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The Classical Continuous Mixed Optimal Control of Couple Nonlinear Parabolic Partial Differential Equations with State Constraints

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Abstract
In this work, the classical continuous mixed optimal control vector (CCMOPCV) problem of couple nonlinear partial differential equations of parabolic (CNLPPDEs) type with state constraints (STCO) is studied. The existence and uniqueness theorem (EXUNTh) of the state vector solution (SVES) of the CNLPPDEs for a given CCMCV is demonstrated via the method of Galerkin (MGA). The EXUNTh of the CCMOPCV ruled with the CNLPPDEs is proved. The Frechet derivative (FÉDE) is obtained. Finally, both the necessary and the sufficient theorem conditions for optimality (NOPC and SOPC) of the CCMOPCV with state constraints (STCOs) are proved through using the Kuhn-Tucker-Lagrange (KUTULA) multipliers theorem (KUTULATH).

Keyword: Mixed Classical Optimal Control, Frechet Derivative, Necessary and Sufficient Conditions for Optimality

مسألة مزيج السيطرة الامثلية التقليدية المستمرة لزوج من المعادلات التفاضلية المكافئة الغير خطية مع قيود الحالة

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الخلاصة
في هذا العمل تم دراسة مسالة مزيج السيطرة الامثلية التقليدية المستمرة لزوج من المعادلات التفاضلية المكافئة غير الخطية مع قيود الحالة , تم برهان مبرهنة وجود ووحدانية الحل لمتجه الحالة باتستخدام طريقة كاليركن عندما يكون متجه مزيج السيطرة معلوما", تم برهان مبرهنة وجود متجه مزيج سيطرة امثلية تقليدية مستمرة مسيطر بواسطة زوج المعادلات التفاضلية المكافئة غير الخطية، تم ايجاد مشتقة فريشيه، تم برهان مبرهنتي الشروط الضرورية والكافية للسيطرة لمزيج السيطرة الامثلية التقليدية مع وجود قيود الحالة باستخدام مبرهنة كهان –تاكر لاكرانج.

1. Introduction

The optimal control problem (OPCPR) is one of the important topics in applied mathematics and in several areas related to it, such as biology, economics, ecology, engineering, finance, management, medicine and many others. The associated mathematical models are formulated

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for example, as ordinary or partial systems [1]. The OPCPR of partial differential equations with state constraints have been intensively studied since the eighties starting with the work by Bonnans [2] and Abergel and Temam [3]. Later, from 2014 to 2016, the classical continuous optimal control problems (CCOPC) of coupled nonlinear partial equations (CNLPDEs) of hyperbolic, elliptic and parabolic types of equations were studied in [4], [5] and [6] respectively. While during 2017-2019, the classical continuous boundary optimal control problem of CNLPDEs of elliptic, hyperbolic, and parabolic type were studied in [7], [8] and [9] respectively.

In this paper, the EXUNTH of the SVES for the CNLPDEs of parabolic type (CNLPPDEs) for a given CCMCV is demonstrated. The theorem of existence of a CCMOPCV ruled by a CNLPPDEs type is demonstrated. Also the derivation of the FÉDE is achieved and the EXUNTH of the vector adjoint solution of the adjoint equations ADVEQ related to the SVES is studied. The KUTULATH are developed and utilized to demonstrate both the NOPC and the SOPC theorems of the CCMOPCV with STCOs.

2. Description of the problem

Let $I = \{t: 0 < t < T\}, T < \infty, \Omega \subset \mathbb{R}^2$ be a bounded open region with $\Gamma = \partial\Omega$, then the CNLPPDEs is:

$$s_{1t} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial s_1}{\partial x_j}) + k_1(x,t)s_1 - k(x,t)s_2 = \mathcal{F}_1(x,t,s_1) \quad \text{in} \quad Q = \Omega \times I \tag{1}$$

$$s_{2t} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(b_{ij}(x,t)\frac{\partial s_2}{\partial x_j}) + k_2(x,t)s_2 + k(x,t)s_1 = \mathcal{F}_2(x,t,s_2,\mu_2) \text{ in } Q = \Omega \times I \tag{2}$$

$$\frac{\partial s_1}{\partial n} = \sum_{i,j=1}^{2} a_{ij}(x,t)\frac{\partial s_1}{\partial x_j}\cos\left(n_1, x_j\right) = \mu_1(x,t) \quad \text{on } \Sigma = \Gamma \times I \tag{3}$$

$$s_1(x,0) = s_{01}(x), \quad \text{in } \Omega \tag{4}$$

$$s_2(x,t) = 0, \quad \text{on } \Sigma = \Gamma \times I \tag{5}$$

$$s_2(x,0) = s_{02}(x) \quad \text{in } \Omega \tag{6}$$

where n_1 is a normal vector on Σ, $x = (x_1, x_2) \in \Omega$, $(s_1, s_2) = (s_1(\mathrm{x,t}), s_2(\mathrm{x,t})) \in \left(H^1(\mathrm{Q})\right)^2$ is the STVES, and μ ⃗=(μ_1 (x,t),μ_2 (x,t))∈〖L^2 (Σ)×L^2 (Q)〗^ is the CCMCV, (F_1 (x,t,s_1),F_2 (x,t,s_2 〖,μ〗_2))∈(L^2 (Q))^2, $a_{ij}(x,t)$, $b_{ij}(x,t), k(x,t)$, $k_1(x,t)$ and $k_2(x,t) \in C^\infty(Q)$.

The set of the CCMCV is

$$\vec{\mu} \in \vec{\mathcal{N}} = \{ \vec{\mu} = (\mu_1, \mu_2) \in L^2(\Sigma) \times L^2(Q) | \vec{\mu} \in \vec{\mathcal{U}}, \text{ a.e. in Q}\}, \text{ with } \vec{\mathcal{U}} \text{ is convex.} \tag{7}$$

Let $\vec{V} = V_1 \times V_2 = \{\vec{v}: \vec{v} = (v_1(\mathrm{x}), v_2(\mathrm{x})) \in H^1(\Omega) \times H_0^1(\Omega)\}$. Let the set of admissible CCMCV be

$\vec{\mathcal{N}}_A = \{\vec{\mu} \in \vec{\mathcal{N}} | \mathcal{H}_1(\vec{\mu}) = 0, \mathcal{H}_2(\vec{\mu}) \le 0\}$, where the cost functions (CF) and the STCOS are given respectively by

$$\mathcal{H}_0(\vec{\mu}) = \int_Q g_{01}(x,t,s_1)dxdt + \int_Q g_{02}(x,t,s_2,\mu_2)dxdt + \int_\Sigma h_{01}(x,t,\mu_1)\,d\sigma \tag{8}$$

$$\mathcal{H}_1(\vec{\mu}) = \int_Q g_{11}(x,t,s_1)dxdt + \int_Q g_{12}(x,t,s_2,\mu_2)dxdt + \int_\Sigma h_{11}(x,t,\mu_1)\,d\sigma = 0 \tag{9}$$

$$\mathcal{H}_2(\vec{\mu}) = \int_Q g_{21}(x,t,s_1)dxdt + \int_Q g_{22}(x,t,s_2,\mu_2)dxdt + \int_\Sigma h_{21}(x,t,\mu_1)\,d\sigma \le 0 \tag{10}$$

Lemma 2.1[10]: Let $A, B, \acute{A}$ be three Hilbert spaces. If a function f and its derivative $\acute{f}$ belong to $L_2(0,T;A)$ and $L_2(0,T;\acute{A})$, then f is a.e. equal to a continuous function from $[0,T]$ into B and satisfies: $\frac{d}{dt}\|f\|^2 = 2\langle \acute{f}, f\rangle$.

Proposition 2.1[11]**:** Suppose that $\mathrm{W} \subset R^2$. Let $k: \mathrm{W} \times R^n \to R^m$ be of a Carathéodory type, that satisfies $\|K(u,v)\| \le \varrho(u) + \vartheta(u)\|v\|^\gamma$, $\forall (u,v) \in \mathrm{W} \times R^n$, where v∈L_d (W×R^n), ϱ(x)∈L_1 (W× R), ϑ∈L^□(d/(d-c)) (W× R) with c∈[0,d], c∈N if d∈[1,∞) , and ϑ≡0 if d=∞. Then the functional K(v)=∫_W^ ▒k(u,v(u))du is continuous.

Theorem 2.1[12]: Alaoglu's theorem (AlaTh): A bounded sequence {a_n} of a Hilbert space A has a subsequence which converges weakly to some a∈A.

Theorem 2.2 [10]**:** Let $A_0 \subset A \subset A_1$ be Banach spaces, where the injections being continuous, A_l is reflexive for $l = 0,1$, and the injection of A_0 into A is compact. Let $S > 0$ be a fixed finite number and let γ_0, $\gamma_1$be two finite numbers such that $\gamma_l > 1$, $l = 0,1$. We consider the Banach space $Y = \{\mathcal{v} \in C = L^{\gamma_0}(0,S;A_0), \acute{\mathcal{v}} \in D = L^{\gamma_1}(0,S;A_1)\}$ with $\|\mathcal{v}\|_B = \sqrt{\|\mathcal{v}\|_C^2 + \|\acute{\mathcal{v}}\|_D^2}$, $\forall \mathcal{v} \in B$. Then the injection is continuous and compact from Y into C.

Definition 2.1 [11]: A sequence $\{x_n\}$ of vectors in an inner product space V is called strongly convergent to a vector x in V if $\| x_n - x \| \to 0$ as $n \to \infty$.

3. Weak Formulation of the SVES

The weak form (WEKFM) of (1-6) when $\vec{s} \in (H^1(\Omega))^2$ is given by (∀v_1,v_2∈V):

$$\langle s_{1t}, \mathcal{v}_1\rangle + a_1(t,s_1,\mathcal{v}_1) + (k_1(t)s_1,\mathcal{v}_1)_\Omega - (k(t)s_2,\mathcal{v}_1)_\Omega = (\mathcal{F}_1(s_1),\mathcal{v}_1)_\Omega + (\mu_1,\mathcal{v}_1)_\Gamma \quad (11a)$$

$$(s_1^0,\mathcal{v}_1)_\Omega = (s_1(0),\mathcal{v}_1)_\Omega \quad (11b)$$

$$\langle s_{2t}, \mathcal{v}_2\rangle + a_2(t,s_2,\mathcal{v}_2) + (k_2(t)s_2,\mathcal{v}_2)_\Omega + (k(t)s_1,\mathcal{v}_2)_\Omega = (\mathcal{F}_2(s_2,\mu_2),\mathcal{v}_2)_\Omega \quad (12a)$$

$$(s_2^0,\mathcal{v}_2)_\Omega = (s_2(0),\mathcal{v}_2)_\Omega \quad (12b)$$

where $a_1(t,s_1,\mathcal{v}_1) = \int_\Omega \sum_{i,j=1}^n a_{ij} \frac{\partial s_1}{\partial x_i}\frac{\partial \mathcal{v}_1}{\partial x_j} dx$, $a_2(t,s_2,\mathcal{v}_2) = \int_\Omega \sum_{i,j=1}^n b_{ij} \frac{\partial s_2}{\partial x_i}\frac{\partial \mathcal{v}_2}{\partial x_j} dx$

The following assumptions are very important to prove the EXUN solution of the WEKFM.

Assumptions (I):

(i) $\mathcal{F}_1, \mathcal{F}_2$ are of a Carathéodory type (CATHT) on $Q \times \mathbb{R}$ and $Q \times Q \times \mathbb{R}^2$, respectively, that satisfies the following conditions for s_1 and (s_2,u_2), i.e.

$|\mathcal{F}_1(x,t,s_1)| \le \eta_1(x,t) + c_1|s_1|$ and $\left|\mathcal{F}_2(x,t,s_2,\mu_2)\right| \le \eta_2(x,t) + c_2|s_2| + \acute{c}_2|\mu_2|$,

where $(x,t) \in Q$, $s_i \in \mathbb{R}$, $c_i, \acute{c}_2 > 0$ and $\eta_i \in L^2(Q)$, $\forall i = 1,2$.

(ii) $\mathcal{F}_i$ is Lipschitz with s_i, $\forall i = 1,2$, i.e.

$|\mathcal{F}_1(x,t,s_1) - \mathcal{F}_i(x,t,\hat{s}_1)| \le L_1|s_1 - \hat{s}_1|$, $|\mathcal{F}_2(x,t,s_2,\mu_2) - \mathcal{F}_2(x,t,\hat{s}_2,\mu_2)| \le L_2|s_2 - \hat{s}_2|$,

where $(x,t) \in Q$, $s_i, \hat{s}_i \in \mathbb{R}$ and $L_i > 0$, $\forall i = 1,2$.

(iii) $\mathcal{D}(t,\vec{s},\vec{\mathcal{v}}) = a_1(t,s_1,\mathcal{v}_1) + (k_1(t)s_1,\mathcal{v}_1)_\Omega + a_2(t,s_2,\mathcal{v}_2) + (k_2(t)s_2,\mathcal{v}_1)_\Omega$,

$|\mathcal{D}(t,\vec{s},\vec{\mathcal{v}})| \le \alpha\|\vec{s}\|_1\|\vec{\mathcal{v}}\|_1$ and $\mathcal{D}(t,\vec{s},\vec{s}) \ge \bar{\alpha}\|\vec{s}\|_1^2$, where α & $\bar{\alpha}$ are real positive constants.

Main Results

Theorem (3.1): (The EXUN of SVES)

For each fixed CCMCV$\vec{\mu} \in L^2(\Sigma) \times L^2(Q)$, the WEKFM (11-12) has a unique solution $\vec{s} \in \left(L^2(I,V)\right)^2$ and $\vec{s}_t \in \left(L^2(I,V^*)\right)^2$.

Proof

Consider that $\vec{V}_n \subset \vec{V}$ is the of functions continuous on Ω , which has the basis $\{\vec{\mathcal{v}}_1, \vec{\mathcal{v}}_2, \dots, \vec{\mathcal{v}}_n\}$, then the solution $\vec{s}$ of (10-11) is approximated by $\vec{s}_n = (s_{1n}, s_{2n})$, such that , for each n

$$s_{1n} = \sum_{j=1}^n c_{1j}(t)\mathcal{v}_{1j}(x) \quad (12a)$$

$$s_{2n} = \sum_{j=1}^n c_{2j}(t)\mathcal{v}_{2j}(x) \quad (12b)$$

Using the MGA , the WEKFM of the (10) -(11) becomes

$$\langle s_{1nt}, \mathcal{v}_1\rangle + a_1(t,s_{1n},\mathcal{v}_1) + (k_1(t)s_{1n},\mathcal{v}_1)_\Omega - (k(t)s_{2n},\mathcal{v}_1)_\Omega = (\mathcal{F}_1(s_1),\mathcal{v}_1)_\Omega + (\mu_1,\mathcal{v}_1)_\Gamma \quad (13a)$$

$$(s_{1n}^0,\mathcal{v}_1)_\Omega = (s_1^0,\mathcal{v}_1)_\Omega, \quad \text{for any } \mathcal{v}_1 \in V_n \quad (13b)$$

$$\langle s_{2nt}, \mathcal{v}_2\rangle + a_2(t,s_{2n},\mathcal{v}_2) + (k_2(t)s_{2n},\mathcal{v}_2)_\Omega + (k(t)s_{1n},\mathcal{v}_2)_\Omega = (\mathcal{F}_2(s_2,\mu_2),\mathcal{v}_2)_\Omega \quad (14a)$$

$$(s_{2n}^0,\mathcal{v}_2)_\Omega = (s_2^0,\mathcal{v}_2)_\Omega , \quad \text{for any } \mathcal{v}_2 \in V_n \quad (14b)$$

with $s_{in}^0 = s_{in}(x,0)$ belongs in V_n , which satisfies, for any $\mathcal{v}_i \in V_n$ and $\forall i = 1,2$, that

$(s_{in}^0,\mathcal{v}_i)_\Omega = (s_i^0,\mathcal{v}_i)_\Omega \Leftrightarrow \left\|s_{in}^0 - s_i^0\right\|_0 \le \left\|s_i^0 - \mathcal{v}_i\right\|_0$

Then, there is a sequnece {w ⃗_n^0} with w ⃗_n^0∈V ⃗_n, for which w ⃗_n^0⟶s ⃗^0 strongly (ST) in 〖(L^2 (Ω))〗^2, therefore and from the above inequality norm, once get s ⃗_n^0⟶s ⃗^0 ST in 〖(L^2 (Ω))〗^2 with ‖s ⃗_n^0 ‖_0≤ b_1.

Now, using 12 (a & b) in 13-14 gives

$$A_1C_1^{'}(t)+D_1C_1(t)-E_1C_2(t) = b_1\,(\bar{V}_1^T(x)C_1(t)) \qquad (12'a)$$

$$A_1C_1(0)=b_1^0 \qquad (12'b)$$

$$A_2C_2^{'}(t)+D_2C_2(t)+E_2C_1(t) = b_2\,(\bar{V}_2^T(x)C_2(t)) \qquad (13'a)$$

$$BC_2(0)=b_2^0 \qquad (13'b)$$

where $A_1=\left(a_{ij}\right)_{n\times n}$, $a_{ij}=\left(v_{1j},v_{1i}\right)_\Omega$, $D_1=\left(d_{ij}\right)_{n\times n}$, $d_{ij}=[a_1\left(t,v_{1j},v_{1i}\right)+\left(k_1(t)v_{1j},v_{1i}\right)_\Omega]$, $E_1=\left(e_{ij}\right)_{n\times n}$, $e_{ij}=\left(b(t)v_{2j},v_{1i}\right)_\Omega$, $C_\ell(t)=\left(c_{\ell j}(t)\right)_{n\times 1}$, $C_\ell^{'}(t)=\left(c_{\ell j}^{'}(t)\right)_{n\times 1}$,$C_\ell(0)=\left(c_{\ell j}(0)\right)_{n\times 1}$, $b_\ell=(b_{li})_{n\times 1}$, $b_{1i}=(\mathcal{F}_1(\bar{v}_1^TC_1(t)),v_{1i})_\Omega+(\mu_1,v_{1i})_\Gamma b_{2i}=\left(\mathcal{F}_2\left(\bar{v}_2^TC_2(t)\right),\mu_2),v_{2i}\right)_\Omega$

, $\overline{w_\ell}=(e_\ell)_{n\times 1}$, $b_\ell^0=\left(b_{\ell i}^0\right)$,$b_{\ell i}^0=\left(y_\ell^0,v_{\ell i}\right)_\Omega$, and $A_2=\left(b_{ij}\right)_{n\times n}$, $b_{ij}=\left(v_{2j},v_{2i}\right)_\Omega$, $D_2=\left(f_{ij}\right)_{n\times n}$, $f_{ij}=[a_2\left(t,v_{2j},v_{2i}\right)+(k_2(t)v_{2j},v_{2i})_\Omega]$, $E_2=\left(h_{ij}\right)_{n\times n}$, $h_{ij}=(k(t)v_{1i},v_{2i})_\Omega$, $\ell=1,2$.

From assumption (I), system (12′-13′) has a unique solution.

The boundedness $\|\vec{s}_n(t)\|_{L^\infty\left(I,L^2(\Omega)\right)}$ ***&*** $\|\vec{s}_n(t)\|_Q$***:***

Putting $v_1=s_{1n}$ and $v_2=s_{2n}$ in 13a & 14a, integrating both sides (i.e. INBS) on $[0,T]$, and collecting them, using Assumption (I, iii), yield

$$\int_0^T\langle\vec{s}_{nt},\vec{s}_n\rangle dt+\int_0^T\|\vec{s}_n(t)\|_1^2dt = \int_0^T(\mathcal{F}_1(s_{1n}),s_{1n})_\Omega dt+\int_0^T(\mathcal{F}_2(s_{2n},\mu_2),s_{2n})_\Omega dt+\int_0^T(\mu_1,s_{1n})_\Gamma dt \qquad (15)$$

Since s ⃗_nt∈ 〖(L〗^2 (I,V^*))^2=(L^2 (I,V))^2 and s ⃗_n∈(L^2 (I,V))^2 in the 1^st term of the L.HN.S. of (15), then applying Lemma 2.1, but with the 2^nd term is nonnegative, letting T=t∈[0,T] , finally from the 1^st two terms in the R.HN.S of (15), and Assumption (I-i), one has $\int_0^t\frac{d}{dt}\|\vec{s}_n(t)\|_0^2\,dt\le\|\eta_1\|_Q^2+\|\eta_2\|_Q^2+\|\mu_1\|_\Sigma^2+\acute{c}_2\|\mu_2\|_Q^2+c_5\int_0^t\|\vec{s}_n\|_0^2\,dt$

$$\Rightarrow\|\vec{s}_n(t)\|_0^2-\|\vec{s}_n(0)\|_0^2\le m_3+c_5\int_0^t\|\vec{s}_n\|_0^2\,dt\,,\ m_3=m_1+m_2+c_1+c_2$$

$$\Rightarrow\|\vec{s}_n(t)\|_0^2\le m_4+c_5\int_0^t\|\vec{s}_n\|_0^2\,dt,\quad m_4=b+m_3.$$

By using the classical Bellman-Gronwall inequality (B.G), one gets

$\Rightarrow\|\vec{s}_n(t)\|_{L^\infty\left(I,L^2(\Omega)\right)}\le h_9$, hence

$$\|\vec{s}_n(t)\|_Q^2=\int_0^T\|\vec{s}_n\|_0^2dt\le T\max_{t\in[0,T]}\|\vec{s}_n(t)\|_0^2\le Th_8=h_{10}^2=h_{10}.$$

The boundedness of $\|\vec{s}_n(t)\|_{L^2(I,V)}$

Also, we apply Lemma 2.1 on the 1^{st} term in the L.HN.S. of (15) . Then, by utilizing the same steps above on its R.HN.S., with setting $t=T$, and $\|\vec{s}_n(T)\|_0^2\ge 0$, it becomes

$$\|\vec{s}_n(T)\|_0^2+2\bar{\alpha}\int_0^T\|\vec{s}_n\|_1^2dt\le\|\eta_1\|_Q^2+\|\eta_2\|_Q^2+\|\mu_1\|_\Sigma^2+\acute{c}_2\|\mu_2\|_Q^2+c_5\|\vec{s}_n\|_Q^2+\|\vec{s}_n(0)\|_0^2$$

$$\Rightarrow\|\vec{s}_n\|_{L^2(I,V)}\le h_{11}.$$

The convergent solution

Assume the sequence of subspaces $\{\vec{V}_n\}_{n=1}^\infty$ of $\vec{V}$, with the assumption that, for any $\vec{v}=(v_1,v_2)$ in $\vec{V}$, there is a sequence {v ⃗_n } of $\{\vec{V}_n\}_{n=1}^\infty$, s.t. $\vec{v}_n\longrightarrow\vec{v}$ ST in $\vec{V}\Rightarrow\vec{v}_n\longrightarrow\vec{v}$ ST in $\left(L^2(\Omega)\right)^2$.

Hence, corresponding to $\{\vec{V}_n\}_{n=1}^{\infty}$, for any $v_{1n}, v_{2n} \in V_n$ and $s_{1n}, s_{2n} \in L^2(I, V_n)$ a.e in I ($n = 1,2, ...$), the WEKF

$$\langle s_{1nt}, v_{1n}\rangle + a_1(t, s_{1n}, v_{1n}) + (k_1(t)s_{1n}, v_{1n})_\Omega - (k(t)s_{2n}, v_{1n})_\Omega$$
$$= (\mathcal{F}_1(s_{1n}), v_{1n})_\Omega + (\mu_1, v_{1n})_\Gamma, \quad (16a)$$
$$(s_{1n}^0, v_{1n})_\Omega = (s_1^0, v_{1n})_\Omega \quad (16b)$$
$$\langle s_{2nt}, v_{2n}\rangle + a_2(t, s_{2n}, v_{2n}) + (k_2(t)s_{2n}, v_{2n})_\Omega + (k(t)s_{1n}, v_{2n})_\Omega$$
$$= (\mathcal{F}_2(s_{2n}, \mu_2)v_{2n})_\Omega \quad (17a)$$
$$(s_{2n}^0, v_{2n})_\Omega = (s_2^0, v_{2n})_\Omega \quad (17b)$$

has a sequence of unique SVES $\{\vec{s}_n\}_{n=1}^{\infty}$. Then, from Theorem 2.1, $\{\vec{s}_n\}_{n\in N}$ has a subsequence claim again $\{\vec{s}_n\}_{n\in N}$ for which $\vec{s}_n \longrightarrow \vec{s}$ WK in $(L^2(Q))^2$ & $(L^2(I,V))^2$ (from the boundedness of $\|\vec{s}_n\|_{L^2(Q)}$ and $\|\vec{s}_n\|_{L^2(I,V)}$).

Then, by utilizing the theorem of compactness, Assumption (I-i), and the norms are bounded, one obtains $\vec{s}_n \longrightarrow \vec{s}$ ST in $(L^2(Q))^2$.

By multiplying (16a) and (17a) by $\psi_i(t) \in C^1[0,T]$, respectively, with $\psi_i(T) = 0$, $\forall i = 1,2$, integrating w.r.t. t on $[0,T]$, and then using the integrating by parts (IBPS) formula for the 1^{st} term in the L.HN.S., we get

$$-\int_0^T (s_{1n}, v_{1n})\psi_1'(t)dt +$$
$$\int_0^T [a_1(t, s_{1n}, v_{1n}) + (k_1(t)s_{1n}, v_{1n})_\Omega - (k(t)s_{2n}, v_{1n})_\Omega]\psi_1(t)dt =$$
$$\int_0^T (\mathcal{F}_1(s_{1n}), v_{1n})_\Omega \psi_1(t)dt + \int_0^T (\mu_1, v_{1n})_\Gamma \psi_1(t)dt + (s_{1n}^0, v_{1n})_\Omega \psi_1(0) \quad (18)$$
$$-\int_0^T (s_{2n}, v_{2n})\psi_2'(t)dt +$$
$$\int_0^T [a_2(t, s_{2n}, v_{2n}) + (k_2(t)s_{2n}, v_{2n})_\Omega + (k(t)s_{1n}, v_{2n})_\Omega]\psi_2(t)dt =$$
$$\int_0^T (\mathcal{F}_2(s_{2n}, \mu_2), v_{2n})_\Omega \psi_2(t)dt + (s_{2n}^0, v_{2n})_\Omega \psi_2(0) \quad (19)$$

But, $s_{in} \longrightarrow s_i$ WK in $L^2(Q)$, $s_{in}^0 \longrightarrow s_i^0$ ST in $L^2(\Omega)$, and

$$v_{in} \longrightarrow v_i \text{ ST in } L^2(\Omega) \;\&\; V\} \Rightarrow \begin{cases} v_{in}\varphi_i' \longrightarrow v_i\psi_i' \text{ ST in } L^2(Q) \\ v_{in}\varphi_i \longrightarrow v_i\psi_i \text{ ST in } L^2(I,V) \end{cases}$$

Then, the following convergences are held

$$\int_0^T (s_{1n}, v_{1n})\psi_1'(t)dt +$$
$$\int_0^T [a_1(t, s_{1n}, v_{1n}) + (k_1(t)s_{1n}, v_{1n})_\Omega - (k(t)s_{2n}, v_{1n})_\Omega]\psi_1(t)dt \longrightarrow \int_0^T (s_1, v_1)\varphi_1'(t)dt +$$
$$\int_0^T [a_1(t, s_1, v_1) + (k_1(t)s_1, v_1)_\Omega - (k(t)s_2, v_1)_\Omega]\psi_1(t)dt \quad (20)$$
$$(s_{1n}^0, v_{1n})_\Omega \psi_1(0) \longrightarrow (s_1^0, v_1)_\Omega \psi_1(0) \quad (21)$$
$$\int_0^T (s_{2n}, v_{2n})\psi_2'(t)dt +$$
$$\int_0^T [a_2(t, s_{2n}, v_{2n}) + (k_2(t)s_{2n}, v_{2n})_\Omega + (k(t)s_{1n}, v_{2n})_\Omega]\psi_2(t)dt \longrightarrow \int_0^T (s_2, v_2)\psi_2'(t)dt +$$
$$\int_0^T [a_2(t, s_2, v_2) + (k_2(t)s_2, v_2)_\Omega + (k(t)s_1, v_2)_\Omega]\psi_2(t)dt \quad (22)$$
$$(s_{2n}^0, v_{2n})_\Omega \psi_2(0) \longrightarrow (s_2^0, v_2)_\Omega \psi_2(0) \quad (23)$$

Now, we set $p_{in} = v_{in}\psi_i$ and $p_i = v_i\psi_i$, hence $p_{1n} \longrightarrow p_1$ ST in $L^2(Q)$ and therefore p_{1n} is measurable w.r.t. (x,t). Utilizing Assumption (I-i), with employing Proposition 2.1 gives that $\int_Q \mathcal{F}_1(x, t, s_{1n})p_{1n}dxdt$ is continuous w.r.t. (s_{1n}, p_{1n}), but $s_{1n} \longrightarrow s_1$ ST in $L^2(Q)$, then

$$\int_0^T (\mathcal{F}_1(s_{1n}), v_{1n})_\Omega \psi_1(t)dt \longrightarrow \int_0^T (\mathcal{F}_1(s_1), v_1)_\Omega \psi_1(t)dt$$

Using the same way, we get

$$\int_0^T (\mathcal{F}_2(s_{2n}, \mu_2), v_{2n})_\Omega \psi_2(t)dt \longrightarrow \int_0^T (\mathcal{F}_2(s_2, \mu_2), v_2)_\Omega \psi_2(t)dt$$

From this convergence and (20-23), (18-19) become

$$-\int_0^T (s_1, v_1)\psi_1'(t)dt + \int_0^T [a_1(t, s_1, v_1) + (k_1(t)s_1, v_1)_\Omega - (k(t)s_2, v_1)_\Omega]\psi_1(t)dt = \int_0^T (\mathcal{F}_1(s_1), v_1)_\Omega \psi_1(t)dt + \int_0^T (\mu_1, v_1)_\Gamma \psi_1(t)dt + (s_1^0, v_1)_\Omega \psi_1(0) \quad (24)$$

$$-\int_0^T (s_2, v_2)\psi_2'(t)dt + \int_0^T [a_2(t, s_2, v_2) + (k_2(t)s_2, v_2)_\Omega + (k(t)s_1, v_2)_\Omega]\psi_2(t)dt = \int_0^T (\mathcal{F}_2(s_2, \mu_2), v_2)_\Omega \psi_2(t)dt + (s_2^0, v_2)_\Omega \psi_2(0) \quad (25)$$

Therefore, we consider the following cases:

Case1: Select $\psi_i \in D[0, T]$, by setting $\psi_i(0) = \psi_i(T) = 0$, $\forall i = 1,2$ in (24) - (25), and for the first terms in the L.HN.S. of each one equation, the integration by parts formula is used to get

$$\int_0^T (s_{1t}, v_1)\psi_1(t)dt + \int_0^T [a_1(t, s_1, v_1) + (k_1(t)s_1, v_1)_\Omega - (k(t)s_2, v_1)_\Omega \psi_1(t)]dt = \int_0^T (\mathcal{F}_1(s_1), v_1)_\Omega \psi_1(t)dt + \int_0^T (\mu_1, v_1)_\Gamma \psi_1(t)dt \quad (26)$$

and

$$\int_0^T (s_{2t}, v_2)\psi_2(t)dt + \int_0^T [a_2(t, s_2, v_2) + (k_2(t)s_2, v_2)_\Omega + (k(t)s_1, v_2)_\Omega \psi_2(t)]dt = \int_0^T (\mathcal{F}_2(s_2, \mu_2), v_2)_\Omega \psi_2(t)dt \quad (27)$$

i.e. $\vec{s}$ is a solution of the WEKFM (10a) - (11a).

Case 2: Select $\forall i = 1,2, \psi_i \in C^1[0, T]$ with $\psi_i(T) = 0$ and $\psi_i(0) \neq 0$.

Using IBPS in the L.HN.S. of (26) & (27) , then subtracting the obtained equations from (24) and (25) respectively, we get

$$(s_i^0, v_i)_\Omega \psi_i(0) = (s_i(0), v_i)_\Omega \psi_i(0) \Rightarrow (s_i^0, v_i)_\Omega = (s_i(0), v_i)_\Omega, \ \forall i = 1,2.$$

The strong convergence in $L^2(I, V)$

By setting $v_1 = s_1$ and $v_1 = s_{1n}$ in (10a) & (13a) and $v_2 = s_2$, $v_2 = s_{2n}$ in (11a) & (14a), integrating the resulting equations on $[0, T]$, collecting the equation resulting from (10a) with that resulting from (13a) together, and doing the same for (11a) & (14a), we get

$$\int_0^T \langle \vec{s}_t, \vec{s} \rangle \, dt + \int_0^T c(t, \vec{s}, \vec{s})dt = \int_0^T [(\mathcal{F}_1(s_1), s_1)_\Omega + (\mathcal{F}_2(s_2, \mu_2), s_2)_\Omega]dt + \int_0^T (\mu_1, s_1)_\Gamma dt \quad (28a)$$

$$\int_0^T \langle \vec{s}_{nt}, \vec{s}_n \rangle \, dt + \int_0^T \mathcal{D}(t, \vec{s}_n, \vec{s}_n)dt = \int_0^T [(\mathcal{F}_1(s_{1n}), s_{1n}) + (\mathcal{F}_2(s_{2n}, \mu_2), s_{2n})]dt + \int_0^T (\mu_1, s_{1n})_\Gamma dt \quad (28b)$$

By employing Lemma 2.1 on the L.HN.S. of (28a&b), one obtains

$$\tfrac{1}{2}\|\vec{s}(T)\|_0^2 - \tfrac{1}{2}\|\vec{s}(0)\|_0^2 + \int_0^T \mathcal{D}(t, \vec{s}, \vec{s})dt = \int_0^T [(\mathcal{F}_1(s_1), s_1)_\Omega + (\mathcal{F}_2(s_2, \mu_2), s_2)_\Omega] \, dt + + \int_0^T (\mu_1, s_1)_\Gamma dt \quad (29a)$$

and

$$\tfrac{1}{2}\|\vec{s}_n(T)\|_0^2 - \tfrac{1}{2}\|\vec{s}_n(0)\|_0^2 + \int_0^T \mathcal{D}(t, \vec{s}_n, \vec{s}_n)dt = \int_0^T [(\mathcal{F}_1(s_{1n}), s_{1n})_\Omega + (\mathcal{F}_2(s_{2n}, \mu_2), s_{2n})_\Omega] \, dt + \int_0^T (\mu_1, s_{1n})_\Gamma dt \quad (29b)$$

Now, consider the following equality:

$$\tfrac{1}{2}(\|\vec{s}_n(T) - \vec{s}(T)\|_0^2 - \|\vec{s}_n(0) - \vec{s}(0)\|_0^2) + \int_0^T \mathcal{D}(t, \vec{s}_n - \vec{s}, \vec{s}_n - \vec{s}) \, dt = B_1 - B_2 - B_3 \quad (30)$$

where

$$B_1 = \tfrac{1}{2}(\|\vec{s}_n(T)\|_0^2 - \|\vec{s}_n(0)\|_0^2) + \int_0^T \mathcal{D}\big(t, \vec{s}_n(T), \vec{s}_n(T)\big)dt$$

$$B_2 = \tfrac{1}{2}\big(\vec{s}_n(T), \vec{s}(T)\big) - \tfrac{1}{2}\big(\vec{s}_n(0), \vec{s}(0)\big) + \int_0^T \mathcal{D}\big(t, \vec{s}_n(T), \vec{s}(T)\big) \, dt$$

$$B_3 = \tfrac{1}{2}\big(\vec{s}(T), \vec{s}_n(T) - \vec{s}(T)\big) - \tfrac{1}{2}\big(\vec{s}(0), \vec{s}_n(0) - \vec{s}(0)\big) + \int_0^T \mathcal{D}\big(t, \vec{s}(T), \vec{s}_n(T) - \vec{s}(T)\big)dt$$

But

$$\vec{s}_n^0 = \vec{s}_n(0) \longrightarrow \vec{s}^0 = \vec{s}(0) \text{ ST in } \big(L^2(\Omega)\big)^2 \quad (31a)$$

$$\vec{s}_n(T) \longrightarrow \vec{s}(T) \text{ ST in } \big(L^2(\Omega)\big)^2 \quad (31b)$$

Which gives

$$\big(\vec{s}(0), \vec{s}_n(0) - \vec{s}(0)\big) \longrightarrow 0 \ \& \big(\vec{s}(T), \vec{s}_n(T) - \vec{s}(T)\big) \longrightarrow 0 \quad (31c)$$

$$\|\vec{s}_n(0)-\vec{s}(0)\|_0^2 \longrightarrow 0 \ \&\|\vec{s}_n(T)-\vec{s}(T)\|_0^2 \longrightarrow 0 \tag{31d}$$

Since $\vec{s}_n \longrightarrow \vec{s}$ WK in $\left(L^2(I,V)\right)^2$, then

$$\int_0^T c\big(t,\vec{s}(T),\vec{s}_n(T)-\vec{s}(T)\big)dt \longrightarrow 0 \tag{31e}$$

Since $s_{in} \longrightarrow s_i$ ST in $L^2(Q), \forall i=1,2$, then from Proposition 2.1, the integrals $\int_0^T (\mathcal{F}_1(s_{1n}),s_{1n})\,dt, \int_0^T (\mathcal{F}_2(s_{2n},\mu_2),s_{2n})\,dt$ are continuous w.r.t. s_{1n},s_{2n} respectively. Therefore

$$\int_0^T [(\mathcal{F}_1(s_{1n}),s_{1n})+(\mathcal{F}_2(s_{2n},\mu_2),s_{2n})]\,dt \longrightarrow \int_0^T [(\mathcal{F}_1(s_1),s_1)+(\mathcal{F}_2(s_2,\mu_2),s_2)]dt \tag{31f}$$

Now, when $n\to\infty$ in (30), the following results are obtained:

1) From (31d), we have $\frac{1}{2}\|\vec{s}_n(T)-\vec{s}(T)\|_0^2 \longrightarrow 0$ and $\frac{1}{2}\|\vec{s}_n(0)-\vec{s}(0)\|_0^2 \longrightarrow 0$

2) From (29b) & (31f) , we have

Eq.(A_1)

$= \int_0^T [(\mathcal{F}_1(s_{1n}),s_{1n})+(\mathcal{F}_2(s_{2n},\mu_2),s_{2n})]\,dt + \int_0^T [(\mu_1,s_{1n})_\Gamma dt \longrightarrow \int_0^T [(\mathcal{F}_1(s_1),s_1)_\Omega +$ $(\mathcal{F}_2(s_2,\mu_2),s_2)_\Omega]\,dt + \int_0^T (\mu_1,s_1)_\Gamma dt$

3) From (29a) , we have Eq.(A_2)

$= \int_0^T [(\mathcal{F}_1(s_1),s_1)_\Omega + (\mathcal{F}_2(s_2,\mu_2),s_2)_\Omega]\,dt + \int_0^T [(\mu_1,s_{1n})_\Gamma\,dt$

4) Through (31c) and c(31e), all the terms are tending to zero in (A_3).

Now, the above steps, and (30), give

$$\int_0^T \mathcal{D}(t,\vec{s}_n-\vec{s},\vec{s}_n-\vec{s})dt \longrightarrow 0 \Rightarrow \bar{\alpha}\int_0^T \|\vec{s}_n-\vec{s}\|_1^2\,dt \longrightarrow 0 \Rightarrow \vec{s}_n \longrightarrow \vec{s} \text{ ST in } \left(L^2(I,V)\right)^2.$$

Uniqueness of the solution

Let $\vec{s}=(s_1,s_2)$, $\vec{\hat{s}}=(\hat{s}_1,\hat{s}_2)$ be two SVES of (10)-(11), i.e. from (10a) we have

$$\langle s_{1t},v_1\rangle + a_1(t,s_1,v_1)+(k_1(t)s_1,v_1)_\Omega-(k(t)s_2,v_1)_\Omega = (\mathcal{F}_1(s_1),v_1)_\Omega+(\mu_1,v_1)_\Gamma$$

$$\langle \hat{s}_{1t},v_1\rangle + a_1(t,\hat{s}_1,v_1)+(k_1(t)\hat{s}_1,v_1)_\Omega-(k(t)\hat{s}_2,v_1)_\Omega = (\mathcal{F}_1(\hat{s}_1),v_1)_\Omega+(\mu_1,v_1)_\Gamma$$

Subtracting the second equation from the first one , then setting $v_1 = s_1-\hat{s}_1$, yield

$$\langle (s_1-\hat{s}_1)_t,s_1-\hat{s}_1\rangle + a_1(t,s_1-\hat{s}_1,s_1-\hat{s}_1)+(k_1(t)s_1-\hat{s}_1),s_1-\hat{s}_1)_\Omega-(k(t)(s_2-\hat{s}_2),s_1-\hat{s}_1)_\Omega = (\mathcal{F}_1(s_1)-\mathcal{F}_1(\hat{s}_1),s_1-\hat{s}_1)_\Omega \tag{32}$$

Also applying the same steps for (11a), yields

$$\langle (s_2-\hat{s}_2)_t,s_2-\hat{s}_2\rangle + a_2(t,s_2-\hat{s}_2,s_2-\hat{s}_2)+(k_2(t)s_2-\hat{s}_2),s_2-\hat{s}_2)_\Omega+(k(t)(s_1-\hat{s}_1),s_2-\hat{s}_2)_\Omega = (\mathcal{F}_2(s_2,\mu_2)-\mathcal{F}_2(\hat{s}_2,\mu_2),s_2-\hat{s}_2)_\Omega, \tag{33}$$

By collecting (32) and (33), employing Lemma 2.1 for the L.HN.S of the resulting equality and applying assumption A-iii, we have

$$\frac{1}{2}\frac{d}{dt}\left\|\vec{s}-\vec{\hat{s}}\right\|_0^2+\bar{\alpha}\left\|\vec{s}-\vec{\hat{s}}\right\|_1^2 \le |(\mathcal{F}_1(s_1)-\mathcal{F}_1(\hat{s}_1),s_1-\hat{s}_1)_\Omega+(\mathcal{F}_2(s_2,\mu_2)-\mathcal{F}_2(\hat{s}_2,\mu_2),s_2-\hat{s}_2)_\Omega| \tag{34}$$

But the second term in the L.HN.S. of (34) is nonnegative. INBS of (34) on $[0,t]$, then employing assumptions (A-ii) for the R.HN.S, using the B.G , one obtains

$$\left\|\vec{s}(t)-\vec{\hat{s}}(t)\right\|_0^2 = 0, \forall t \Rightarrow \left\|\vec{s}-\vec{\hat{s}}\right\|_{L^2(I,V)} = 0 \Rightarrow \vec{s}=\vec{\hat{s}}\,.$$

4. Existence of the CCMOPCV

In this part, the following theorem and lemma are useful in studding the EXUNTh for the CCMOPCV.

Theorem (4.1):

(a) If assumption (I) is held and if $\vec{\mu}$ and $\vec{\mu}+\overrightarrow{\Delta\mu}$ are bounded CCMCVs in $L^2(\Sigma)\times L^2(\Omega)$ and their corresponding SVES, then

$\left\|\overrightarrow{\Delta s}\right\|_{L^\infty\left(I,L^2(\Omega)\right)} \le \mathcal{K}_1\left\|\overrightarrow{\Delta\mu}\right\|_{L^2(\Sigma)\times L^2(\Omega)}$, $\left\|\overrightarrow{\Delta s}\right\|_{L^2(Q)} \le \mathcal{K}_2\left\|\overrightarrow{\Delta\mu}\right\|_{L^2(\Sigma)\times L^2(\Omega)}$ and

$\left\|\overrightarrow{\Delta s}\right\|_{L^2(I,V)} \le \mathcal{K}_3\left\|\overrightarrow{\Delta\mu}\right\|_{L^2(\Sigma)\times L^2(\Omega).}$

(b) If assumption (I) is held, then the operator $\vec{\mu} \longmapsto \vec{s}_{\vec{\mu}}$ from $L^2(\Sigma) \times L^2(\Omega)$ into $(L^\infty(I, L^2(\Omega)))^2$, $(L^2(I,V))^2$ and $(L^2(Q))^2$ is Lipschitz continuous (LC).

Proof:

(a) Take $\vec{\mu}, \vec{\hat{\mu}} \in L^2(\Sigma) \times L^2(\Omega)$. Hence, from theorem (3.1), $\vec{s}_{\vec{\mu}}$ and $\vec{\hat{s}}_{\vec{\hat{\mu}}}$ are their corresponding SVES, which satisfies the WEKFM (10) - (11), $\forall \upsilon_1, \upsilon_2 \in V$, i.e.

$$\langle \hat{s}_{1t}, \upsilon_1 \rangle + a_1(t, \hat{s}_1, \upsilon_1) + (k_1(t)\hat{s}_1, \upsilon_1)_\Omega - (k(t)\hat{s}_2, \upsilon_1)_\Omega = (\mathcal{F}_1(\hat{s}_1), \upsilon_1)_\Omega + (\hat{\mu}_1, \upsilon_1)_\Gamma \quad (35a)$$

$$(\hat{s}_1(0), \upsilon_1)_\Omega = (s_1^0, \upsilon_1)_\Omega \quad (35b)$$

$$\langle \hat{s}_{2t}, \upsilon_2 \rangle + a_2(t, \hat{s}_2, \upsilon_2) + (k_2(t)\hat{s}_2, \upsilon_2)_\Omega + (k(t)\hat{s}_1, \upsilon_2)_\Omega = (\mathcal{F}_1(\hat{s}_2, \mu_2), \upsilon_2)_\Omega \quad (36a)$$

$$(\hat{s}_2(0), \upsilon_2)_\Omega = (s_2^0, \upsilon_2)_\Omega \quad (36b)$$

Subtract (10) from (35) and (11) from (36) and put $\Delta s_i = \hat{s}_i - s_i$, $\Delta \mu_i = \hat{\mu}_i - \mu_i$, $\forall i = 1,2$ in the two obtained equations, to get

$$\langle \Delta s_{1t}, \upsilon_1 \rangle + a_1(t, \Delta s_1, \upsilon_1) + (k_1(t)\Delta s_1, \upsilon_1)_\Omega - (k(t)\Delta s_2, \upsilon_1)_\Omega = (\mathcal{F}_1(s_1 + \Delta s_1), \upsilon_1)_\Omega - (\mathcal{F}_1(s_1), \upsilon_1)_\Omega + (\Delta\mu_1, \upsilon_1)_\Gamma \quad (37a)$$

$$(\Delta s_1(0), \upsilon_1)_\Omega = 0 \quad (37b)$$

and

$$\langle \Delta s_{2t}, \upsilon_2 \rangle + a_2(t, \Delta s_2, \upsilon_2) + (k_2(t)\Delta s_2, \upsilon_2)_\Omega + (k(t)\Delta s_1, \upsilon_2)_\Omega = (\mathcal{F}_2(s_2 + \Delta s_2, \Delta\mu_2), \upsilon_2)_\Omega - (\mathcal{F}_2(s_2, \mu_2), \upsilon_2)_\Omega, \quad (38a)$$

$$(\Delta s_2(0), \upsilon_2)_\Omega = 0 \quad (38b)$$

Using $\upsilon_1 = \Delta s_1$, $\upsilon_2 = \Delta s_2$ in (37a) and (38a), then collecting them, employing Lemma 2.1 for the 1^{st} term in the L.HN.S. and utilizing Assumption (I-iii), one gets

$$\frac{1}{2}\frac{d}{dt}\|\overrightarrow{\Delta s}\|_0^2 + \bar{\alpha}\|\overrightarrow{\Delta s}\|_1^2 \le |(\mathcal{F}_1(s_1 + \Delta s_1) - \mathcal{F}_1(s_1), \Delta s_1)| + |(\mathcal{F}_2(s_2 + \Delta s_2, \Delta\mu_2) - \mathcal{F}_2(s_2, \Delta\mu_2), \Delta s_2)| + |(\Delta\mu_1, \Delta s_1)| \quad (39)$$

The 2^{nd} term of L.HN.S. of the inequality is nonnegative. Hence, INBS w.r.t. t on $[0, t]$, then by employing Assumptions I-ii and the inequality of Cauchy-Schwarz for the R.HN.S. and then employing the Trace theorem, we obtain

$$\|\overrightarrow{\Delta s}(t)\|_0^2 \le \|\overrightarrow{\Delta\mu}\|_{L^2(\Sigma)\times L^2(\Omega)}^2 + L_3 \int_0^t \|\overrightarrow{\Delta s}\|_0^2 \, dt \quad \text{, where } L_3 = \sum constant$$

Applying the B.G gives

$$\|\overrightarrow{\Delta s}(t)\|_0 \le \mathcal{K}_1 \|\overrightarrow{\Delta\mu}\|_{L^2(\Sigma)\times L^2(\Omega)} \ , \ t \in [0, T]$$

$$\Rightarrow \|\overrightarrow{\Delta s}\|_{L^\infty(I, L^2(\Omega))} \le \mathcal{K}_1 \|\overrightarrow{\Delta\mu}\|_{L^2(\Sigma)\times L^2(\Omega)} \ , \ t \in [0, T]$$

From this result, one easily obtains that

$$\|\overrightarrow{\Delta s}\|_{L^2(Q)} \le \mathcal{K}_2 \|\overrightarrow{\Delta\mu}\|_{L^2(\Sigma)\times L^2(\Omega)} \ , \ \text{and} \ \|\overrightarrow{\Delta s}\|_{L^2(I,V)} \le \mathcal{K}_3 \|\overrightarrow{\Delta\mu}\|_{L^2(\Sigma)\times L^2(\Omega)}$$

(b) From part (a), one directly obtains that the operator $\vec{\mu} \longmapsto \vec{s}$ is LC from $L^2(\Sigma) \times L^2(\Omega)$ into the spaces $(L^\infty(I, L^2(\Omega)))^2$, $(L^2(I,V))^2$, $(L^2(Q))^2$.

Assumption (II)

Consider that ($\forall l = 0,1,2$) g_{l1} , g_{l2} , h_{l1} are of a CATHT on $(Q \times \mathbb{R})$, $(Q \times \mathbb{R}^2)$, $(\Sigma \times \mathbb{R})$ respectively, and satisfy:

$|g_{l1}(x, t, s_1, \mu_1)| \le \gamma_{l1}(x, t) + c_{l1}(s_1)^2 + c_{l1}(\mu_1)^2$

$|g_{l2}(x, t, s_2)| \le \gamma_{l2}(x, t) + c_{l2}(s_2)^2$, $|h_{l2}(x, t, \mu_2)| \le \delta_{l2}(x, t) + d_{l2}(\mu_2)^2$

where $s_i, \mu_i \in \mathbb{R}$ with $\gamma_{li} \in L^1(Q)$, $\delta_{l1} \in L^1(\Sigma)$, $\delta_{l2} \in L^1(Q)$, $i = 1,2$.

Lemma (4.2)

If assumption (II) is held, then $\mathcal{H}_l(\vec{\mu})$ ($\forall l = 0,1,2$) is continuous on $L^2(\Sigma) \times L^2(\Omega)$.

Proof

From the given assumptions, with utilizing Proposition 2.1, we have $\int_Q g_{l1}(x, t, s_1) \, dxdt$ and

$\int_Q g_{l2}(x,t,s_2,\mu_2)\,dxdt$ are continuous on $L^2(Q)$ and $\int_\Sigma h_{l1}(x,t,\mu_1)\,d\sigma$ is on $L^2(\Sigma)$ $\forall l = 0,1,2$.
Then $\mathcal{H}_l(\vec{\mu})$ is continuous on $L^2(\Sigma)\times L^2(Q)$, $\forall l = 0,1,2$.
Theorem (4.3)
Consider that assumptions (I) and (II) are held and that $\vec{\mathcal{U}}$ is compact. Consider that $\vec{\mathcal{N}_A} \neq \emptyset$, if for fixed $(x,t,\vec{s})$, $\mathcal{H}_0(\vec{\mu})$ and $\mathcal{H}_2(\vec{\mu})$ are convex w.r.t. $\vec{\mu}$ and that $\mathcal{H}_1(\vec{\mu}\,)$ is independent of $\vec{\mu}$. Then there is a CCMOPCV.
Proof:
Since $\vec{\mathcal{U}}$ is compact and convex, then $\vec{\mathcal{N}}$ is WK compact. Since $\vec{\mathcal{N}_A} \neq \emptyset$, then $\exists \vec{\vec{u}} \in \vec{\mathcal{N}_A}$ and there is a minimum sequence$\{\vec{\mu}_k\}$, $\vec{\mu}_k \in \vec{\mathcal{N}_A}$, $\forall k$ that satisfies
$\lim_{k\to\infty} \mathcal{H}_0\,(\vec{\mu}_k) = \inf_{\vec{\vec{u}}\in \vec{W}_A} \mathcal{H}_0(\,\vec{\vec{\mu}})$.
But $\vec{\mathcal{N}}$ is WK compact, then $\{\vec{\mu}_k\}$ has a subsequence claim again $\{\vec{\mu}_k\}$, which converges WK to some element $\vec{\mu}$ in $\vec{\mathcal{N}}$, or $\vec{\mu}_k \longrightarrow \vec{\mu}$ WK in $L^2(\Sigma)\times L^2(\Omega)$, then $\{\vec{\mu}_k\}$ is bounded $\forall k$.
By theorem (3.2), the WEKFM has a unique SVES $\vec{s}_k = \vec{s}_{\vec{u}_k}$ for each CCMCV $\vec{\mu}_k$, with $\|\vec{s}_k\|_{L^\infty(I,L^2(\Omega))}$, $\|\vec{s}_k\|_{L^2(Q)}$, $\|\vec{s}_k\|_{L^2(I,V)}$ are bounded. Then, by employing the AlaTh, $\{\vec{s}_k\}$ has subsequence claim again $\{\vec{s}_k\}$, such that $\vec{s}_k \longrightarrow \vec{s}$ WK in the spaces $(L^\infty(I,L^2(\Omega)))^2$, $(L^2(Q))^2$ and $(L^2(I,V))^2$.
Also, since $\|\vec{s}_k\|_{L^2(I,V^*)}$ is bounded, from theorem (3.2), and
$(L^2(I,V))^2 \subset (L^2(Q))^2 \cong ((L^2(Q))^*)^2 \subset (L^2(I,V^*))^2$
hence by utilizing theorem 2.2, $\{\vec{s}_k\}$ has a subsequence claim again $\{\vec{s}_k\}$ s.t. $\vec{s}_k \longrightarrow \vec{s}$ ST in $(L^2(Q))^2$.
Since $\forall k$, $\vec{s}_k$ is the corresponding SVES to the CCMCV $\vec{\mu}_k$, then

$$\langle s_{1kt}, v_1\rangle + a_1(t,s_{1k},v_1) + (k_1(t)s_{1k},v_1)_\Omega - (k(t)s_{2k},v_1)_\Omega = (\mathcal{F}_1(x,t,s_{1k}),v_1)_\Omega + (\mu_{1k},v_1)_\Gamma \qquad (40)$$

and

$$\langle s_{2kt}, v_2\rangle + a_2(t,s_{2k},v_2) + (k_2(t)s_{2k},v_2)_\Omega + (k(t)s_{1k},v_2)_\Omega = (\mathcal{F}_2(x,t,s_{2k},\mu_{2k}),v_2)_\Omega \qquad (41)$$

Let $\psi_i \in C^1[I]$, $\forall i = 1,2$, for which $\psi_i(T) = 0$. Multiplying (40) and (41) by $\psi_1(t)$ and $\psi_2(t)$, respectively, then INBS w.r.t. t from $[0,T]$, and using IBPS formula for the 1^{st} terms in the L.HN.S., yield

$$-\int_0^T (s_{1k},v_1)\acute{\psi}_1(t)dt + \int_0^T [a_1(t,s_{1k},v_1) + (k_1(t)s_{1k},v_1)_\Omega - (k(t)s_{2k},v_1)_\Omega]\psi_1(t)dt = \int_0^T (\mathcal{F}_1(x,t,s_{1k}),v_1)_\Omega\psi_1(t)dt + \int_0^T (\mu_{1k},v_1)_\Gamma\psi_1(t)\,dt + (s_{1k}(0),v_1)_\Omega\psi_1(0) \qquad (42)$$

and

$$-\int_0^T (s_{2k},v_2)\acute{\psi}_2(t)dt + \int_0^T [a_2(t,s_{2k},v_2) + (k_2(t)s_{2k},v_2)_\Omega + (k(t)s_{1k},v_2)_\Omega]\psi_2(t)dt = \int_0^T (\mathcal{F}_2(x,t,s_{2k},\mu_{2k}),v_2)_\Omega\psi_2(t)dt + (s_{2k}(0),v_2)_\Omega\psi_2(0), \qquad (43)$$

Since $\vec{s}_k \longrightarrow \vec{s}$ WK in the spaces $(L^2(Q))^2$ and $(L^2(I,V))^2$, then

$$-\int_0^T (s_{1k},v_1)\acute{\psi}_1(t)dt + \int_0^T [a_1(t,s_{1k},v_1) + (k_1(t)s_{1k},v_1)_\Omega - (\mathrm{k(t)}s_{2k},v_1)_\Omega]\psi_1(t)dt \longrightarrow -\int_0^T (s_1,v_1)\acute{\psi}_1(t)dt + \int_0^T [a_1(t,s_1,v_1) + (k_1(t)s_1,v_1)_\Omega - (k(t)s_2,v_1)_\Omega]\psi_1(t)dt \qquad (44a)$$

and

$$-\int_0^T (s_{2k},v_2)\acute{\psi}_2(t)dt + \int_0^T [a_2(t,s_{2k},v_2) + (k_2(t)s_{2k},v_2)_\Omega + (\mathrm{k(t)}s_{1k},v_2)_\Omega]\psi_1(t)dt \to -\int_0^T (s_2,v_2)\acute{\psi}_2(t)dt + \int_0^T [a_2(t,s_2,v_2) + (k_2(t)s_2,v_2)_\Omega + (k(t)s_1,v_2)_\Omega]\psi_2(t)dt \qquad (45a)$$

Since $s_{1k}(0), s_{2k}(0)$ are bounded in $L^2(\Omega)$ and by theorem 3.1, we get

$$(s_{ik}(0),v_i)_\Omega\psi_1(0) \longrightarrow (s_i^0,v_i)_\Omega\psi(0).\ i = 1,2 \qquad (45b)$$

Let $p_1 = v_1\psi_1(t)$, and it is fixed for any fixed $(x,t)\in Q$. Hence, $p_1 \in L^\infty(I,V) \subset L^2(Q)$. Let $v_1 \in C[\bar{\Omega}]$ then it is measurable w.r.t. (x,t). Hence set $\bar{\mathcal{F}}_1(s_{1k}) = \mathcal{F}_1(s_{1k})p_1$ then

$\bar{\mathcal{F}}_1 : Q \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous w.r.t. s_{1k} for fixed $(x,t) \in Q$, then
$\|\bar{\mathcal{F}}_1(x,t,s_{1k}(X)\| \leq \eta_1|p_1| + c_1|\, s_{1k}||p_1| = \bar{\eta}_1^2 + \bar{c}_2\|s_{1k}\|^2$, where $\overline{\eta_1^2} = \frac{1}{2}(\eta_1^2 + \bar{c}_1|v_1|^2)$.
By employing proposition 2.1, we get $\int_Q \mathcal{F}_1(\, s_{1k}) p_1 dxdt$ is continuous w.r.t. s_{1k} but $s_{1k} \longrightarrow s_1$ ST in $L^2(Q)$, thus

$$\int_Q \mathcal{F}_1(\, s_{1k}) p_1 dxdt \longrightarrow \int_Q \mathcal{F}_1(\, s_1) p_1 dxdt \forall p_1 \in C[\bar{Q}] \qquad (44c)$$

We can use the same way to get that the integral $\int_Q \mathcal{F}_2(\, s_{2k}, \mu_{2k}) p_2 dxdt$ is continuous w.r.t. (s_{2k}, μ_{2k}), but $\mu_{2k} \longrightarrow \mu_2$ WK in $L^2(Q)$, then

$$\int_Q \mathcal{F}_2(\, s_{2k}, \mu_{2k}) p_2 dxdt \longrightarrow \int_Q \mathcal{F}_2(\, s_{2k}, \mu_{2k}) p_2 dxdt \qquad (44d)$$

on the other hand, since $\mu_{1k} \longrightarrow \mu_1$ WK in $L^2(\Sigma)$, then

$$\int_\Gamma (\mu_{1k}, v_1) \psi_1(t)\, d\Gamma dt \longrightarrow \int_\Gamma (\, \mu_1, v_1) \psi_1(t) d\Gamma dt \qquad (44h)$$

Eventually, utilizing (44) & (45b) in (42)-(43) gives

$$-\int_0^T (s_1, v_1) \acute{\psi}_1\,(t) dt + \int_0^T [a_1(t, s_1, v_1) + (k_1(t) s_1, v_1)_\Omega - (k(t) s_2, v_1)_\Omega] \psi_1(t) dt \;=\; \int_0^T (\mathcal{F}_1(x,t,s_1), v_1)_\Omega \psi_1(t) dt + \int_0^T (\mu_1, v_1)_\Gamma\, \psi_1(t) dt + (s_1^0, v_1)_\Omega \psi_1(0) \qquad (46)$$

$$-\int_0^T (s_2, v_2) \acute{\psi}_2\,(t) dt + \int_0^T [a_2(t, s_2, v_2) + (k_2(t) s_2, v_2)_\Omega + (k(t) s_1, v_2)_\Omega] \psi_2(t) dt = \int_0^T (\mathcal{F}_2(x,t,s_2,\mu_2), v_2)_\Omega \psi_2(t) dt + (s_2^0, v_2)_\Omega \psi_2(0) \qquad (47)$$

Of course (46) - (47) are also satisfied for any $v_i \in V$, $\forall i = 1,2$.
Same steps can be used here, like those that were used in Cases 1 and 2 in the proof of theorem 3.1 to obtain that $\vec{s}$ is a SVES of the WEKFM.
From the continuity of $g_{l1}(x,t,s_{1k})$, $g_{l2}(x,t,s_{2k},\mu_{2k})$ $(\forall l = 0,1,2)$ w.r.t s_{1k} , s_{2k}, and the proof of Lemma 4.2, we get that $\int_Q g_{l1}(x,t,s_{1k})\, dxdt$, $\int_Q g_{l2}(x,t,s_{2k},\mu_{2k})\, dxdt$ are continuous w.r.t s_{1k} and s_{2k} respectively. Then we have the following convergence:
Since $\mathcal{H}_1$ is independent of $\vec{u}$ and since $\vec{s}_k \longrightarrow \vec{s}$ ST in $\left(L^2(Q)\right)^2$, then
$\mathcal{H}_1(\vec{\mu}) = \lim_{k\to\infty} \mathcal{H}_1(\vec{\mu}_k) = 0$.
And

$$\int_Q g_{l1}(x,t,s_{1k})\, dxdt \longrightarrow \int_Q g_{l1}(x,t,s_1)\, dxdt, \int_Q g_{l2}(x,t,s_{2k},\mu_2)\, dxdt \longrightarrow \int_Q g_{l2}(x,t,s_2,\mu_2)\, dxdt \qquad (48)$$

From the hypotheses $h_{l1}(x,t,\mu_1)$ is WK lower semi continuous w.r.t. μ_1 for each $l = 0,2$, then from (48) one has

$$\int_Q g_{l1}(x,t,s_{1k})\, dxdt + \int_Q g_{l2}(x,t,s_{2k},\mu_2)\, dxdt + \int_\Sigma h_{l1}(x,t,\mu_1)\, d\sigma \leq$$
$$\lim_{k\to\infty} \inf[\int_\Sigma h_{l1}(x,t,\mu_1)\, d\sigma] + \int_Q g_{l1}(x,t,s_1)\, dxdt + \int_Q g_{l2}(x,t,s_2,\mu_2)\, dxdt$$
$$= \lim_{k\to\infty} \inf\, [\int_\Sigma h_{l1}(x,t,\mu_1)\, d\sigma] + \lim_{k\to\infty} \int_Q (g_{l1}(x,t,s_1) - g_{l1}(x,t,s_{1k})) dxdt \;+$$
$$\lim_{k\to\infty} \int_Q g_{l1}(x,t,s_{1k})\, dxdt + \lim_{k\to\infty} \int_Q (g_{l2}(x,t,s_2,\mu_2) - g_{l2}(x,t,s_{2k},\mu_2)) dxdt \;+$$
$$\lim_{k\to\infty} \int_Q g_{l2}(x,t,s_{2k},\mu_2)\, dxdt$$
$$= \lim_{k\to\infty} \inf\{[\int_\Sigma h_{l1}(x,t,\mu_1)\, d\sigma] + \int_Q (g_{l1}(x,t,s_{1k}) + g_{l2}(x,t,s_{2k},\mu_2)) dxdt]$$

Then
$\mathcal{H}_l(\vec{\mu}) \leq \lim_{k\to\infty} \inf \mathcal{H}_l(\vec{\mu}_k)$, (for each $l = 0,2$) .
But $\mathcal{H}_2(\vec{\mu}_k) \leq 0$, $\forall k$, then $\mathcal{H}_2(\,\vec{\mu}) \leq 0$ and one gets that $\vec{\mu} \in \vec{N}_A$ and that
$\mathcal{H}_0(\vec{\mu}) \leq \lim_{k\to\infty} \inf \mathcal{H}_0(\vec{\mu}_k) = \lim_{k\to\infty} \mathcal{H}_0(\vec{\mu}_k) = \inf_{\vec{u}\in\vec{W}_A} \mathcal{H}_0(\vec{\vec{\mu}}_k)$
$\Rightarrow \mathcal{H}_0(\vec{\mu}) = \min_{\vec{\vec{u}}\in\vec{W}_A} \mathcal{H}_0(\vec{\vec{\mu}}_k) \Rightarrow \vec{\mu}$ is a CCMOPCV .

5. The NOPC for Optimality

In this section, and under appropriate assumptions, the derivation of the FÉDE is obtained. The theorem of NOPC as well as the theorem of SOPC is demonstrated. Therefore it is necessary to start with the following assumptions, since they will be needed later.

Assumptions (III): If $\mathcal{F}_{1y_1}, g_{l_1y_1}, h_{l_1u_1}$ $(l = 0,1,2)$ are of a CATHT on $Q \times \mathbb{R}$, $Q \times \mathbb{R}$, $\Sigma \times \mathbb{R}$, respectively, then $\mathcal{F}_{2y_2}, \mathcal{F}_{2\mu_2}, g_{l_2y_2}, g_{l_2\mu_2}$, $(l = 0,1,2)$ are of a CATHT on $Q \times \mathbb{R}^2$,

$|\mathcal{F}_{1y_1}(x,t,y_1)| \leq \acute{L}_1, |\mathcal{F}_{2y_2}(x,t,y_2,\mu_2)| \leq \acute{L}_2$

$|g_{l_1s_1}(x,t,s_1)| \leq \zeta_{l1}(x,t) + e_{l1}|s_1|$, $|h_{l_1u_1}(x,t,\mu_1)| \leq \eta_{l1}(x,t) + f_{l1}|\mu_1|$,

$|g_{l_2u_2}(x,t,s_2,\mu_2)| \leq \zeta_{l2}(x,t) + e_{l1}|s_2| + f_{l1}|\mu_2|$

$|g_{l_2s_2}(x,t,s_2,\mu_2)| \leq \zeta_{l3}(x,t) + e_{l1}|s_2| + f_{l1}|\mu_2|$

where$(x,t) \in Q$, $s_i, \mu_1, \mu_2 \in \mathbb{R}$, $\zeta_{li}(x,t) \in L^2(Q)$,$\eta_{l1}(x,t) \in L^2(\Sigma)$,$e_{l1}(x,t), f_{l1}(x,t) \in L^2(Q)$

Theorem (5.1)

By dropping the index l, the Hamiltonian H is defined by

$H(x,t,\vec{s},\vec{z},\vec{\mu}) = [g_1(x,t,s_1) + h_1(x,t,\mu_1) + g_2(x,t,s_2,\mu_2)] + z_1\mathcal{F}_1(x,t,s_1) + z_2\mathcal{F}_2(x,t,s_2,\mu_2)$

Also, the adjoint state equation $z_i = z_{iu}$(where $s_i = s_{ui}$) satisfies

$-z_{1t} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial z_1}{\partial x_i}) + k_1(x,t)z_1 + k(x,t)z_2 = z_1\mathcal{F}_{1s_1}(x,t,s_1) + g_{1s_1}(x,t,s_1)$ on Ω

$-z_{2t} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(b_{ij}(x,t)\frac{\partial z_2}{\partial x_i}) + k_2(x,t)z_2 - k(x,t)z_1 = z_2\mathcal{F}_{2s_2}(x,t,s_2,\mu_2) + g_{2s_2}(x,t,s_2,\mu_2)$ on Ω

$z_1(x,T) = 0$, on Ω

$z_2(x,T) = 0$, on Ω

$\frac{\partial z_1}{\partial n} = 0$, on Σ

$\frac{\partial z_2}{\partial n} = 0$, on Σ

Then, the FÉDE of $\mathcal{H}$ is given by

$\acute{\mathcal{H}}(\vec{\mu})\overrightarrow{\Delta\mu} = \int_\Sigma (z_1 + h_{\mu_1}).\Delta\mu_1 d\sigma + \int_Q (z_2\mathcal{F}_{2\mu_2} + g_{\mu 2}).\Delta\mu_2 dxdt = (H_{\vec{u}}(x,t,\vec{s},\vec{z},\vec{\mu}), \overrightarrow{\Delta\mu})_{\Sigma\times Q}$

where $H_{\vec{\mu}}(x,t,\vec{s},\vec{z},\vec{\mu}) = ((z_1 + h_{\mu_1}), (z_2\mathcal{F}_{2\mu_2} + g_{2\mu_2}))$.

Proof

The WEKFM of the ADVEQ is

$$-\langle z_{1t}, v_1\rangle + a_1(t,z_1,v_1) + (k_1(t)z_1,v_1)_\Omega + (k(t)z_2,v_1)_\Omega = (z_1\mathcal{F}_{1s_1},v_1)_\Omega + (g_{1s_1},v_1)_\Omega \quad (49)$$

$$-\langle z_{2t}, v_2\rangle + a_2(t,z_2,v_2) + (k_2(t)z_2,v_2)_\Omega - (k(t)z_1,v_2)_\Omega = (z_2\mathcal{F}_{2s_2},v_2)_\Omega + (g_{2s_2},v_2)_\Omega \quad (50)$$

Now, by setting $v_1 = z_1$, $v_2 = z_2$ in (37) and (38), INBSw.r.t. t on $[0,T]$, then collecting the obtained equalities, one obtains

$$\int_0^T \langle\overrightarrow{\Delta s}_t, \vec{z}\rangle dt + \int_0^T [a_1(t,\Delta s_1,z_1) + (k_1(t)\Delta s_1,z_1)_\Omega - (k(t)\Delta s_2,z_1)_\Omega + a_2(t,\Delta s_2,z_2) + (k_2(t)\Delta s_2,z_2)_\Omega + (k(t)\Delta s_1,z_2)_\Omega]dt = \int_0^T (\mathcal{F}_1(s_1+\Delta s_1),z_1)_\Omega dt - \int_0^T (\mathcal{F}_1(s_1),z_1)_\Omega dt + \int_0^T (\Delta\mu_1,z_1)_\Gamma dt + \int_0^T (\mathcal{F}_2(s_2+\Delta s_2),\mu_2),z_2)_\Omega dt - \int_0^T (\mathcal{F}_2(s_2,u_2),z_2)_\Omega dt \quad (51)$$

The FÉDE of $\mathcal{F}_1, \mathcal{F}_2$ exist for each $\forall s_i \in L^2(Q)$ (from Assumption (I)-ii and proposition (3.1) in [11]),

after utilizing the outcome of Theorem (4.1), they are

$$\int_0^T (\mathcal{F}_1(x,t,s_1+\Delta s_1) - \mathcal{F}_1(x,t,s_1),z_1)_\Omega dt = \int_0^T (\mathcal{F}_{1s_1}\Delta s_1,z_1)dt + \varepsilon_1(\overrightarrow{\Delta\mu})\|\overrightarrow{\Delta\mu}\|_{\Sigma\times Q}, \quad (52)$$

$$\int_0^T (\mathcal{F}_2(x,t,s_2+\Delta s_2,\mu_2) - \mathcal{F}_2(x,t,s_2,\mu_2),z_2)_\Omega dt = \int_0^T (\mathcal{F}_{2s_2}\Delta s_2,z_2)dt + \int_0^T (\mathcal{F}_{2\mu_2}\Delta\mu_2,z_2)dt + \varepsilon_2(\overrightarrow{\Delta\mu})\|\overrightarrow{\Delta\mu}\|_{\Sigma\times Q} \quad (52b)$$

where $\varepsilon_i(\overrightarrow{\Delta\mu}) \longrightarrow 0$ as $\|\overrightarrow{\Delta\mu}\|_{\Sigma\times Q} \longrightarrow 0$, $i = 1,2$
Using (52 a&b) in R.HN.S. of (51) yields

$$\int_0^T \langle\overrightarrow{\Delta s_t}, \vec{z}\rangle dt + \int_0^T [a_1(t, \Delta s_1, z_1) + (k_1(t)\Delta s_1, z_1)_\Omega - (k(t)\Delta s_2, z_1)_\Omega + a_2(t, \Delta s_2, z_2) + (k_2(t)\Delta s_2, z_2)_\Omega + (k(t)\Delta s_1, z_2)_\Omega]dt = \int_0^T (\mathcal{F}_{1s_1}\Delta s_1, z_1)_\Omega dt + \int_0^T (\mathcal{F}_{2s_2}\Delta s_2, z_2)_\Omega dt + \int_0^T (\mathcal{F}_{2\mu_2}\Delta\mu_2, z_2)dt + \int_0^T (\Delta\mu_1, z_1)_\Gamma dt + \varepsilon_3(\overrightarrow{\Delta\mu})\|\overrightarrow{\Delta\mu}\|_{\Sigma\times Q} \quad (53)$$

Now, substituting $v_1 = \Delta s_1$ and $v_2 = \Delta s_2$ in (49) and (50), respectively, INBSw.r.t t on $[0, T]$, using the integrating part formula for the first term of each obtained equality, and then collecting the outcomes, gives

$$\int_0^T \langle\overrightarrow{\Delta s_t}, \vec{z}\rangle dt + \int_0^T [a_1(t, z_1, \Delta s_1) + (k_1(t)z_1, \Delta s_1)_\Omega + (k(t)z_2, \Delta s_1)_\Omega + a_2(t, z_2, \Delta s_2) + (k_2(t)z_2, \Delta s_2)_\Omega - (k(t)z_1, \Delta s_2)_\Omega]dt = \int_0^T (z_1\mathcal{F}_{1s_1}, \Delta s_1)_\Omega dt + \int_0^T (g_{1s_1}, \Delta s_1)_\Omega dt + \int_0^T (z_2\mathcal{F}_{2s_2}, \Delta s_2)_\Omega dt + \int_0^T (g_{2s_2}, \Delta s_2)_\Omega dt \quad (54)$$

By subtracting (54) from (53), one gets

$$\int_0^T (g_{1s_1}, \Delta s_1)_\Omega dt + \int_0^T (g_{2s_2}, \Delta s_2)_\Omega dt = \int_0^T (\mathcal{F}_{2u_2}\Delta\mu_2, z_2)dt + \int_0^T (\Delta\mu_1, z_1)_\Omega dt + \varepsilon_3(\overrightarrow{\Delta\mu})\|\overrightarrow{\Delta\mu}\|_{\Sigma\times Q} \quad (55)$$

Now, let $\mathcal{H}_A(\vec{\mu}) = \int_Q g_1(x, t, s_1)dxdt + \int_\Sigma h_1(x, t, \mu_1)d\sigma$

$\mathcal{H}_B(\vec{\mu}) = \int_Q g_2(x, t, s_2, \mu_2)dxdt.$

From the FÉDE and the result of theorem (4.1), one has

$$\mathcal{H}_A(\vec{\mu} + \overrightarrow{\Delta\mu}) - \mathcal{H}_A(\vec{\mu}) = \int_Q (g_{1s_1}\Delta s_1 dxdt + \int_\Sigma h_{1\mu_1}\Delta\mu_1 d\sigma + \varepsilon_4(\overrightarrow{\Delta\mu})\|\overrightarrow{\Delta\mu}\|_{\Sigma\times Q} \quad (56)$$

$$\mathcal{H}_B(\vec{\mu} + \overrightarrow{\Delta\mu}) - \mathcal{H}_B(\vec{\mu}) = \int_Q g_{2s_2}\Delta s_2 dxdt + \int_Q g_{2\mu_2}\Delta\mu_2 dxdt + \varepsilon_5(\overrightarrow{\Delta\mu})\|\overrightarrow{\Delta\mu}\|_{\Sigma\times Q} \quad (57)$$

Collecting (56) and (57) leads to

$$\mathcal{H}(\vec{\mu} + \overrightarrow{\Delta\mu}) - \mathcal{H}(\vec{\mu}) = \int_Q (g_{1s_1}\Delta s_1 + g_{2s_2}\Delta s_2)\, dxdt + \int_Q g_{2\mu_2}\Delta\mu_2 dxdt + \int_\Sigma h_{1\mu_1}\Delta\mu_1 d\sigma + \varepsilon_6(\overrightarrow{\Delta\mu})\|\overrightarrow{\Delta\mu}\|_{\Sigma\times Q} \quad (58)$$

Substituting (55) in (58) gives

$$\mathcal{H}(\vec{\mu} + \overrightarrow{\Delta\mu}) - \mathcal{H}(\vec{\mu}) = \int_0^T (\mathcal{F}_{2\mu_2}\Delta\mu_2, z_2)dt + \int_Q g_{2\mu_2}\Delta\mu_2 dxdt + \int_0^T (\Delta\mu_1, z_1)_\Omega dt + \int_\Sigma h_{1\mu_1}\Delta\mu_1 d\sigma + \varepsilon_6(\overrightarrow{\Delta\mu})\|\overrightarrow{\Delta\mu}\|_{\Sigma\times Q}$$

where $\varepsilon_6(\overrightarrow{\Delta\mu}) \to 0$ as $\|\overrightarrow{\Delta\mu}\|_{\Sigma\times Q} \to 0$

Using Proposition (3.2) in [11], the FÉDE of $\mathcal{H}$ is

$$(\acute{\mathcal{H}}(\vec{\mu}), \overrightarrow{\Delta\mu}) = \int_\Sigma (z_1 + h_{1\mu_1}).\Delta\mu_1 d\sigma + \int_Q (z_2\mathcal{F}_{2\mu_2} + g_{2\mu_2}).\Delta\mu_2 dxdt = (H_{\vec{\mu}}(x, t, \vec{s}, \vec{z}, \vec{\mu}), \overrightarrow{\Delta\mu})_{\Sigma\times Q}$$

where $H_{\vec{\mu}}(x, t, \vec{s}, \vec{z}, \vec{\mu}) = ((z_1 + h_{1\mu_1}), (z_2\mathcal{F}_{2\mu_2} + g_{2\mu_2}))$.

Theorem (5.2): The NOPC for *Optimality*

If $\vec{\mu} \in \vec{\mathcal{N}}_A$ is a CCMOPCV, i.e. there exists multipliers $\xi_l \in R$, $l = 0,1,2$ with $\xi_0 \geq 0$,$\xi_2 \geq 0$, $\sum_{l=0}^{2}|\xi_l| = 1$, such that

$$\sum_{l=0}^{2} \xi_l \acute{\mathcal{H}}_l(\vec{\mu})(\vec{\acute{\mu}} - \vec{\mu}) \geq 0 , \forall \vec{\acute{\mu}} \in \vec{\mathcal{N}} \quad (59)$$

&

$$\xi_2\mathcal{H}_2(\vec{\mu}) = 0 \quad (60)$$

Also, (59) is equivalent to the following minimum principle

$$H_{\vec{\mu}}(x, t, \vec{s}, \vec{z}, \vec{\mu})\vec{\mu} = \min_{\vec{\acute{\mu}}\in U} H_{\vec{\mu}}(x, t, \vec{s}, \vec{z}, \vec{\mu})\vec{\acute{\mu}} \quad \text{a.e on } \Sigma\times Q \quad (61)$$

where $H_{\vec{\mu}}(x, t, \vec{s}, \vec{z}, \vec{\mu}) = ((z_1 + h_{1\mu_1}), (z_2\mathcal{F}_{2\mu_2} + g_{2\mu_2}))$.

Proof

From assumptions (I), (II) and (III), the functions $\mathcal{H}_l(\vec{\mu})$ and $\acute{\mathcal{H}}_l(\vec{\mu})$ are continuous (for l=0,1,2) and are linear w.r.t. $(\vec{\vec{\mu}} - \vec{\mu})$. Therefore, $\mathcal{H}_l(\vec{\mu})$ is ρ −differentiable at every $\vec{\mu} \in \overrightarrow{\mathcal{N}}$, $\forall\rho$. Hence, by applying the KUTULATH, there exists multipliers $\xi_l \in \mathbb{R}$, $l = 0,1,2$, with $\xi_0 \geq 0$, $\xi_2 \geq 0$, $\sum_{l=0}^{2}|\xi_l| = 1$, such that (60) -(61) are held, or

$\left(\xi_0\acute{\mathcal{H}}_0(\vec{\mu}) + \xi_1\acute{\mathcal{H}}_1(\vec{\mu}) + \xi_2\acute{\mathcal{H}}_2(\vec{\mu})\right) \cdot \left(\vec{\vec{\mu}} - \vec{\mu}\right) \geq 0$, $\forall\vec{\vec{\mu}} \in \overrightarrow{\mathcal{N}}$.

By utilizing Theorem (5.1), putting $\overrightarrow{\Delta\mu} = \vec{\vec{\mu}} - \vec{\mu}$, and employing the FÉDE of $\mathcal{H}_l$, $\forall l = 0,1,2$ in (58), we obtain

$\sum_{l=0}^{2} \xi_l \left(\int_\Sigma (z_1 + h_{1\mu_1})\Delta\mu_1 d\sigma + \int_Q (z_2\mathcal{F}_{2\mu_2} + g_{2\mu_2})\Delta\mu_2 dxdt\right) \geq 0.$

Let $z_1 = \sum_{l=0}^{2} \xi_l\, z_{l1}$, $h_{1\mu_1} = \sum_{l=0}^{2} \xi_l\, h_{l1_{\mu_1}}$, $z_2 = \sum_{l=0}^{2} \xi_l\, (z_{l2}\mathcal{F}_{2\mu_2})$ and $g_{2\mu_1} = \sum_{l=0}^{2} \xi_l g_{l2\mu_1}$

$$\Rightarrow \int_{\Sigma\times Q} H_{\vec{\mu}}(x,t,\vec{s},\vec{z},\vec{\mu}) \cdot \overrightarrow{\Delta\mu}\, d\sigma \geq 0 \qquad (62)$$

Consider that $\overrightarrow{\mathcal{N}}_{\vec{U}} = \{\vec{\mu} \in (L^2(\Sigma,\mathbb{R}))^2 | \vec{\mu}(x,t) \in \vec{\mathcal{U}} \text{ a.e. in} \Sigma \times Q\}$, $\vec{\mathcal{U}} \subset \mathbb{R}^2$, $\{\vec{\vec{\mu}}\}$ is a "dense" sequence in $\overrightarrow{\mathcal{N}}_{\vec{U}}$ and ϖ is Lebesgue measure on $\Sigma \times Q$. Let $S \subset \Sigma \times Q$ be a measurable set which has the property

$$\vec{\vec{\mu}}(x,t) = \begin{cases} \vec{\mu}_k(x,t), \text{if } (x,t) \in S \\ \vec{\mu}(x,t), \text{if } (x,t) \notin S. \end{cases}$$

Therefore, (62) becomes

$\int_S H_{\vec{\mu}}(x,t,\vec{s},\vec{z},\vec{\mu}) \cdot (\vec{\mu}_k - \vec{\mu}) dS \geq 0$, $\forall S$

Using theorem (3.1), we get

$H_{\vec{\mu}}(x,t,\vec{s},\vec{z},\vec{\mu}).(\vec{\mu}_k - \vec{\mu}) \geq 0$, a.e. in $\Sigma \times Q$.

Therefore, the inequality holds everywhere on the boundary $\Sigma \times Q$ of Q, except in a subset Σ_k for which $(\Sigma_k) = 0$, $\forall k$, or it hold everywhere on the boundary $\Sigma \times Q$, except in $\bigcup_k \Sigma_k$ with $\varpi(\bigcup_k \Sigma_k) = 0$. Since $\{\vec{\vec{\mu}}_k\}$ is dense in $\overrightarrow{\mathcal{N}}$, then there exists $\vec{\vec{\mu}} \in \overrightarrow{\mathcal{N}}$ such that $H_{\vec{\mu}}(x,t,\vec{s},\vec{z},\vec{\mu})\vec{\mu} = \min_{\vec{w}\in\vec{U}} H_{\vec{\mu}}(x,t,\vec{s},\vec{z},\vec{\mu})\vec{\vec{\mu}}$, a.e. in $\Sigma \times Q$, $\forall\vec{\vec{\mu}} \in \overrightarrow{\mathcal{N}}$.

The converse is clear.

6. The SOPC for Optimality

Theorem (6.1): The SOPC for *Optimality*

Consider that the assumptions (I), (II), and (III) are held, $\overrightarrow{\mathcal{N}} = \overrightarrow{\mathcal{N}}_{\vec{U}}$ is convex, $\mathcal{F}_1$, $g_{11,}g_{21,}$ and g_{01} are affine w.r.t. s_1 , $\forall(x,t)$, $\mathcal{F}_2, g_{22}, g_{12}, g_{02}$ are convex w.r.t. (s_2,μ_2), and h_{01}, h_{11}, h_{21} are convex w.r.t. μ_1, $\forall(x,t)$. Then, the NOPC in Theorem (5.2) are sufficient, if $\xi_0 > 0$.

Proof

Assume that $\vec{\mu}$ satisfies the KUTULA condition (59) with $\vec{\mu} \in \overrightarrow{\mathcal{N}}_A$, i.e.

$[\int_\Sigma (z_1 + h_{1\mu_1})\, d\sigma + \int_Q (z_2\mathcal{F}_{2\mu_2} + g_{2\mu_2})\, dxdt] \geq 0$, $\forall\vec{\vec{\mu}} \in \overrightarrow{\mathcal{N}}$

$\&\xi_2\mathcal{H}_2(\vec{\mu}) = 0.$

Let $\mathcal{H}(\vec{\mu}) = \sum_{l=0}^{2} \lambda_l \mathcal{H}_l(\vec{\mu})$, then

$\acute{\mathcal{H}}(\vec{\mu}) \cdot \overrightarrow{\Delta\mu} = \sum_{l=0}^{2} \lambda_l \acute{\mathcal{H}}_l(\vec{\mu}) \cdot \overrightarrow{\Delta\mu} = \xi_0[\int_\Sigma \left(z_{01} + h_{01\mu_1}\right) d\sigma + \int_Q \left(z_{02}\mathcal{F}_{2\mu_2} + g_{02\mu_2}\right) dxdt +$

$\xi_1 \, [\int_\Sigma \left(z_{11} + h_{11\mu_1}\right) d\sigma +$

$\int_Q (z_{12}\mathcal{F}_{2\mu_2} + g_{12\mu_2})\, dxdt + \xi_2[\int_\Sigma \left(z_{21} + h_{21\mu_1}\right) d\sigma + \int_Q \left(z_{22}\mathcal{F}_{2\mu_2} + g_{22\mu_2}\right) dxdt \geq 0,$

since $\mathcal{F}_1$,$\mathcal{F}_2$ in the R.HN.S. of (1)-(2) are affine w.r.t. s_1, s_2 $\forall(x,t) \in Q$, respectively, i.e.

$\mathcal{F}_1(x,t,s_1) = \mathcal{F}_{11}(x,t)s_1 + \mathcal{F}_{12}(x,t)$ & $\mathcal{F}_2(x,t,s_2,\mu_2) = \mathcal{F}_{21}(x,t)s_2 + \mathcal{F}_{22}(x,t)\mu_2 + \mathcal{F}_{23}(x,t)$.

Let $\vec{\mu} = (\mu_1,\mu_2)\&\vec{\vec{\mu}} = (\bar{\mu}_1,\bar{\mu}_2)$ be two CCMCVs and (by Theorem (3.1)) $\vec{s} = \left(s_{\mu_1}, s_{\mu_2}\right) = (s_1,s_2)$ & $\vec{\vec{y}} = \left(\bar{s}_{\bar{\mu}_1}, \bar{s}_{\bar{\mu}_2}\right) = (\bar{s}_1,\bar{s}_2)$ are their SVES, i.e.

$s_{1t} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial s_1}{\partial x_j}) + k_1(x,t)s_1 - k(x,t)s_2 = \mathcal{F}_{11}(x,t)s_1 + \mathcal{F}_{12}(x,t)$
$\sum_{i,j=1}^{2} a_{ij}\frac{\partial s_1}{\partial n} = \mu_1(x,t)$,
$s_1(x,0) = s_1^0(x)$
and
$\bar{s}_{1t} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial \bar{s}_1}{\partial x_j}) + k_1(x,t)\bar{s}_1 - k(x,t)\bar{s}_2 = \mathcal{F}_{11}(x,t)\bar{s}_1 + \mathcal{F}_{12}(x,t)$
$\sum_{i,j=1}^{2} a_{ij}\frac{\partial \bar{s}_1}{\partial n} = \bar{\mu}_1\,(x,t)$
$\bar{s}_1(x,0) = s_1^0(x)$.
Multiply the first above equality and its initial condition by $\theta \in [0,1]$, and the second one and its initial condition by $(1-\theta)$, then collect the outcome equalities and their initial conditions, to get

$$(\theta s_1 + (1-\theta)\bar{s}_1)_t - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial(\theta s_1+(1-\theta)\bar{s}_1)}{\partial x_j}) + k_1(x,t)(\theta s_1 + (1-\theta)\bar{s}_1) - k(x,t)(\theta s_2 + (1-\theta)\bar{s}_2) = \mathcal{F}_{11}(x,t)(\theta s_1 + (1-\theta)\bar{s}_1) + \mathcal{F}_{12}(x,t) \quad (63a)$$

$$\theta s_1(x,0) + (1-\theta)\bar{s}_1(x,0) = s_1^0(x) \quad (63b)$$

$$\sum_{i,j=1}^{2} a_{ij}\frac{\partial(\theta s_1+(1-\theta)\bar{s}_1)}{\partial n} = (\theta\mu_1 + (1-\theta\bar{\mu}_1) \text{ , on } \Sigma \quad (63c)$$

$(\theta s_2 + (1-\theta)\bar{s}_2)_t - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(b_{ij}(x,t)\frac{\partial(\theta s_2+(1-\theta)\bar{s}_2)}{\partial x_j}) + k_2(x,t)(\theta s_{21} + (1-\theta)\bar{s}_2) +$
$k(x,t)(\theta s_1 + (1-\theta)\bar{s}_1) = \mathcal{F}_{21}(t)(\theta s_2 + (1-\theta)\bar{s}_2) + \mathcal{F}_{22}(x,t)(\theta u_2 + (1-\theta\bar{\mu}_2) +$
$\mathcal{F}_{23}(x,t)$ (64a)
$\theta s_2(x,0) + (1-\theta)\bar{s}_2(x,0) = s_2^0(x)$
(64b)

$$\sum_{i,j=1}^{2} b_{ij}\frac{\partial(\theta s_2+(1-\theta)\bar{s}_2)}{\partial n} = 0 \text{ , on Q} \quad (64c)$$

Equations (63)-(64) explain that the CCMCV$\vec{\tilde{\mu}} = (\tilde{\mu}_1, \tilde{\mu}_2)$, with $\vec{\tilde{\mu}} = \theta\vec{\mu} + (1-\theta)\vec{\bar{\mu}}$, has the corresponding SEVS, $\vec{\tilde{s}} = (\tilde{s}_1, \tilde{s}_2)$, with $\vec{\tilde{s}} = \theta\vec{s} + (1-\theta)\,\vec{\bar{s}}$. Hence, $\vec{\mu} \longmapsto \vec{s}_{\vec{\mu}}$ is convex – linear w.r.t. $(\vec{s},\vec{u})$, $\forall(x,t)$.
Now, since 〖 g〗 _11 (x,t,s_1), g_12 (x,t,s_2,μ_2) are affine w.r.t 〖 s〗 _1 〖(s〗 _2,μ_2) and h_11 (x,t,μ_1) is affine w.r.t. μ_i, ∀(x,t)∈Σ , respectively, i.e.
$g_{11}(x,t,s_1) = I_{11}(x,t)s_1 + I_{21}(x,t)$, $h_{11}(x,t,s_1) = I_{11}(x,t)\mu_1 + I_{31}(x,t)$ and
$g_{12}(x,t,s_2,\mu_2) = I_{12}(x,t)s_2 + I_{22}(x,t)\mu_2 + I_{32}(x,t)$.
Let $\vec{\mu}$ & $\vec{\bar{\mu}}$ be two CCMCVs and $\vec{s} = \vec{s}_{\vec{\mu}}$ & $\vec{\bar{y}} = \vec{\bar{s}}_{\vec{\bar{\mu}}}$ are their corresponding SEVS. Then,
$\mathcal{H}_1(\theta\vec{\mu} + (1-\theta)\,\vec{\bar{\mu}}) =$
$\int_Q g_{11}(x,t,s_{1(\theta\mu_1+(1-\theta)\bar{\mu}_1})dxdt + \int_Q g_{12}(x,t,s_{2(\theta\mu_2+(1-\theta)\bar{\mu}_2)},(\theta\mu_2 + (1-\theta)\bar{\mu}_2)dxdt +$
$\int_\Sigma h_{11}(x,t,(\theta\mu_1 + (1-\theta)\bar{\mu}_1)\,d\sigma$.
Since the operator $\vec{\mu} \longmapsto \vec{s}_{\vec{\mu}}$ is convex – linear, then
$\mathcal{H}_1(\theta\vec{\mu} + (1-\theta)\,\vec{\bar{\mu}}) = \theta\mathcal{H}_1(\vec{\mu}) + (1-\theta)\mathcal{H}_1(\vec{\bar{\mu}})$
$\Rightarrow \mathcal{H}_1(\vec{\mu})$ is convex – linear w.r.t. $(\vec{s},\vec{\mu})$, $\forall(x,t) \in Q$.
From the Assumptions, $\int_Q g_{01}\,dxdt$ is convex w.r.t. y_1 ,$\int_Q g_{02}\,dxdt$ is convex w.r.t. (s_2,μ_2), and $\int_\Sigma h_{01}\,d\sigma$ is convex w.r.t. μ_1. Then, $\mathcal{H}_0(\vec{\mu})$&$\mathcal{H}_2(\vec{\mu})$ are convex w.r.t. $(\vec{s},\vec{\mu})$ $(\forall(x,t) \in Q, \forall(x,t) \in \Sigma)$, i.e. $\mathcal{H}(\vec{\mu})$ is convex w.r.t. $(\vec{s},\vec{\mu})$ $(\forall(x,t) \in Q, \forall(x,t) \in \Sigma)$. Also since $\vec{\mathcal{N}} = \vec{\mathcal{N}}_{\vec{U}}$ is convex and $\mathcal{H}_l(\vec{\mu})$ $(\forall l = 0,1,2)$ has a continuous FÉDE for each $\vec{\mu} \in \vec{\mathcal{N}}$ (by Theorem (5.1) and Assumptions (I), (II) and (C)) , then it satisfies $\acute{\mathcal{H}}(\vec{\mu})\overrightarrow{\Delta\mu} \geq 0$. Thus $\mathcal{H}(\vec{\mu})$ has a minimum at $\vec{\mu}$, i.e.

$$\xi_0\mathcal{H}_0(\vec{\mu}) + \xi_1\mathcal{H}_1(\vec{\mu}) + \xi_2\mathcal{H}_2(\vec{\mu}) \leq \xi_0\mathcal{H}_0(\vec{e}) + \xi_1\mathcal{H}_1(\vec{e}) + \xi_2\mathcal{H}_2(\vec{e}) \quad (65)$$

Let $\vec{e} \in \overrightarrow{\mathcal{N}}_A$, with $\xi_2 \geq 0$, then from (64) we have
$\Rightarrow \mathcal{H}_0(\vec{\mu}) \leq \mathcal{H}_0(\vec{e}), \forall \vec{e} \in \overrightarrow{\mathcal{N}}$, since $(\xi_0 > 0)$.
$\therefore \vec{\mu}$ is a CCMOPCV .

7. Conclusions

The EXUNTh of a CCMOPCV that is ruling by the considered CNLPPDEs with the STCOs is demonstrated using the MGA. The existence of a CCMOPCV is demonstrated under appropriate conditions, whilst the EXUNTh for the couple of ADVEQ related with the considered CNLPPDEs is considered and the derivation of the FÉDE of the Hamiltonian is obtained. Lastly the theorems of the NOPC and the SOPC of the CNLPPDEs with the STCOs are demonstrated.

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