_3) + (\beta_3 y_3 - \beta_6 y_3 + \beta_5 y_1, v_3)_\Omega]\varphi_3(t)dt$$
$$= \int_0^T [(h_3(y_3), v_3)_\Omega + (u_3, v_3)_\Gamma]\varphi_3(t)dt \quad (42)$$

Which gives that $\vec{y}$ is a solution of ((11a) , (12a) , (13a)) a.e. on I.
Case2: By choosing $\varphi_i \in C^2[0,T]$, s.t. $\varphi_i(T) \neq 0$ & $\varphi_i(0) \neq 0$, $\forall i = 1,2,3$. MBS of (11a) , (12a), and (13a) by $\varphi_1(t)$, $\varphi_2(t)$ and $\varphi_3(t)$, respectively, and integrating on $[0,T]$ then IBP the 1^{st} term in the LHS of each one of these equations, then subtracting each one of these obtained equations from those correspond in (34) , (36) and (38) respectively, we obtain
$(y_i^1, v_i)\varphi_i(0) = (y_{it}(0), v_i)\varphi_i(0)$, $\forall i = 1,2,3$
Case3: By choosing $\varphi_i \in C^2[0,T]$, s.t. $\varphi_i(0) = \varphi_i(T) = \acute{\varphi}_i(T) = 0$, $\acute{\varphi}_i(0) \neq 0$, $\forall i = 1,2,3$. MBS of (11a), (12a), and (13a) by $\varphi_1(t)$, $\varphi_2(t)$ and $\varphi_3(t)$, respectively, and integrating on $[0,T]$, then IBP twice the 1^{st} term in the LHS of each one of these equations, then subtracting each one of these obtained equations from those correspond in (35) , (37), and (39), respectively, we have
$(y_i^0, v_i)\acute{\varphi}_i(0) = (y_i(0), v_i)\acute{\varphi}_i(0)$, $\forall i = 1,2,3$.
From Case2 and 3, one obtains the initial conditions (11b), (12b) & (13b).
To prove that $\vec{y}_n \longrightarrow \vec{y}$ strongly in $\boldsymbol{L^2(I, V)}$, we begin with integrating (26) on $[0,T]$, to get

$$\|\vec{y}_{nt}(T)\|_{\boldsymbol{Q}}^2 - \|\vec{y}_{nt}(0)\|_{\boldsymbol{Q}}^2 + s(t, \vec{y}_n, \vec{y}_n)(T) - s(t, \vec{y}_n, \vec{y}_n)(0) - \int_0^T s_t(t, \vec{y}_n, \vec{y}_n)dt$$
$$= \int_0^T [(43a) + (43b)]dt$$

(43) $(43a) = 2((\beta_4 y_{2n} + \beta_5 y_{3n}, y_{1nt})_\Omega - (\beta_4 y_{1n} + \beta_6 y_{3n}, y_{2nt})_\Omega + (\beta_6 y_{2n} - \beta_5 y_{1n}, y_{3nt})_\Omega$
$(43b) = 2((h_1(y_{1n}), y_{1nt})_\Omega + (u_1, y_{1nt})_\Gamma + (h_2(y_{2n}), y_{2nt})_\Omega + (u_2, y_{2nt})_\Gamma + (h_3(y_{3n}), y_{3nt})_\Omega + (u_3, y_{3nt})_\Gamma)$
The same steps utilized to obtain (26 & 43) can be also utilize here with $\vec{y}, \vec{y}_t$ instead of $\vec{y}_n, \vec{y}_{nt}$, i.e.

$$\|\vec{y}_t(T)\|_{\boldsymbol{Q}}^2 - \|\vec{y}_t(0)\|_{\boldsymbol{Q}}^2 + s(t, \vec{y}, \vec{y})(T) - s(t, \vec{y}, \vec{y})(0) - \int_0^T s_t(t, \vec{y}, \vec{y})dt = \int_0^T [(44a) + (44b)\,]dt \quad (44)$$

$(44a) = 2((\beta_4 y_2 + \beta_5 y_3, y_{1t})_\Omega - (\beta_4 y_1 + \beta_6 y_3, y_{2t})_\Omega + (\beta_6 y_2 - \beta_5 y_1, y_{3t})_\Omega$
$(44b) = 2((h_1(y_1), y_{1t})_\Omega + (u_1, y_{1t})_\Gamma + (h_2(y_2), y_{2t})_\Omega + (u_2, y_{2t})_\Gamma + (h_3(y_3), y_{3t})_\Omega + (u_3, y_{3t})_\Gamma)$
Since
$\|\vec{y}_{nt}(T) - \vec{y}_t(T)\|_0^2 - \|\vec{y}_{nt}(0) - \vec{y}_t(0)\|_0^2 + s(t, \vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})(T) - s(t, \vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})(0) - \int_0^T s_t(t, \vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})dt = (45a) - (45b) - (45c)$ (45)
where
$(45a) = \|\vec{y}_{nt}(T)\|_0^2 - \|\vec{y}_{nt}(0)\|_0^2 + s(t, \vec{y}_n, \vec{y}_n)(T) - s(t, \vec{y}_n, \vec{y}_n)(0) - \int_0^T s_t(t, \vec{y}_n, \vec{y}_n)dt$
$(45b) = (\vec{y}_{nt}(T), \vec{y}_t(T)) - (\vec{y}_{nt}(0), \vec{y}_t(0)) + s(t, \vec{y}_n, \vec{y})(T) - s(t, \vec{y}_n, \vec{y})(0) - \int_0^T s_t(t, \vec{y}_n, \vec{y})dt$
$(45c) = (\vec{y}_t(T), \vec{y}_{nt}(T) - \vec{y}_t(T)) - (\vec{y}_t(0), \vec{y}_{nt}(0) - \vec{y}_t(0)) + s(t, \vec{y}, \vec{y}_n - \vec{y})(T) - s(t, \vec{y}, \vec{y}_n - \vec{y})(0)$
$- \int_0^T s_t(t, \vec{y}, \vec{y}_n - \vec{y})dt$
Since $\vec{y}_n \longrightarrow \vec{y}$ strongly in $\boldsymbol{L^2(Q)}$, $\vec{y}_n \longrightarrow \vec{y}$ weakly in $\boldsymbol{L^2(I, V)}$ and $\vec{y}_{nt} \longrightarrow \vec{y}_t$ weakly in $\boldsymbol{L^2(Q)}$, then from $(43b)$ and the Assum. (A-i), the following is obtained
$\int_0^T (43b)dt = 2\int_0^T ((h_1(y_{1n}), y_{1n})_\Omega + (h_1(y_{2n}), y_{2n})_\Omega + (h_1(y_{3n}), y_{3n})_\Omega + (u_1, y_{1n})_\Gamma + (u_2, y_{2n})_\Gamma + (u_3, y_{3n})_\Gamma)dt \longrightarrow 2\int_0^T ((h_1(y_1), y_1)_\Omega + (h_1(y_2), y_2)_\Omega + (h_1(y_3), y_3)_\Omega) + (u_1, y_1)_\Gamma)dt + \int_0^T ((u_2, y_2)_\Gamma + (u_3, y_3)_\Gamma)\, dt = \int_0^T (44b)dt$
(43c)

Also, since $\vec{y}_n \longrightarrow \vec{y}$ strongly in $\boldsymbol{L^2(Q)}$ and $\vec{y}_{nt} \longrightarrow \vec{y}_t$ weakly in $\boldsymbol{L^2(Q)}$, and from (43c), we obtain
$(45a) = \int_0^T [(43a) + (43b)]dt \longrightarrow \int_0^T [(44a) + (44b)]dt$.
The same manner utilized to obtain (19) can be utilized also to obtain
$\vec{y}_{nt}(T) \to \vec{y}_t(T)$ strongly in $\boldsymbol{L(\Omega)^2}$. (46)
On the other hand, since $\vec{y}_n \longrightarrow \vec{y}$ weakly in $\boldsymbol{L^2(I, V)}$, then we use (19 & 46) to get
$(45b) \to \int_0^T [(44a) + (44b)]dt$
All the terms in (45c) imply to zero, as well as the 1st two terms in the LHS of (45), hence (45) gives
$\bar{a}\|\vec{y}_n - \vec{y}\|_1^2 \leq \int_0^T s_t(t, \vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})dt \to 0$ as $n \to \infty$, so we get that $\vec{y}_n \longrightarrow \vec{y}$ strongly in $\boldsymbol{L^2(I, V)}$.

Uniqueness of the solution: Let $\vec{y} = (y_1, y_2, y_2)$ and $\vec{\bar{y}} = (\bar{y}_1, \bar{y}_2, \bar{y}_2)$ be two solutions of the WKF (11-13). By subtracting each equation from the other, setting $v_i = (y_i - \bar{y}_i)_t$, for each $i = 1,2,3$, then adding the obtained equalities, using Lemma 2.1 for the 1st term in the L.H.S and assum. (A- ii) for the term in the RHS, it becomes
$\frac{d}{dt}\left[\|(\vec{y} - \vec{\bar{y}})_t\|_0^2 + s(t, \vec{y} - \vec{\bar{y}}, \vec{y} - \vec{\bar{y}})\right] \leq s_t(t, \vec{y} - \vec{\bar{y}}, \vec{y} - \vec{\bar{y}}) + L(\|(\vec{y} - \vec{\bar{y}})\|_{\boldsymbol{1}}^2 + \left\|(\vec{y} - \vec{\bar{y}})_t\right\|_{\boldsymbol{0}}^2)$
where $L = \max(L_1, L_2, L_3)$
IBS from 0 to t, considering the ICs, then utilizing the Assum. (A-iii) , we obtain
$\int_0^t \frac{d}{dt}\, [\|(\vec{y} - \vec{\bar{y}})_t(t)\|_{\boldsymbol{Q}}^2 + \bar{a}\|(\vec{y} - \vec{\bar{y}})\|_{\boldsymbol{1}}^2 \leq L_5 \int_0^t [\bar{a}\, \|(\vec{y} - \vec{\bar{y}})\|_{\boldsymbol{1}}^2\, dt + \left\|(\vec{y} - \vec{\bar{y}})_t\right\|_{\boldsymbol{0}}^2]dt$
where $L_4 = \mathrm{b} + \mathrm{L}$, $L_5 = \max(\frac{L_4}{\bar{a}}, L)$.
After utilizing the BGI on the above inequality, it becomes
$\|(\vec{y} - \vec{\bar{y}})_t(t)\|_{\boldsymbol{Q}}^2 + \bar{a}\|(\vec{y} - \vec{\bar{y}})\|_{\boldsymbol{1}}^2(t) \leq 0$, $\forall t \in I. \Rightarrow \|(\vec{y} - \vec{\bar{y}})(t)\|_{\boldsymbol{L^2(I,V)}} = 0$
Thus the solution is unique.

Lemma 3.1: In addition to assum. (A), if the BCV is bounded, then the operator $\vec{u} \longmapsto \vec{y}_{\vec{u}}$ from $\boldsymbol{L^2(\Sigma)}$ into $\boldsymbol{L^\infty}\left(\boldsymbol{I}, \boldsymbol{L^2}(\Omega)\right)$, into $\boldsymbol{L^2(I, V)}$, or into $\boldsymbol{L^2(Q)}$ is continuous.

Proof: $Let\ u^{\rightarrow} = (u\_1, u\_2, u\_3\ ), \bar{u}^{\rightarrow} = (\bar{u}\_1, \bar{u}\_2, \bar{u}\_3\ ) \in\ L\hat{}2\ (\Sigma). Set\ (\delta u)^{\rightarrow} = \bar{u}^{\rightarrow} - u^{\rightarrow}, then\ for\ \epsilon > 0\ , u^{\rightarrow}\_\varepsilon = u^{\rightarrow} + \varepsilon(\delta u)^{\rightarrow} \in L\hat{}2\ (\Sigma), Applying\ Theorem\ 3.1, y^{\rightarrow} = y^{\rightarrow}\_u^{\rightarrow} = (y\_1, y\_2, y\_3)\ and\ y^{\rightarrow}\_\varepsilon = y^{\rightarrow}\_(u^{\rightarrow}\_\varepsilon\ ) =$

$(y\_1\varepsilon, y\_2\varepsilon, y\_3\varepsilon)$ *are the SVS (corresponding to* $u^{\to}$ *and* $u^{\to}\_\varepsilon$ *resp.) which satisfy the WKF* $(10-11)$, *setting* $(\delta y)^{\to}\_\varepsilon = (\delta y\_1\varepsilon, ⟦\delta y⟧\_2\varepsilon, ⟦\delta y⟧\_3\varepsilon) = y^{\to}\_\varepsilon - y^{\to}$, *then* $(10-11)$, *give* $\langle \delta y_{1\varepsilon tt}, v_1\rangle + \alpha_1(t, \delta y_{1\varepsilon}, v_1) + (\beta_1 \delta y_{1\epsilon} - \beta_4 \delta y_{2\epsilon} - \beta_5 \delta y_{3\epsilon}, v_1)_\Omega =$

$((h_1(y_1 + \delta y_{1\varepsilon}) - h_1(y_1), v_1)_\Omega + (\varepsilon \delta u_1, v_1)_\Gamma$ (47a)

$\delta y_{1\varepsilon}(x,0) = 0$ and $\delta y_{1\varepsilon t}(x,0) = 0$ (47b)

$\langle \delta y_{2\varepsilon tt}, v_2\rangle + \alpha_2(t, \delta y_{2\varepsilon}, v_2) + (\beta_2 \delta y_{2\epsilon} + \beta_4 \delta y_{1\epsilon} + \beta_6 \delta y_{3\epsilon}, v_2)_\Omega =$

$(h_2(y_2 + \delta y_{2\varepsilon}) - h_2(y_2), v_2)_\Omega + (\varepsilon \delta u_2, v_2)_\Gamma$ (48a)

$\delta y_{2\varepsilon}(x,0) = 0$ and $y_{2\varepsilon t}(x,0) = 0,$ (48b)

$\langle \delta y_{3\varepsilon tt}, v_3\rangle + \alpha_3(t, \delta y_{3\epsilon}, v_3) + (\beta_3 \delta y_{3\epsilon} - \beta_6 \delta y_{3\epsilon} + \beta_5 \delta y_{1\epsilon}, v_3)_\Omega =$

$(h_3(y_3 + \delta y_{3\varepsilon}) - h_3(y_3), v_3)_\Omega + (\varepsilon \delta u_3, v_3)_\Gamma$ (49a)

$\delta y_{3\epsilon}(0) = 0$ and $\delta y_{3\epsilon t}(0) = 0$, $\forall\, v_3 \in V_3$ (49b)

By replacing $v_i = \delta y_{i\epsilon t}$ for $i = 1,2,3$ in (47a), (48a) & (49a), respectively, and adding these three equations, utilizing the same steps utilized to get (27), a similar equation can be obtained but with $\overrightarrow{\delta y_\epsilon}$ instead of $\vec{y}_n$. By utilizing assum. (A-iii) for the second term in the LHS of (26) and taking absolute value for both sides, then utilizing assum. (A-i) for the RHS of the obtained equation, we obtain

$\frac{d}{dt}\left[\left\|\overrightarrow{\delta y}_{\varepsilon t}(t)\right\|_0^2 + \bar{a}\left\|\overrightarrow{\delta y_\varepsilon}\right\|_1^2\right] \leq$

$b\left\|\overrightarrow{\delta y_\varepsilon}\right\|_1^2 + 2(|(\beta_4 \delta y_{2\varepsilon} + \beta_5 \delta y_{3\varepsilon}, \delta y_{1\varepsilon t})_\Omega| + |(\beta_4 \delta y_{1\varepsilon} + \beta_6 \delta y_{3\varepsilon}, \delta y_{2\varepsilon t})_\Omega| +$
$|(\beta_6 \delta y_{2\varepsilon} - \beta_5 \delta y_{1\varepsilon}, \delta y_{3\varepsilon})_\Omega| + L_1|(\delta y_{1\varepsilon}, \delta y_{1\varepsilon t})_\Omega| + |(\varepsilon \delta u_1, \delta y_{1\varepsilon t})_\Gamma| + L_2|(\delta y_{2\varepsilon}, \delta y_{2\varepsilon t})_\Omega| +$
$|(\varepsilon \delta u_2, \delta y_{2\varepsilon t})_\Gamma| + L_3|(\delta y_{3\varepsilon}, \delta y_{3\varepsilon t})_\Omega| + |(\varepsilon \delta u_3, \delta y_{3\varepsilon t})_\Gamma|$

IBS of the above equality on $[0,t]$, the definitions of the norms and the relations between them, and then using the TTH, we get

$$\left\|\overrightarrow{\delta y}_{\varepsilon t}(t)\right\|_0^2 + \bar{a}\left\|\overrightarrow{\delta y}_\varepsilon(t)\right\|_1^2 \leq b\int_0^t \left\|\overrightarrow{\delta y}_\varepsilon\right\|_1^2 dt + b_3\int_0^t (\left\|\overrightarrow{\delta y}_\varepsilon\right\|_0^2 + \left\|\overrightarrow{\delta y}_{\varepsilon t}\right\|_1^2)\, dt + \varepsilon\int_0^T \left\|\overrightarrow{\delta u}\right\|_\Gamma^2 dt$$
$$+\varepsilon\int_0^t \left\|\overrightarrow{\delta y}_{\varepsilon t}\right\|_\Gamma^2 dt + b_2\int_0^t (\left\|\overrightarrow{\delta y}_\varepsilon\right\|_0^2 + \left\|\overrightarrow{\delta y}_{\varepsilon t}\right\|_1^2)\, dt$$
$$\leq \int_0^t \left(b\left\|\overrightarrow{\delta y}_{\varepsilon t}\right\|_0^2 + b_3\left\|\overrightarrow{\delta y}_\varepsilon\right\|_1^2\right) dt + \varepsilon\left\|\overrightarrow{\delta u}(t)\right\|_\Sigma^2 + \int_0^t (b_4\left\|\overrightarrow{\delta y}_\varepsilon\right\|_1^2 +$$
$$b_2\left\|\overrightarrow{\delta y}_{\varepsilon t}\right\|_0^2)\, dt$$
$$\leq \varepsilon\left\|\overrightarrow{\delta u}(t)\right\|_\Sigma^2 + b_8\int_0^t (\left\|\overrightarrow{\delta y}_{\varepsilon t}\right\|_0^2 + \bar{a}\left\|\overrightarrow{\delta y}_\varepsilon\right\|_1^2)\, dt$$

where $|\beta_i| \leq c_i$ for $i = 4,5,6$, $b_1 = 2\max(c_4, c_5, c_6)$, $b_2 = 2\max(L_1, L_2, L_3)$, $b_3 = b + b_1$, $b_4 = \varepsilon + b_2$, $b_6 = b_3 + b_4$, $b_7 = b_2 + b$, $b_8 = \max(b_7, \frac{b_6}{\bar{a}})$.

Applying the BGI, with $L^2 = \varepsilon e^{b_8}$, gives

$\left\|\overrightarrow{\delta y}_{\varepsilon t}(t)\right\|_0^2 + \bar{a}\left\|\overrightarrow{\delta y}_\varepsilon(t)\right\|_1^2 \leq L^2\left\|\overrightarrow{\delta u}(t)\right\|_\Sigma^2$, $\forall t \in \bar{I} \Rightarrow \left\|\overrightarrow{\delta y}_\varepsilon(t)\right\|_1^2 \leq L^2\left\|\overrightarrow{\delta u}(t)\right\|_\Sigma^2$, $L^2 = \frac{L^2}{\bar{a}}$, $\forall t \in \bar{I} \Rightarrow$
$\left\|\overrightarrow{\delta y}_\varepsilon\right\|_{L^\infty(I, L^2(\Omega))} \leq L\left\|\overrightarrow{\delta u}\right\|_\Sigma$, $\left\|\overrightarrow{\delta y}_\varepsilon\right\|_{L^2(I,V)} \leq L\left\|\overrightarrow{\delta u}\right\|_\Sigma$ and $\left\|\overrightarrow{\delta y}_\varepsilon\right\|_Q \leq L\left\|\overrightarrow{\delta u}\right\|_\Sigma$

Form these three inequalities ,we obtain the continuity of the operator $\vec{u} \longmapsto \vec{y}$.

4. Solvability of BOCV: This section is concerned with the proof of the solvability theorem of BOCV which satisfies the EINESVC. The following assumption and lemma will be useful.

Assums. (B): Consider p_{li} and q_{li} ($\forall l = 0,1,2$ and $\forall i = 1,2,3$) are of CTHDT on $(Q \times \mathbb{R})$ and on$(\Sigma \times \mathbb{R})$, respectively, and satisfy the following, i.e.

$|p(x,t,y_i,w_i)| \leq P_{li}(x,t) + c_{li}y_i^2$,$|q_{li}(x,t,w_i)| \leq Q_{li}(x,t) + d_{li}(u_i)^2$,

where $y_i, u_i \in \mathbb{R}$ with $P_{li} \in L^1(Q)$,$Q_{li} \in L^1(\Sigma)$.

Lemma 4.1: With assums. (B) and $\forall l = 0,1,2$, the functional $\vec{u} \longmapsto J_l(\vec{u})$ is continuous on $\boldsymbol{L^2(\Sigma)}$.

Proof: The result is obtained through employing assums.(B) in proposition 2.1.

Theorem 4.1: In addition to the assums.(A&B), if the set $\vec{U}$ is convex and compact, $\vec{W}_A \neq \emptyset$, g_{1i} is independent of u_i for each $i = 1,2$, and p_{0i} and p_{2i} are convex w.r.t u_i for fixed (x,t,y_i) , then there exists a BOCV.

Proof: Since $\vec{W}_A \neq \emptyset$, then there is $\vec{\vec{u}} \in \vec{W}_A$ and a minimum sequence $\{\vec{u}_k\}$ with $\vec{u}_k \in \vec{W}_A$,$\forall k$, such that $\lim_{n\to\infty} J_0(\vec{u}_k) = \inf_{\vec{\vec{u}} \in \vec{U}_A} J_0(\vec{\vec{u}})$. By utilizing the hypotheses on $\vec{U}$ and the theorem 2.2, $\vec{U}_c$ is weakly compact. Then $\{\vec{u}_k\}$ has a subsequence, let us denote it again $\{\vec{u}_k\}$ for simplicity , for which $\vec{u}_k \longrightarrow \vec{u}$

weakly in $\vec{U}_c$ and $\|\vec{u}_k\|_\Sigma \le c$, $\forall k$. From theorem 3.1, for each $\vec{u}_k$, the WKF of the TNLHBVP has a unique SVS $\vec{y}_k = \vec{y}_{\vec{u}_k}$ and the norms $\|\vec{y}_k\|_{L^2(I,V)}$, $\|\vec{y}_{kt}\|_{L^2(Q)}$ are bounded. Then by ALGTH, there exists a subsequence of $\{\vec{y}_k\}$ and $\{\vec{y}_{kt}\}$, let us denote them again $\{\vec{y}_k\}$ and $\{\vec{y}_{kt}\}$, s.t. $\vec{y}_k \longrightarrow \vec{y}$ weakly in $\boldsymbol{L^2(I,V)}$, and $\vec{y}_{kt} \longrightarrow \vec{y}_t$ weakly in $\boldsymbol{L^2(Q)}$.

Then by utilizing theorem 2.1, there is a subsequence of $\{\vec{y}_k\}$, let us denote it again $\{\vec{y}_k\}$, s.t. $\vec{y}_k \longrightarrow \vec{y}$ strongly in $\boldsymbol{L^2(Q)}$.

Now, since for each k, $\vec{y}_k$ satisfies the WKF (11a),(12a) - (13a), then MBS of each of these equation by $\varphi_i(t)$, $\forall i = 1,2,3$, respectively, (with $\varphi_i \in C^2[0,T]$, s.t. $\varphi_i(T) = \acute{\varphi}_i(T) = 0$, $\varphi_i(0) \neq 0, \acute{\varphi}_i(0) \neq 0$, $\forall i = 1,2,3$), IBS from 0 to T, and finally IBP for these first terms, become

$$\int_0^T \frac{d}{dt}(y_{1kt}, v_1)\varphi_1(t)dt + \int_0^T [\alpha_1(t, y_{1k}, v_1) + (\beta_1 y_{1k} - \beta_4 y_{2k} - \beta_5 y_{3k}, v_1)_\Omega]\varphi_1(t)]td$$
$$= \int_0^T (h_1(y_{1k}), v_1)_\Omega \varphi_1(t)dt + \int_0^T (u_{1k}, v_1)_\Gamma \varphi_1(t)\, dt + (y_{1k}(0), v_1)_\Omega \varphi_1(0)$$
(49)

$$\int_0^T \frac{d}{dt}(y_{2kt}, v_2)\varphi_2(t)dt + \int_0^T [\alpha_2(t, y_{2k}, v_2) + (\beta_4 y_{1k} + \beta_2 y_{2k} + \beta_6 y_{3k}, v_2)_\Omega]\varphi_2(t)]dt$$
$$= \int_0^T (h_2(y_{2k}), v_2)_\Omega \varphi_2(t)dt + \int_0^T (u_{2k}, v_2)_\Gamma \varphi_2(t)\, dt + (y_{2k}(0), v_2)_\Omega \varphi_2(0)$$
(50)

$$\int_0^T \frac{d}{dt}(y_{3kt}, v_3)\varphi_3(t)dt + \int_0^T [\alpha_3(t, y_{3k}, v_3) + (\beta_5 y_{1k} + \beta_3 y_{3k} - \beta_6 y_{2k}, v_3)_\Omega]\varphi_3(t)]td$$
$$= \int_0^T (h_3(y_{3k}), v_3)_\Omega \varphi_3(t)dt + \int_0^T (u_{3k}, v_3)_\Gamma \varphi_3(t)\, dt + (y_{3k}(0), v_3)_\Omega \varphi_3(0)$$
(51)

In this point, we can utilize the same manner utilized in the proof of theorem 3.1 to passage the limits in the LHS of (49), (50), and (51), so it remains to passage the limits in the right hand RHS of these equations, which will be done as follows:

Let v_i∈C[Ω ̅] and w_i=v_i φ_i (t), ∀i=1,2,3. Then w_i∈C[Q ̅]⊂L^∞ (I,U)⊂L^2 (Q). Set h ̅_i1 (y_1k)=h_i1 (y_ik) w_i, then h ̅_i1:Q×R→R is of CTHDT. Now, utilizing proposition 2.1 to give that the integral ∫_Q^ ▒ 〖h_i1 (y_ik) w_i 〗 dxdt is continuous w.r.t. y_ik. But y_ik □(⟶)y_i strongly in L^2 (Q), therefore

$\int_Q h_{i1}(y_{1k})w_i\, dxdt \to \int_Q h_{i1}(y_i)w_i\, dxdt$,$\forall \eta_i \in C[\bar{Q}]$, for $i = 1,2,3$ (52a)

This result also holds for every $v_i \in V$,$\forall i = 1,2,3$, since $C(\bar{\Omega})$ is dense in V.

On the other hand, since $w_{ik} \longrightarrow w_i$ weakly in $L^2(\Sigma)$, then

$\int_\Sigma w_{ik}u_i dxdt \to \int_\Sigma w_i u_i\, dtxdt$, $\forall u_i \in C(\bar{\Omega})]$, for $i = 1,2,3$ (52b)

Hence, $\vec{y}$ is the SVS of the WKF (11a,12a&13a) $\forall v_i \in V$, a.e. on I.

Finally, to passage the limits in the ICs easily, one can utilize the same steps which are utilized in the proof of theorem 3.1 to get that $\vec{y}$ satisfies ICs (11b,12b&13b). Hence, $\vec{y}$ is the SVS of the WKF of the NLHBVP.

On the other hand, since $J_1(\vec{u}_k) = \sum_{i=1}^3 \int_Q p_{1i}(y_{ik})\, dxdt$ is continuous w.r.t. y_{ik} (for = 1,2,3), then by Lemma 4.1, $\int_Q p_{1i}(y_{ik})\, dxdt$ is continuous w.r.t.y_{ik}, but $\vec{y}_k \longrightarrow \vec{y}$ strongly in $\boldsymbol{L^2(Q)}$, then from proposition 2.1:

$J_1(\vec{u}) = \lim_{k\to\infty} J_1(\vec{u}_k) = 0.$

Again, since $\forall i = 1,2,3$ and $\forall l = 0,2$, $p_{li}(y_{ik})$ is continuous w.r.t. y_{ik}, then from the proof of Lemma 4.1, one gets

$\int_Q p_{li}(y_{ik})\, dxdt \longrightarrow \int_Q p_{li}(y_i)\, dxdt$
(53)

Now, from assums. (B), $q_{li}(u_i)$ is a weakly lower semi continuous w.r.t. u_i, $\forall i = 1,2,3$, and $l = 0,2$. Then from (53), one has

$$\int_Q p_{li}(y_i)\, dxdt + \int_\Sigma q_{li}(u_i)\, d\sigma \le \lim_{k\to\infty} \inf \int_\Sigma q_{li}(u_{ik})d\sigma + \int_Q p_{li}(y_i)\, dxdt =$$
$$\lim_{k\to\infty} \inf \int_\Sigma (q_{li}(u_{ik})d\sigma + \lim_{k\to\infty} \int_Q (p_{li}(y_i) - p_{li}(y_{ik}))dxdt + \lim_{k\to\infty} \int_Q p_{li}(y_{ik})\, dxdt$$
$$= \lim_{k\to\infty} \inf \int_\Sigma q_{li}(u_{ik})\, d\sigma + \lim_{k\to\infty} \inf \int_Q p_{li}(y_{ik})\, dxdt$$

i.e. $J_l(\vec{u}) \le \lim_{k\to\infty} \inf J_l(\vec{u}_k)$, (for each $l = 0,2$)

But $J_2(\vec{u}) \le 0$ (since $J_2(\vec{u}_k) \le 0$, $\forall k$), which means $\vec{u} \in \vec{W}_A$ and
$J_0(\vec{u}) \le \lim_{k\to\infty} \inf J_0(\vec{u}_k) = \lim_{k\to\infty} J_0(\vec{u}_k) = \inf_{\vec{\bar{u}}\in\vec{U}_A} J_0(\vec{\bar{u}}_k)$.
Hence, $\vec{u}$ is a BCV.

Assums. (C): If h_{iy_i}, $p_{l_i y_i}$, and $q_{l_i w_i}$, ($\forall l = 0,1,2$ and $\forall i = 1,2$), are of CTHDT on $Q \times (\mathbb{R})$, $Q \times (\mathbb{R})$, and $\Sigma \times (\mathbb{R})$, respectively, such that
$\left|h_{iy_i}(x,t,y_i)\right| \le \acute{L}_i$
$\left|p_{l_i y_i}(x,t,y_i,u_i)\right| \le K_{li}(x,t) + m_{li}|y_i|$, $\left|q_{l_i u_i}(x,t,y_i,w_i)\right| \le L_{li}(x,t) + n_{li}|y_i|$
where $(x,t) \in Q$, $y_i, u_i \in \mathbb{R}$, $K_{li}(x,t) \in L^2(Q)$ $L_{li}(x,t) \in L^2(\Sigma)$, , $\acute{L}_i, m_{li}, n_{li} \ge 0$.

Theorem 4. : By neglecting the indicator l in p_{li} , q_{li}, and $J_l$and considering the CFu $F_0(\vec{c})$ in (10), with the assums. (A), (B), and (C), the following ATHBVP $\vec{z} = (z_1, z_2, z_3)$ of the NTHBVPs (1-9) are given by:

$$z_{1tt} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(\alpha_{1ij} \frac{\partial z_1}{\partial x_j}\right) + \beta_1 z_1 + \beta_4 z_2 + \beta_5 z_3 = z_1 h_{1y_1}(y_1) + p_{1y_1}(y_1), \text{ in } Q \quad (54a)$$

$$\frac{\partial z_1}{\partial v_\alpha} = 0 , \quad \text{on } \Sigma, \quad z_1(x,T) = z_{1t}(x,T) = 0 \quad \text{on } \Omega , \quad (54b)$$

$$z_{2tt} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(\beta_{ij} \frac{\partial z_2}{\partial x_i}\right) + \beta_2 z_2 - \beta_4 z_1 - \beta_6 z_3 = z_2 h_{2y_2}(y_2) + p_{2y_2}(y_2), \text{ in } Q \quad (55a)$$

$$\frac{\partial z_2}{\partial v_\beta} = 0, \quad \text{on } \Sigma, \quad z_2(x,T) = z_{2t}(x,T) = 0 \quad \text{on } \Omega \quad (55b)$$

$$z_{3tt} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(\gamma_{ij} \frac{\partial z_3}{\partial x_j}\right) + \beta_3 z_3 - \beta_6 z_1 + \beta_5 z_2 = z_3 h_{2y_3}(y_3) + p_{2y_3}(3), \text{ in } Q \quad (56a)$$

$$\frac{\partial z_3}{\partial v_\gamma} = 0 \quad \text{on } \Sigma, \quad z_3(x,T) = z_{3t}(x,T) = 0 \quad \text{on } \Omega \quad (56b)$$

where each of v_α, v_β, and v_γ is a unit vector normal outer on the boundary Σ
and the "Hamiltonian" is defined by:
$\mathcal{H}(x,t,y_i,z_i,u_i) = \sum_{i=1}^{3}(z_i h_i(y_i) + p_i(y_i) + q_i(u_i))$
where $J(\vec{u}) = \sum_{i=1}^{3} \int_Q p_i(y_i)dxdt + \int_\Sigma q_i(u_i)d\gamma dt.$
Then for $\vec{u} \in \vec{U}$, the DRD of G is given by

$$DJ(\vec{u}, , \vec{\bar{u}} - \vec{u}) = \lim_{\varepsilon\to 0} \frac{J(\vec{u} + \varepsilon\overrightarrow{\delta u}) - J(\vec{u})}{\varepsilon} = \int_\Sigma \mathcal{H}_{\vec{u}}(x,t,\vec{y},\vec{z},\vec{u}).\delta\vec{u}\, d\sigma,$$

where $\mathcal{H}_{\vec{u}} = (z_1 + q_{1u_1}, z_2 + q_{2u_2}, z_3 + q_{3u_3})^T$ is the DRDH and $\overrightarrow{\delta u} = (\delta u_1, \delta u_2, \delta u_3)^T$.
Proof: At first, let the WKF of the ATHBVP be given as $\forall v_i \in V$, by

$$\langle z_{1tt}, v_1\rangle + \alpha_1(t, z_1, v_1) + (\beta_1 z_1 + \beta_4 z_2 + \beta_5 z_3, v_1)_\Omega = (z_1 h_{1y_1}, v_1)_\Omega + (p_{1y_1}, v_1)_\Omega \text{ , a.e. on } I \quad (57a)$$

$$(z_1(T), v_1)_\Omega = (z_{1t}(T), v_1)_\Omega = 0, \quad (57b)$$

$$\langle z_{2t}, v_2\rangle + \alpha_2(t, z_2, v_2) + (\beta_2 z_2 - \beta_4 z_1 - \beta_6 z_3, v_2)_\Omega = \left(z_2 h_{2y_2}, v_2\right)_\Omega + \left(p_{2y_2}, v_2\right)_\Omega, \text{ a.e. on } I \quad (58a)$$

$$(z_2(T), v_2)_\Omega = (z_{2t}(T), v_2)_\Omega = 0, \quad (58b)$$

$$\langle z_{3t}, v_3\rangle + \alpha_3(t, z_3, v_3) + (\beta_3 z_3 - \beta_6 z_1 + \beta_5 z_2, v_3)_\Omega = \left(z_3 h_{3y_3}, v_3\right)_\Omega + \left(p_{3y_3}, v_3\right)_\Omega, \text{ a.e. on } I \quad (59a)$$

$$(z_3(T), v_3)_\Omega = (z_{3t}(T), v_3)_\Omega = 0 \quad (59b)$$

From the given hypotheses and utilizing the same manner which is applied in the proof of theorem3.1, it can be proved that the WKF (57a, 58a & 59a) has a unique solution $\vec{z} = (z_1, z_2, z_2) \in \boldsymbol{L^2(Q)}$**.** By replacing $v_i = \delta y_{i\varepsilon}$ in (57a), (58a), and in (59a) for i=1,2,3 resp., then IBS on [0,T], yield to
∫_0^T▒〈δy_1ε,z_1tt 〉 dt+∫_0^T▒[α_1 (t,z_1,δy_1ε)_ +(β_1 z_1+β_4 z_2+β_5 z_3,δy_1ε)_Ω]dt=

$$\int_0^T [\left(z_1 h_{1y_1}, \delta y_{1\varepsilon}\right)_\Omega + \left(p_{1y_1}, \delta y_{1\varepsilon}\right)_\Omega]dt \quad (60)$$

$$\int_0^T \langle \delta y_{2\varepsilon}, z_{2tt}\rangle\, dt + \int_0^T [\alpha_2(t, z_2, \delta y_{2\varepsilon}) + (\beta_2 z_2 - \beta_4 z_1 - \beta_6 z_3, \delta y_{2\varepsilon})_\Omega]dt =$$
$$\int_0^T [\left(z_2 h_{2y_2}, \delta y_{2\varepsilon}\right)_\Omega + \left(p_{2y_2}, \delta y_{2\varepsilon}\right)_\Omega]dt \quad (61)$$

$$\int_0^T \langle \delta y_{3\varepsilon}, z_{3tt}\rangle\, dt + \int_0^T [\alpha_3(t, z_3, \delta y_{3\varepsilon}) + (\beta_3 z_3 - \beta_6 z_1 + \beta_5 z_2, \delta y_{3\varepsilon})_\Omega]dt =$$
$$\int_0^T [\left(z_3 h_{3y_3}, \delta y_{3\varepsilon}\right)_\Omega + \left(p_{3y_3}, \delta y_{3\varepsilon}\right)_\Omega]dt \quad (62)$$

Now, let $\vec{u}, \vec{\bar{u}} \in \boldsymbol{L^2(\Sigma)}$, $\overrightarrow{\delta u} = \vec{\bar{u}} - \vec{u}$ for $\varepsilon > 0$, $\vec{u}_\varepsilon = \vec{u} + \varepsilon\overrightarrow{\delta u} \in \boldsymbol{L^2(\Sigma)}$, then by theorem 3.1, their corresponding SVS are $\vec{y} = \vec{y}_{\vec{w}}$, and $\vec{y}_\varepsilon = \vec{y}_{\vec{u}_\varepsilon}$. By putting $\overrightarrow{\delta y}_\varepsilon = (\delta y_{1\varepsilon}, \delta y_{2\varepsilon}) = \vec{y}_\varepsilon - \vec{y}$ and setting $u_i = z_i$ for $i = 1,2,3$ in (47a) , (48a), and (49a), respectively, IBS on $[0,T]$, then the IBP is twice the first term in the LHS of each equation. By finding the DRD of h_i for $i = 1,2,3$ in the RHS of each equality (which exist from the assumptions(C)), then from the result of Lemma 3.1 and the "Minkowiski inequality", we obtain

$$\int_0^T \langle \delta y_{1\varepsilon}, z_{1tt}\rangle\, dt + \int_0^T [\, \alpha_1(t, \delta y_{1\varepsilon}, z_1) + (\beta_1 \delta y_{1\epsilon} - \beta_4 \delta y_{2\epsilon} - \beta_5 \delta y_{3\epsilon}, z_1)_\Omega] dt =$$
$$\int_0^T (\, h_{1y_1} \delta y_{1\varepsilon}, z_1)_\Omega dt + \int_0^T (\, \varepsilon \delta u_1, z_1)_\Gamma dt + O_{11}(\varepsilon) \quad (63)$$
$$\int_0^T \langle \delta y_{2\varepsilon}, z_{2tt}\rangle\, dt + \int_0^T [\, \alpha_2(t, \beta_2 \delta y_{2\epsilon} + \beta_4 \delta y_{1\epsilon} + \beta_6 \delta y_{3\epsilon}, z_2)] dt =$$
$$\int_0^T (\, h_{2y_2} \delta y_{2\varepsilon}, z_2)_\Omega dt + \int_0^T (\, \varepsilon \delta u_2, z_2)_\Gamma dt + O_{12}(\varepsilon) \quad (64)$$
$$\int_0^T \langle \delta y_{3\varepsilon}, z_{3tt}\rangle\, dt + \int_0^T [\, \alpha_3(t, \beta_3 \delta y_{3\epsilon} - \beta_6 \delta y_{3\epsilon} + \beta_5 \delta y_{1\epsilon}, z_3)] dt =$$
$$\int_0^T (\, h_{3y_3} \delta y_{3\varepsilon}, z_3)_\Omega dt + \int_0^T (\, \varepsilon \delta u_3, z_3)_\Gamma dt + O_{13}(\varepsilon) \quad (65)$$

where $O_{1i}(\varepsilon) \longrightarrow 0$, as $\varepsilon \longrightarrow 0$, with$O_{1i}(\varepsilon) = \|\delta y_{i\varepsilon}\|_Q$, for each $i = 1,2,3$

Then we subtract (63), (64), and (65) from (60), (61), and (62), respectively, and add each corresponding pair to obtain

$$\varepsilon \int_0^T \sum_{i=1}^3 (\delta u_i, z_i)_\Gamma\, dt + O_1(\varepsilon) = \int_0^T \sum_{i=1}^3 \big(p_{iy_i}, \delta y_{i\varepsilon}\big)\, dt \quad (66)$$

where $O_1(\varepsilon) = O_{11}(\varepsilon) + O_{12}(\varepsilon) + O_{13}(\varepsilon) \longrightarrow 0$, as $\varepsilon \longrightarrow 0$, with $O_1(\varepsilon) = \left\|\overrightarrow{\delta y_\varepsilon}\right\|_Q$

On the other hand, from the assumptions on p_i, q_i ($i = 1,2,3$), the definition of the DRD, and the result of Lemma 3.1, and then by using "Minkowiski inequality", one gets

$$J_0(\vec{u}_\varepsilon) - J_0(\vec{u}) = \sum_{i=1}^3 \big(\int_Q p_{iy_i} \delta y_{i\varepsilon} dx dt + \varepsilon \int_\Sigma q_{iu_i} \delta u_i\, d\gamma dt\big) + O_2(\varepsilon) \quad (67)$$

where $O_2(\varepsilon) = \left\|\overrightarrow{\delta y_\varepsilon}\right\|_Q + \varepsilon \left\|\overrightarrow{\delta u}\right\|_\Sigma$, $O_2(\varepsilon) \longrightarrow 0$, as $\varepsilon \longrightarrow 0$

Now, using (66) in (67) gives

$$J_0(\vec{u}_\varepsilon) - J_0(\vec{u}) = \varepsilon \sum_{i=1}^3 \int_\Sigma (z_i + q_{iu_i}) \delta u_i\, dx dt + O_3(\varepsilon)$$

where $O_3(\varepsilon) = O_1(\varepsilon) + O_2(\varepsilon) \longrightarrow 0$, as $\varepsilon \longrightarrow 0$, with $O_3(\varepsilon) = 2\left\|\overrightarrow{\delta y_\varepsilon}\right\|_Q + \varepsilon \left\|\overrightarrow{\delta u}\right\|_\Sigma$

Finally, the result is obtained after dividing both sides of this equality by ε, then taking the limit $\varepsilon \longrightarrow 0$, i.e.

$$DJ\big(\vec{u}, \vec{\vec{u}} - \vec{u}\big) = \int_\Sigma \mathcal{H}_{\vec{u}}\, . \overrightarrow{\delta u}\ d\sigma.$$

5. NCOs and SCOs for optimality: In this section, the NCOs and the SCOs theorems for OP under prescribed assumptions are found and proved as follows.

Theorem 5.1: (NCOs for Optimality)

a) With assums. (A), (B) , and (C), if $\vec{U}_c$ is convex, $\vec{u} \in \vec{W}_A$ is a BOCV, then there exist multipliers $\lambda_l \in \mathbb{R}$, $l = 0,1,2$ with $\lambda_0 \geq 0, \lambda_2 \geq 0$, $\sum_{l=0}^{2} |\lambda_l| = 1$, such that the following Kuhn-Tucker-Lagrange (TKL) conditions hold

$$\sum_{l=0}^{2} \lambda_l DJ_l\big(\vec{u}, , \vec{\vec{u}} - \vec{u}\big) \geq 0 \, , \forall \vec{\vec{u}} \in \vec{U} \quad , \quad (68a)$$

$$\lambda_2 J_2(\vec{u}) = 0 \quad \text{, (Transversality condition)} \quad (68b)$$

(b) The inequality (68a) is equivalent to

$$H_{\vec{u}}(x, t, \vec{y}, \vec{z}, \vec{u}). \vec{u}(t) = \min_{\vec{\vec{u}} \in \vec{U}} H_{\vec{u}}(x, t, \vec{y}, \vec{z}, \vec{u}). \vec{\vec{u}}(t) \ \text{ a.e. on } Q \quad (69)$$

where $H_{\vec{u}}(x, t, \vec{y}, \vec{z}, \vec{u})$ is defined as in theorem 3 above,

with $q_i = \sum_{l=0}^{2} \lambda_l q_{li}$ and $z_i = \sum_{l=0}^{2} \lambda_l z_{li}$, (for $i = 1,2$).

Proof: a) From Lemma 4.1, the functional $J_l(\vec{u})$ (for $= 0,1,2$) is continuous and, from theorem 4.2, the functional DJ_l (for $l = 0,1,2$) is continuous w.r.t. $\vec{\vec{u}} - \vec{u}$ and linear in $\vec{\vec{u}} - \vec{u}$. Then, DJ_l is M −differential for every M. Hence, by utilizing theorem 2.3, there exist multipliers $\lambda_l \in \mathbb{R}$, $l = 0,1,2$ with $\lambda_0 \geq 0, \lambda_2 \geq 0$, $\sum_{l=0}^{2} |\lambda_l| = 1$, such that (68a-b) hold. By utilizing theorem 4.2, (42a) gives

$\sum_{l=0}^{2} \int_\Sigma \sum_{i=1}^{2} \lambda_l \big(z_{li} + q_{liu_i}\big) \delta u_i d\gamma dt \geq 0$, which can be rewritten as

$$\int_\Sigma (\vec{z} + \vec{q}_{\vec{u}}). (\vec{\vec{u}} - \vec{u}) d\gamma dt \geq 0, \forall \vec{\vec{u}} \in \vec{U}\ , \quad (70)$$

where $\vec{z}+\vec{q}_{\vec{u}}=(z_1+q_{1u_1}, z_2+q_{2u_2}, z_3+q_{3u_3})$, with $q_i=\sum_{l=0}^{2}\lambda_l q_{li}$, $z_i=\sum_{l=0}^{2}\lambda_l z_{li}$, $\forall i=1,2,3$.

Now, let $\{\vec{\bar{u}}_k\}$ be a dense sequence in $\vec{U}$ and $q\subset Q$ be a measurable set with "*Lebesgue* measure μ" such that $\vec{\bar{u}}(x,t)=\begin{cases}\vec{u}_k(x,t) & , \ if\ (x,t)\in q\\ \vec{u}(x,t) & , \ if\ (x,t)\notin q\end{cases}$

Therefore, (70) becomes

$$\int_q(\vec{z}+\vec{q}_{\vec{u}}).(\vec{\bar{u}}-\vec{u})d\gamma dt\geq 0, \tag{70a}$$

or

$$(\vec{z}+\vec{q}_{\vec{u}}).(\vec{\bar{u}}-\vec{u})\geq 0 \text{ , a.e. on } \Sigma, \tag{70b}$$

which gives that (70b) holds on Σ/S_k,such that $(\mathrm{S}_k)=0$, $\forall k$, i.e. (70b) holds on $\Sigma/\bigcup_k \mathrm{S}_k$ with $\mu(\bigcup_k \mathrm{S}_k)=0$. But $\{\vec{\bar{u}}_k\}$ is dense in $\vec{U}$, therefore there exists $\vec{\bar{u}}\in\vec{U}$ such that

$(\vec{z}+\vec{q}_{\vec{u}}).(\vec{\bar{u}}-\vec{u})\geq 0$, a.e. on Σ, $\forall\vec{\bar{u}}\in\vec{U}$,

i.e. (70a) gives (70). The converse is clear.

Theorem 5.2: (SCOs for Op)

In Addition to the assums. (A), (B), and (C), suppose that $\vec{U}_c$ is convex, with $\vec{U}_c$ convex, and that h_i, p_{1i} (h_{1i}) are affine w.r.t. y_i $(\forall(x,t)\in Q)$ and u_i $(\forall(x,t)\in\Sigma)$. Suppose that p_{0i}, p_{2i} are convex w.r.t. y_i $(\forall(x,t)\in Q)$ and q_{0i}, q_{2i} are convex w.r.t. $u_i(\forall(x,t)\in\Sigma)$, $\forall i=1,2,3$. Then, the NCOs of theorem5.1 with $\lambda_0>0$ are also sufficient.

Proof: Assume that the TKL conditions hold by $\vec{u}\in\vec{W}_A$. Let $J(\vec{u})=\sum_{l=0}^{2}\lambda_l J_l(\vec{u})$, then from theorem 4.2, $DJ(\vec{u},\vec{\bar{u}}-\vec{u})=\sum_{l=0}^{2}\lambda_l\int_\Sigma\sum_{i=1}^{3}(z_{li}+q_{liui})\,\delta u_i\,dxdt\geq 0$.

Consider $h_1(x,t,y_1)=h_{11}(x,t)y_1+h_{12}(x,t)=h_{11}y_1+h_{12}$ and

$h_2(x,t,y_2,u_2)=h_{21}(x,t)y_2+h_{22}(x,t)=h_{21}y_2+h_{22}$

Let $\vec{u}=(u_1,u_2,u_3)$ and $\vec{\bar{u}}=(\bar{u}_1,\bar{u}_2,\bar{u}_3)$ be two given BCV, then $\vec{y}=(y_{u1},y_{u2},y_{u3})=(y_1,y_2,y_3)$ and $\vec{\bar{y}}=(\bar{y}_{\bar{u}1},\bar{y}_{\bar{u}2},\bar{y}_{\bar{u}3})=(\bar{y}_1,\bar{y}_2,\bar{y}_3)$ are their corresponding SVS. By MBS of (1-9) by $\gamma\in[0,1]$ once, and once again by $\gamma_1=(1-\gamma)$ after replacing $\vec{u}$ and $\vec{y}$ by $\vec{\bar{u}}$ and $\vec{\bar{y}}$, respectively, in (1-9), then finally adding each resulting pair of equations together, we obtain:

$$\tilde{y}_{1tt}-\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}(\alpha_{ij}\frac{\partial\tilde{y}_1}{\partial x_j})+\beta_1\tilde{y}_1-\beta_4\tilde{y}_2-\beta_5\tilde{y}_3=h_{11}(\tilde{y}_1)+h_{12} \tag{71a}$$

$$\frac{\partial\tilde{y}_1}{\partial n_\alpha}=\tilde{u}_1,\ \text{on }\Sigma \tag{71b}$$

$$\tilde{y}_1(x,0)=y_1^0(x),\quad \tilde{y}_{1t}(x,0)=y_1^1(x) \tag{71c}$$

$$\tilde{y}_{2tt}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(\beta_{ij}\frac{\partial\tilde{y}_2}{\partial x_j})+\beta_4\tilde{y}_1+\beta_2\tilde{y}_2+\beta_6\tilde{y}_3=h_{21}(\tilde{y}_2)+h_{22} \tag{72a}$$

$$\frac{\partial\tilde{y}_2}{\partial\, n_\beta}=\tilde{u}_2,\ \text{on }\Sigma \tag{72b}$$

$$\tilde{y}_2(x,0)=y_2^0(x),\ \tilde{y}_{2t}(x,0)=y_2^1(x) \tag{72c}$$

$$\tilde{y}_{3tt}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(\gamma_{ij}\frac{\partial\tilde{y}_3}{\partial x_j})+\beta_5\tilde{y}_1-\beta_6\tilde{y}_2+\beta_3\tilde{y}_3=h_{31}(\tilde{y}_3)+h_{32} \tag{73a}$$

$$\frac{\partial\tilde{y}_3}{\partial\, n_\gamma}=\tilde{u}_3,\ \text{on }\Sigma \tag{73b}$$

$$\tilde{y}_3(x,0)=y_3^0(x),\ \tilde{y}_{3t}(x,0)=y_3^1(x) \tag{73c}$$

Equations (71), (72), and (73) show that if the BCV is $\vec{\tilde{u}}=(\tilde{u}_1,\tilde{u}_2,\tilde{u}_3)$ with $\vec{\tilde{u}}=\gamma\vec{u}+\gamma_1\vec{\bar{u}}$, then its corresponding SVS is $\vec{\tilde{y}}=(\tilde{y}_1,\tilde{y}_2,\tilde{y}_3)$ with $\tilde{y}_i=y_{i\tilde{u}_i}=y_{i(\gamma u_i+\gamma_1\bar{u}_i)}=\gamma y_i+\gamma_1\bar{y}_i$, $\forall i=1,2,3$. Thus the operator $\vec{u}\longmapsto\vec{y}_{\vec{u}}$ is convex linear (CL) w.r.t. $(\vec{y},\vec{u})$, $\forall(x,t)\in Q$.

On the other hand, the function $J_1(\vec{u})$ is CL w.r.t $(\vec{y},\vec{u})$, $\forall(x,t)\in Q$ (since the sum of two affine functions is affine and the operator $\vec{u}\longmapsto\vec{y}_{\vec{u}}$ is CL). The functions $J_0(\vec{u})$, $J_2(\vec{u})$ are convex w.r.t. $(\vec{y},\vec{u})$, for each $(x,t)\in Q$ (from the assumptions on the functions p_{li} , q_{li} and since the sum of two integrals of convex function is also convex).

Hence $J(\vec{u})$ is convex w.r.t. $(\vec{y},\vec{u})$, $\forall(x,t)\in Q$ in the convex set $\vec{U}$, and has a continuous DRD that satisfies

$DJ(\vec{u},,\vec{\bar{u}}-\vec{u})\geq 0\Rightarrow J(\vec{u})$ and has a minimum at $\vec{u}\Rightarrow J(\vec{u})\leq J(\vec{\bar{u}})$, $\forall\vec{\bar{u}}\in\vec{U}$, or

$\sum_{l=0}^{2} \lambda_l J_l(\vec{u}) \leq \sum_{l=0}^{2} \lambda_l J_l(\vec{\vec{u}}),\ \forall \vec{\vec{u}} \in \vec{U}.$
Let $\vec{\vec{u}} \in \vec{W}_A$, but $\lambda_2 \geq 0$, then from (68b), this inequality gives
$\lambda_0 J_0(\vec{u}) \leq \lambda_0 J_0(\vec{\vec{u}})$, $\forall \vec{\vec{u}} \in \vec{U} \Rightarrow\ J_0(\vec{u}) \leq J_0(\vec{\vec{u}}),\ \ \forall \vec{\vec{u}} \in \vec{U} \Rightarrow \therefore \vec{u}$ is a BOCV.

Conclusions: The solvability theorem for the SVS of the TNLHBVP when the BCV is given, utilizing the GAM with the AUTH, is proved successfully. The solvability theorem (existence theorem) of a BOCV governed by the TNLHBVP with EINESVC is proved. The solvability solution of the ATHBVP associated with the TNLHBVP is studied. The DRDH is derived. The theorems of the NCOs and the SCOs for the optimality of the constrained problem are generalized and proved.

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