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Involutive Gamma Derivations on n-Gamma Lie Algebra and 3- Pre Gamma -Lie Algebra

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Abstract

In this paper, the structure of $n - \Gamma - Lie\, Algebra$ and $3 - \Gamma - Pre\, Lie\, Algebra$ have been introduced and studied. We also obtain that a $\Gamma -$ $Lie\, algebra\, V$ is $one\ \lambda - dimentional\ extension$ of a $\Gamma - Lie\, algebra$ if and only if there exists an $involutive\ \lambda - derivation\ D_\lambda$ on V such that $dimV_1 = 1\ or\ dimV_{-1} = 1$. In addition, we obtain that $two - \lambda - dimensional\ extension$ of Γ – $Lie\ algebras$ if and only if there is an $involutive - \lambda - derivation\ D_\lambda$ on $U = U_1\ , U = U_{-1}$ such that $U_1 = 2or\ dimU_{-1} = 2$, where $U_1\ andU_{-1}$ are subspaces of U with eigenvalues 1 and -1, respectively. We also find t that the existence of $involutive - \lambda - derivation\ D_\lambda$ on $3 - \Gamma - Lie\ algebra$ implies that there exists a compatible $3 - \Gamma - Pre\ Lie\ algebra$ under appropriate condition.

Keywords: $Algebra, Lie\ Algebra\ , Derivation\ , Gamma\ Lie\ algebra$.

اشتقاقات كاما اللاارادية علىn -كاما جبر لي و3-كاما جبر لي العكسي

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الخلاصة

في هذا البحث ،قدمت ودرست بنية n –كاما- جبر لي و 3-كاما- جبر لي العكسي واستنتج ان كاما -جبر لي V هو توسيع اول ذو بعد λ ل كاما- جبر لي اذا وفقط اذا وجد λ–اشتقاق لا ارادي D_λ على V حيث بعد V_1=1او بعد V_{-1}=1.وكذلك استنتج توسيع ثاني ذو بعد λ ل كاما- جبور لي اذا وفقط اذا وجد λ–اشتقاق لا ارادي D_λ على $U = U_1\ , U = U_{-1}$ بحيث بعد U_1=2 اوبعد U_{-1} =2 عندما U_1وU_{-1} فضاءات جزئية من U مع قيم ذاتية 1 و -1 ،على التوالي .واستنتج ان وجود λ–اشتقاق لا ارادي D_λ على 3-كاما- جبر لي يؤدي الى وجود 3-كاما جبر لي العكسي تحت شروط مناسبة .

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Introduction

The notion of $n - Lie\ algebra$ was introduced by Filippov [1]. Derivation have also a relation with the extensions of $n - Lie\ algebra$.The concept of $3 - Lieclassical\ Yang\ Baxter$ equations was introduce in [2], as well as $Involutive\ Derivation$ is an important concept in $3 - Lie\ algebra$. In[3] authors investigated the existence of $involutive\ derivations$ and studied its properties on $n - Lie\ algebra$ .They also investigated a class of $3 - Lie\ algebras$ with $involutive\ derivations$ which are $two - dimensional\ extension$ of $Lie\ algebra$ .A. H. Rezaei and B. Davvaz. in [4] introduced the notion of Construction of $\Gamma\ - algebra$ and $\Gamma - lie\ admissible\ algebras$. The concept of compatible with$3 - pre\ Liealgebra\ (A, \{,,\}_D)$ such that A is$adjacent$ $3 - Lie\ algebra$ in particular is introduced in [5]. For more results on Gamma $- derivations$ can be found in [6,7] .

We study the structure of n-Gamma Lie Algebra and 3-Gamma Pre-Algebra, and the algebra $D_\lambda(V)$ is a Lie $\lambda - subalgebra$ of $gl_\lambda(V)$ has been obtained . We also show that if $n = 2r\ \ r \geq 1$ then there is an $involutive\ \lambda - derivation\ D\ on\ V$ if and only if V is $abelian$. Furthermore, if $n = 2r + 1, r \geq 1$ then there is an $involutive\ \lambda - derivation$ on V if and only if V has the $decomposition\ V = A + B$, so that $A = V_1$ and $= V_{-1}$ as well as if V $3 - \lambda - Lie\ Algebras$ then V is $one\ dimensional\ extension$ of a $\lambda - Lie\ Algebras$ $(V, [\,,]_\lambda)$ if and only if the exists an $involutive\ \lambda - derivation\ D_\lambda$ on V such that $dimV_1 = 1$,or $dimV_{-1} = 1$. Moreover if (U, [, ,]) is a $3 - \lambda - Lie\ Algebras$ then U has a $two\ dimensional\ extension$
$3 - \lambda - Lie\ Algebras\ of\ \lambda - Lie\ Algebras$ if and onlyeif there is an $involutive\ - \lambda - derivation\ D$ on U such that $dimU_1 = 2\ or\ dimU_{-1} = 2$, where$U_1 and\ U_{-1}$ are subspaces of U with eigenvalues 1 and -1, respectively. The existence of $involutive\ \lambda - derivation\ D_\lambda$ on $3 - \Gamma - Lie\ algebra$ is obtained, it implies that there exists a compatible $3 - Pre - \Gamma - Lie\ \ algebra\ \ (V, \{\ ,\ ,\}_{\lambda D})$ where $\{u_1, u_2, u_3\}_{\lambda D} = [Du_1, Du_2, u_3]_\lambda, \forall\, u_1\ , u_2, u_3 \in V$. This is done under appropriate condition .

1-Prelimainaries

In this section, we introduce the basic definitions and examples which are used throughout this paper.

Definition 1.1 :- [4] Let Γ be a groupoid and V be a $vector\ space$ over a field F. Then ,V is called a $\Gamma - algebra$ over the field F if there exists a mapping $V \times \Gamma \times V \rightarrow V$ (the image is denoted by $u_1 \lambda\, u_2, for\ u_1, u_2, u_3 \in V\ and\ \lambda \in \Gamma$) such that the following conditions hold:

(1)$(u_1 + u_2)\lambda u_3 = u_1\lambda u_3 + u_2 \lambda u_3$, $u_1\lambda(u_2 + u_3) = u_1\lambda u_2 + u_1 \lambda u_3$

(2) $u_1(\lambda + \beta)u_2 = u_1\lambda u_2 + u_1\beta u_2$

(3) $(cu_1)\lambda u_2 = c(u_1\lambda u_2) = u_1\lambda(cu_2), for\ all\ u_1, u_2, u_3 \in V, c \in F\ and\ \lambda, \beta \in \Gamma.$

Moreover, $\Gamma - algebr$ is called $associative$ if

(4) $(u_1\lambda u_2)\beta u_3 = u_1\lambda(u_2\beta u_3)$

Example 1.2 :- Let V be the set of 2×3 matrices over the field of real numbers R and $\left\{ \Gamma = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \\ 0 & 0 \end{bmatrix} \quad \alpha, \beta \in R \right\}$. Then V is an associative $\Gamma -\ algebra$.

Definition 1.3:- [4] Let V be an $associative\ \Gamma - algebra$ over a fi$eld\ F$.Then, for every $\lambda \in \Gamma$ one can construct an $\lambda - Lie\ algebra\ L_\lambda(V)$ as a vector space, $L_\lambda(V)$, which is the same as V . The Lie bracket of two elements of $L_\lambda(V)$ is defined to be their commutator in V , $[u, v]_\lambda = u\lambda v - v\lambda u$. Note that $[u, v]_\lambda = -[v, u]_\lambda$for every $u, v \in V$ and $\lambda \in \Gamma$. Also, $L_\lambda(V)$ is abelian if either char $(F) = 2$ or char $(F) \neq 2$ then $[u, v]_\lambda$= 0 for every $, v \in V$.

Example 1.4:- Let V be the set of all real 3 × 5 matrices of the form

$$\begin{pmatrix} 0 & a & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 \end{pmatrix}$$

and Γ b is the set of all real 5× 3 matrices. Then, $\forall\, \lambda \in \Gamma$ of the shape

$$\begin{pmatrix} \alpha & \beta & \delta \\ 0 & 0 & 0 \\ \mu & \rho & \sigma \\ \theta & \vartheta & \tau \\ 0 & 0 & 0 \end{pmatrix}$$

Thus for every $A, B \in V$, we have $[A, B]_\lambda = 0$, so that $L_\lambda(V)$ is $abelian$,and the $\lambda - dimension$ of V is zero.

Definition 1.5:- [4] Let V and U be two $associative\ \Gamma - algebras$ over a field F and $\lambda \in \Gamma$. A linear transformation $\varphi^\lambda : V \to U$ is called an λ-homomorphism if $\varphi^\lambda([v, u]_\lambda) = [\varphi^\lambda(v), \varphi^\lambda(u)]_\lambda$ for all $v, u \in V$, and if $Ker(\varphi^\lambda) = 0$, then φ^λ is called an $\lambda - monomorphism$, while it is called $\lambda - epimorphism$ if $Im(\varphi^\lambda) = U$. φ^λ is called an $\lambda - isomorphism$ if both $\lambda - monomorphism$ and $\lambda - epimorphism$ are satisfied. If $\varphi^\lambda(v) = 0$, then $Ker(\varphi^\lambda)$ is an $\lambda - ideal$ of $L_\lambda(V)$ certainly, and if $u \in V$ is arbitrary, then $\varphi^\lambda([v\, u]_\lambda) = [\varphi^\lambda(v), \varphi^\lambda(u)]_\lambda = 0$. It is also apparent that $Im(\varphi^\lambda)$ is an $\lambda - Lie\ subalgebra\ of\ L_\lambda\ (U)$.

Definition 1.6:- [1] An $n - Lie\ algebra$ is a $vector\ space$ V over a field F endowed with a $linear\ multiplication$ $[\ ,\ldots,\ ]:\wedge^n V \to V$ satisfying for all $v_1, \ldots, v_n, u_2, \ldots, u_n \in V$ $[[v_1, \ldots . v_n], u_2, \ldots . u_n] = \sum_{i=1}^{n} [v_1, \ldots, [v_i, u_2, \ldots, u_n], \ldots, v_n]$.This equation is usually called the generalized $Jacobi\ identity, or\ Filippov$ identity .The $Lie\ sub\ algebra$ generated by the vectors$[v_1, \ldots, v_n]$ for any $v_1, \ldots, v_n \in V$ is called the $derived\ algebra$ of V, which is denoted by V^1 . If $V^1 = 0$, V is called an $abelian\ algebra$.

Definition 1.7:- [1] The derived algebra of an $n - Lie\ algabra\ V$ is a $subalgabra$ of V generated by $[v_1, \ldots, v_n]$ for all $v_1, \ldots . v_n \in V$ and is a $linear\ transformation$ $D : V \to V$. Satisfying , $D([v_1, \ldots, v_n]) = \sum_{i=1}^{n} [v_1 \ldots , D(v_i), \ldots v_n]$ for all $v_1, \ldots, v_n \in V$ and the set of all $derivation$ is denoted by $Der(V)$for all $v_1, \ldots, v_n \in V$. The map $ad(v_1, \ldots, v_{n-1}): V \to V$ is given by $ad(v_1, \ldots . v_{n-1})(u) = [v_1, \ldots v_{n-1}, u]$ $for\ all\ u \in V$.

2-$Involutive\ Gamma\ Derivation\ on\ n - Gamma\ Lie\ algebra$

In this section, we study $involutive\ \lambda - derivations$ on $n - \lambda -\ Lie\ algebras$

Definition 2.1:- Let V be an $associative\ \Gamma - algebra$ over a field F, then for all $\lambda \in \Gamma$,$n - \lambda - Lie\ algebra$ $L_\lambda(V)$can be defined with a linear multiplication $[\ , \ldots,]_\lambda : \wedge^n V \to V$ satisfies for all $v_1, \ldots . v_n, u_2, \ldots, u_n \in V$. $[[v_1, \ldots v_n]_\lambda, u_2, \ldots, u_n]_\lambda = \sum_{i=1}^{n} [v_1 \ldots, [v_i, u_2, \ldots, u_n]_\lambda, \ldots, v_n\ ]_\lambda$, then A is an $n - \lambda - Lie\ subalgebra$ of (V,$[\ , \ldots,]_\lambda$) if it is closed under the bracket , that means if $[A, A, \ldots, A, A,]_\lambda \subseteq A$, and subspace $\mathcal{J}$ of V is called an $ideal$ if $[\mathcal{J}, V, V, \ldots, V, V]_\lambda) \subseteq \mathcal{J}$, and the center of (V, $[\ , \ldots,]_\lambda$) is denoted by $Z(V) = \{ v \in V: [v, v_1, \ldots, v_n]_\lambda = 0$ for all $v_1, \ldots, v_n \in V\}$, $Z(V)$ is an $abelian\ ideal$ of V .

Definition 2.2:- $Let\ V$ be an $n - \lambda - Lie\ algabra\ over\ F$, a transformation linear $D: V \to V$ satisfies $D([v_1, \ldots, v_n]_\lambda) = \sum_{i=1}^{n} [v_1 \ldots, D(v_i), \ldots, v_n]_\lambda$ is $\lambda - derivation$ of V for all $v_1, \ldots, v_n \in V$. The set of all $\lambda - derivation\ D$ is defined by $Der_\lambda(V)$, and if a $\lambda - derivation$ D satisfies $D^2 = I_d$, then D is called $aninvolutive$ $\lambda - derivation\ on\ V$, and if V is a finite dimensional $vector\ space$ over F, and D is an λ-$endomorphism$ of V with $D^2 = I_d$, then V can be $decomposed$ into the $direct\ sum$ of

$subspaces\ \ V = V_1 + V_{-1}$ (1) where $V_1 = \{v \in V | Dv = v\}, and\ V_{-1} = \{v \in V | D\, v = -v\}$. And if D is an $involutive\ \lambda - derivation$ on V.

Then $D([v_1, \dots, v_n]_\lambda) = \sum_{i=1}^{n} [v_1 \dots, D(v_i), \dots, v_n]_\lambda = n[v_1, \dots, v_n]_\lambda, \forall v_1, \dots, v_n \in V$.

Example 2.3 :- Let V be a $3-dimensional\ 3-\lambda-Lie\ algebra$ with the multiplication of V in the basis $\{e_1, e_2, e_3\}$ be as follows, $[e_1, e_2, e_3]_\lambda = e_1$. A linear mapping D : V → V defined by $D(e_i) = e_i$ for $1 \le i \le 2$,and $D(e_3) = -e_3$ is an $involutive\ \lambda - derivation$ on V, and it satisfies $e_1, e_2 \in V_1$ and $e_3 \in V_{-1}$.

Theorem 2.4:- For any $n - \lambda - Lie\ algabra\ V$ the algebra $D_\lambda(V)$ is a λ-Lie $subalgebra$ of $gl_\lambda(V)$.

Proof : Since $D([v_1, \dots, v_n]_\lambda) = \sum_{i=1}^{n} [v_1 \dots, D(v_i), \dots, v_n]_\lambda$,

then for all $D_1, D_2 \in D_\lambda(V)$ and $v_1, \dots, v_n \in V$ we have

$D_1 D_2([v_1, \dots, v_n]_\lambda) = D_1(\sum_{i=1}^{n} [v_1 \dots, D_2(v_i), \dots, v_n]_\lambda$

$= \sum_{i=1}^{n} [v_1 \dots, D_1 D_2(v_i), \dots v_n]_\lambda + \sum_{1 \le s \ne i \le n}^{n} [v_1 \dots D_1(v_s) \dots, D_2(v_i), \dots v_n]_\lambda$

Similarly, we get $D_2 D_1([v_1, \dots, v_n]_\lambda) = D_2(\sum_{i=1}^{n} [v_1 \dots, D_1(v_i), \dots, v_n]_\lambda$

$= \sum_{i=1}^{n} [v_1 \dots, D_2 D_1(v_i), \dots, v_n]_\lambda + \sum_{1 \le s \ne i \le n}^{n} [v_1, \dots, D_2(v_s) \dots, D_1(v_i), \dots, v_n]_\lambda$

Hence, it implies

$$(D_1 D_2 - D_2 D_1)([v_1, \dots, v_n]_\lambda) = \sum_{i=1}^{n} [v_1 \dots, (D_1 D_2 - D_2 D_1)(v_i), \dots, v_n]_\lambda$$
$$= \sum_{i=1}^{n} [v_1, \dots, [D_1\, D_2]_\lambda(v_i), \dots, v_n]_\lambda = [D_1\, D_2]_\lambda([v_1, \dots, v_n]_\lambda) .$$

Therefore the result is obtained . ■

Lemma 2.5 :- Let V be an $n - \lambda - Lie\ algabra$ over F. If $D \in D_\lambda(V)$ is an $involutive\ \lambda - derivation$ then for all $v_1, \dots, v_n \in V$

$$[v_1, \dots, v_n]_\lambda = \frac{-2}{n-1} \sum_{i=1}^{n} \left[v_1, \dots, v_{i-1}, D(v_i), v_{i+1}, \dots, D(v_j), v_{j+1}, \dots, v_n\right]_\lambda$$

And

$$[D(v_1), \dots, D(v_n)]_\lambda = \frac{-2}{n-1} \sum_{i=1}^{n} \left[D(v_1), \dots, D(v_{i-1}), v_i, D(v_{i+1}), \dots, D(v_{j-1}), v_j, D(v_{j+1}), \dots, D(v_n)\right]_\lambda$$

Proof:-If D is an $involutive\ \lambda - derivation$ on V then for all $v_1, \dots. v_n \in V$ we have

$[v_1, \dots, v_n]_\lambda = D^2([v_1, \dots, v_n]_\lambda) = D(D([v_1, \dots, v_n]_\lambda))$

$= D(\sum_{i=1}^{n} [v_1, \dots, D(v_i), \dots, v_n]_\lambda) = \sum_{i=1}^{n} [v_1, \dots, D(D(v_i)), \dots, v_n]_\lambda$

$+ \sum_{i<j}^{n} \left[v_1, \dots, D(v_i), \dots, D(v_j), \dots, v_n\right]_\lambda + \sum_{j<n} \left[v_1, \dots, D(v_i), \dots, D(v_j), \dots, v_n\right]_\lambda$

$= \sum_{i=1}^{n} [v_1, \dots, v_i, \dots, v_n]_\lambda + 2n \sum_{1 \le i<j}^{n} \left[v_1, \dots, D(v_i), \dots, D(v_j), \dots, v_n\right]_\lambda$

Then $(n-1)[v_1, \dots, v_n]_\lambda = -2n \sum_{1 \le i<j}^{n} \left[v_1, \dots, D(v_i), \dots, D(v_j), \dots, v_n\right]_\lambda$

$[v_1, \dots, v_n]_\lambda = \frac{-2}{n-1} \sum_{i=1}^{n} \left[v_1, \dots, v_{i-1}, D(v_i), \dots, v_{j-1}, D(v_j), \dots, v_n\right]_\lambda$

And

$[D(v_1), \dots, D(v_n)]_\lambda =$

$\frac{-2}{n-1} \sum_{i=1}^{n} \left[D(v_1), \dots, D(v_{i-1}), v_i, D(v_{i+1}), \dots, D(v_{j-1}), v_j, D(v_{j+1}), \dots, D(v_n)\right]_\lambda$

Because $^2 = Id$. ■

Theorem 2.6 :- Let V be a finite dimensional $n - \lambda - Lie\ algebra$ with $n = 2r, r \ge 1$. Then there is an $involutive\ \lambda - derivation\ D\ on\ V$ if and only if V is $abelian.$

Proof:- If V is $abelian$ then $[u_1, \dots\dots, u_i, v_1, \dots\dots, v_{n-i}]_\lambda = 0$, hence D is an $involutive\ \lambda - derivation$ D on V . Conversely, let D be an $involutive\ \lambda - derivation$ on V, then Vcan be $decomposed$ into the $direct\ sum$ of $subspaces$ $V = V_1 + V_{-1}$.
Hence , for any $i \in \mathbb{Z}, 1 \le i \le n$, $u_1, \dots\dots, u_n \in V_1$,and $v_1, \dots\dots, v_n \in V_{-1}$
$D([u_1, \dots u_i, v_1, \dots, v_{n-i}]_\lambda) = i[u_1, \dots u_i, v_1, \dots, v_{n-i}]_\lambda - (n-i)[u_1, \dots u_i, v_1, \dots, v_{n-i}]_\lambda$
$= (2i - 2r)[u_1, \dots\dots u_i, v_1, \dots\dots, v_{n-i}]_\lambda \in V_{2i-2r}$. Then
$D([u_1, \dots\dots, u_n]_\lambda) = 2r[u_1, \dots\dots, u_n]_\lambda$, and $D([v_1, \dots\dots, v_n]_\lambda) = -2r[v_1, \dots\dots, v_n]_\lambda$.
Then $\pm 2r \ne 1 and\ 2i - 2r \ne \pm 1$, $V_{2i-2r}, V_{\pm 2r} = 0$. Therefore V is . ■

Theorem 2.7 :- Let V be a finite dimensional $n - \lambda -$ $Lie\ algebra$ with n=2r+1 ,r ≥1, and D be an $involutive - \lambda - derivation$ on V , then V_1 and V_{-1} are $abelian\ subalgebras$, and

$$\Big[\underbrace{V_1, \dots\dots, V_1}_{j}, \underbrace{V_{-1}, \dots\dots, V_{-1}}_{2r+1-j}\Big]_\lambda = 0 \ , \forall 1 \le j \le 2r, j \ne r, r+1$$

$$\Big[\underbrace{V_1, \dots\dots, V_1}_{r+1}, \underbrace{V_{-1}, \dots\dots, V_{-1}}_{r}\Big]_\lambda \subseteq V_1, \qquad \Big[\underbrace{V_1, \dots\dots, V_1}_{r}, \underbrace{V_{-1}, \dots\dots, V_{-1}}_{r+1}\Big]_\lambda \subseteq V_{-1}$$

proof. Since $D \in D_\lambda(V)$

$$\Big[\underbrace{V_1, \dots\dots, V_1}_{j}, \underbrace{V_{-1}, \dots\dots, V_{-1}}_{2r+1-j}\Big] \subseteq V_{2j-2r-1}, 0 \le j \le 2r+1$$

If $\Big[\underbrace{V_1, \dots\dots, V_1}_{j}, \underbrace{V_{-1}, \dots\dots, V_{-1}}_{2r+1-j}\Big]_\lambda \ne 0$ then $2r + 1 - j = \mp 1$ that is $r + 1 = j$ .Therefore
$[V_1, \dots\dots, V_1]_\lambda = [V_{-1}, \dots\dots, V_{-1}]_\lambda = 0$

Theorem 2.8 :- Let V be an $m - dimensional\ n - \lambda -$ $Lie\ algebra$ with
$n = 2r + 1, r \ge 1$. Then there is an $involutive\ \lambda - derivation$ on V if and only if V has the $decomposition$ $V = A + B$ such that

$$\Big[\underbrace{A, \dots\dots, A}_{i}, \underbrace{B, \dots\dots, B}_{2r+1-i}\Big]_\lambda = 0 \ \forall 1 \le i \le 2r, i \ne r, r+1$$

(2)

$$\Big[\underbrace{A, \dots\dots, A}_{r}, \underbrace{B, \dots\dots, B}_{r+1}\Big]_\lambda \subseteq B \ , \ \Big[\underbrace{A, \dots\dots, A}_{r+1}, \underbrace{B, \dots\dots, B}_{r}\Big]_\lambda \subseteq A$$

(3)

Proof : If D is an $involutive\ \lambda - derivation$ on V , then by Theorem 2.7 we have $A = V_1$, and $B = V_{-1}$ satisfy

$$\Big[\underbrace{A, \dots\dots, A}_{i}, \underbrace{B, \dots\dots, B}_{2r+1-i}\Big]_\lambda = 0 \ \forall 1 \le i \le 2r, i \ne r, r+1$$

$$\Big[\underbrace{A, \dots\dots, A}_{r}, \underbrace{B, \dots\dots, B}_{r+1}\Big]_\lambda \subseteq B \ , \ \Big[\underbrace{A, \dots\dots, A}_{r+1}, \underbrace{B, \dots\dots, B}_{r}\Big]_\lambda \subseteq A$$

Now, let D be an endomorphism of V defined by $D(u) = u, D(v) = -v, for\ all\ u \in A, v \in B$. Then $D^2 = Id$, $A = V_1$,and $B = V_{-1}$ satisfy (2) and (3) . Therefore D is an $involutive\ \lambda - derivation$ on V .

Corollary 2.9 : Let A be a $(2r+1)$-dimensional , $(2r+1)-\lambda-Lie\ algebra$ with the multiplication $[e_1, \dots \dots e_{2r+1}]_\lambda = e_1$, where $\{e_1, \dots \dots e_{2r+1}\}$ is a basis of V . Then the $linear\ mapping$ $D : V \to V$. Now by $D(e_i) = e_i$, $1 \le i \le r+1$ $D(e_j) = -e_j$, $(r+1) \le j \le (2r+1)$is an $involutive\ \lambda - derivation$ on V .

Proof. Since an $endomorphism$ D of V defined by $D(e_i) = e_i$, $1 \le i \le r+1$, $D(e_j) = -e_j$, $(r+1)$ $j \le (2r+1)$, however by Theorem 2.7 we get $D^2 = Id$, so that there is an $involutive\ \lambda - derivation$ on V .

3- Involutive Gamma Derivations with $3 - \lambda - Lie\ Algebras$

In this section, we study $involutive\ \lambda - derivations$ on $3 - \lambda - Lie\ Algebras$

Definition 3.1:- Let $(V, [\ ,]_\lambda)$ be an $associative\ \lambda - Lie\ algebra$ over F, such that $\lambda \in \Gamma$ and k is an element which is not contained in V then $U = V + F_k$ is a $3-\lambda-Lie\ Algebras$ in the multiplication.

$$[u, r, h]_\lambda = 0 \tag{4}$$

$[k, u, r]_\lambda = [u, r]_\lambda$, $for\ all$ $u, r, h \in V$. And the $3-\lambda-Lie\ Algebras$ $(U, [,,]_\lambda)$ is called *one-dimensional extension* of V . For example let V be an $abelian\ \lambda - Lie\ algebra$ with the $basis\{e_1, e_2, e_3\}$, and let $U = V + F_k$, $F_k \subseteq Z(U)$, then $[e_1, e_2, e_3]_\lambda = 0$, and for all $k \in F_k, [k, e_i, e_j]_\lambda = [e_i, e_j]_\lambda, 1 \le i, j \le 3, i \le j$. Therefore $(U, [\ ,\ ,]_\lambda)$ is *one-dimensional extension* of V .

Theorem 3.2 :- Let V be $3-\lambda-Lie\ Algebras$ then U is $one\ dimensional\ extension$ of a $\lambda - Lie\ Algebras$ $(V, [\ ,]_\lambda)$ if and only if the exists an $involutive\ \lambda - derivation$ D_λ on V such that either $dimV_1 = 1$, or $dimV_{-1} = 1$.

Proof:- If U is *one-dimensional extension* of a λ −Lie algebra V then $U_\lambda = V_\lambda + F_k$. Since $D_\lambda: U \to U$ is $endomorphism$ which is defined by $D_\lambda(k) = k, (or(-k))\ with\ D_\lambda(r) = r(or(-r)\ r \in V)$. $D_\lambda^2(k) = D_\lambda(D_\lambda(k)) = D_\lambda(k) = k$, and $D_\lambda^2(-k) = -k$ then $D_\lambda^2 = Id$

$D_\lambda([u, r, h]_\lambda) = [D_\lambda(u), r, h]_\lambda + [u, D_\lambda(r), h]_\lambda + [u, r, D_\lambda(h)]_\lambda = 0$

$D_\lambda([k, u, r]_\lambda) = [D_\lambda(k), u, r]_\lambda + [k, D_\lambda(u), r]_\lambda + [k, u, D_\lambda(r)]_\lambda = [k, u, r]_\lambda = [u, r]_\lambda$, for all $u, r \in V$ .Therefore D_λ is an $involutive\ \lambda - derivation$ on V such that $dimV_1 = 1$, or $dimV_{-1} = 1$. Conversely, let D_λ be an $involutive - \lambda - derivation$ on V such that $dimV_1 = 1$, or $dimV_{-1} = 1$. Let $U_{-1} = F_K$, and $U_1 = V$ (or $U_{-1} = V$, and $U_1 = F_K$) ,where $k \in U - V$. Then by Theorem 2.6 , V is an $\lambda - Lie\ algebra$ with the multiplication $[u, r]_\lambda = [k, u, r]_\lambda$ for all $u, r \in V$, and U is $one - dimensional\ extension$ of V .

Let $(V, [,]_{1\lambda})$ and $(V, [,]_{2\lambda})$ be $\lambda - Lie\ algebras$, and$\{v_1, \dots \dots, v_n\}$ is a basis of V . It is easy to define $\lambda - Lie\ algebras$ $(V, [,]_\lambda)$ be V_m , $m = 1{,}2$, and let k_1, k_2 are two distinct elements which are not contained in V, and $3-\lambda-Lie\ Algebras$ $(U_1, [,]_{1\lambda})$ and $(U_2, [,]_{2\lambda})$ are $one - dimentional\ extensio$n of $\lambda - Lie\ algebras\ V_1$, and V_2,respectively such that $U_1 = V_1 + F_{K1}$, $U_2 = V_2 + F_{K2}$, then $D_\lambda(V_1)$ and$D_\lambda(V_2)$ are $sub\ algebras$ of $gl_\lambda(V)$.

Definition 3.3 : Let $U_1 = (V, [,]_{1\lambda})$, and $U_2 = (V, [,]_{2\lambda})$ be two $\lambda - Lie\ algebras$, and k_1, k_2 are two special elements that are not present in V such that $U = V + F_{K1} + F_{K2}$. Then $3-\lambda-Lie\ Algebras\ (U, [\ ,\ ,]_\lambda)$ is called a $two - dimensional\ extension$ of $\lambda - Lie\ Algebras\ V_m$, $m = 1{,}2$ such that $[\ ,\ ,\ ]_\lambda : U \wedge U \wedge U \to U$ defined by

$$[u, r, k_1]_\lambda = [u, r]_{1\lambda} \quad , \quad [u, r, k_2]_\lambda = [u, r]_{2\lambda} \quad , \quad [u, r, h]_\lambda = 0 \tag{5}$$

$[k_1, k_2, u]_\lambda = \alpha_u k_1 + \beta_u k_2 \ \forall\, u, r, h \in V, and\ \alpha_u, \beta_u \in F$
If U is an $3-\lambda-Lie\ Algebras$ then U is called a $two\text{-}dimensional\ extension\ 3-\lambda-Lie\ Algebras$ of $\lambda-Lie\ Algebras\ V_m, m = 1,2$
Let U= V_m+ R be a $two-dimensional\ extension$ of $\lambda-Lie\ Algebras\ V_m, m = 1,2$
And $R = F_{K1} + F_{K2}$. Define linear mappings $3-\lambda-Lie\ Algebras$ as follows
$D_{1\lambda}(u)=ad(k_1, u)$, $D_{2\lambda}(u) = ad(k_2, u)$,
(6)
$D_\lambda(u) = ad(k_1, k_2)(u)\ \forall\, u \in V$ that is, for all $\forall\, r \in V$
$D_{1\lambda}(u)(r) = [u, r, k_1]_\lambda = [u, r]_{1\lambda}$, (7)
$D_{2\lambda}(u)(r) = [u, r, k_2]_\lambda = [u, r]_{2\lambda}, and, D_\lambda(u) = [k_1, k_2, u]_\lambda$

Theorem 3.4 :- Let $3-\lambda-Algebras\ U$ be a $two\text{-}dimensional\ extension$ of $\lambda-Lie\ Algebras\ V_m, m = 1,2$ then U is a $3-\lambda-Lie\ Algebras$ if and only if linear mappings $D_{1\lambda}, D_{2\lambda}$, and D_λ $where$ $D_{1\lambda}: V_1 \to Der_\lambda(V_1)$, $D_{2\lambda}: V_2 \to Der_\lambda(V_2)$ are $\lambda-Lie\ homomorphisms$, and
$D_{1\lambda}(u_3)[u_1, u_2]_{2\lambda} = [D_{1\lambda}(u_3)(u_1), u_2]_{2\lambda} + [(u_1), D_{1\lambda}(u_3)u_2]_{2\lambda}$
(8)
$-\alpha_{u_3}[u_1, u_2]_{1\lambda} - \beta_{u_3}[u_1, u_2]_{2\lambda}$
$D_{2\lambda}(u_3)[u_1, u_2]_{1\lambda} = [D_{2\lambda}(u_3)(u_1), u_2]_{1\lambda} + [(u_1), D_{2\lambda}(u_3)u_2]_{1\lambda}$
(9)
$+\alpha_{u_3}[u_1, u_2]_{1\lambda} + \beta_{u_3}[u_1, u_2]_{2\lambda}$
$D_\lambda([u_1, u_2]_{1\lambda}) = (\beta_{u_1}\alpha_{u_2} - \alpha_{u_1}\beta_{u_2})k_1$
(10)
$D_\lambda([u_1, u_2]_{2\lambda}) = (\beta_{u_1}\alpha_{u_2} - \alpha_{u_1}\beta_{u_2})\, k_2$
(11)
$D_{i\lambda}(u_1), (u_2) = -D_{i\lambda}(u_2), (u_1)$
(12)
$for\ all\ u_1, u_2 \in V, i = 1,2$
Where $u_1, u_2, u_3 \in$ V , $D_\lambda(u_i) = \alpha_{ui}k_1 + \beta_{ui}k_2$ $i = 1,2,3$

Proof : If U is $two-dimensional\ extension\ 3-\lambda-Lie\ Algebras$ then, by definition 3.3 linear mappings $D_{i\lambda}$ satisfy $D_{i\lambda}(V_i) \subseteq Der_\lambda(V_i)$, and $D_{i\lambda}$ are $\lambda-Lie\ homomorphisms$ $i = 1,2$ by (5) we have

$$D_{1\lambda}(u_3)[u_1, u_2]_{2\lambda} = [k_1, u_3, [u_1, u_2]_{2\lambda}]_\lambda = [k_1, u_3[k_2, u_1, u_2]_{2\lambda}]_\lambda$$
$$= [k_2, [k_1, u_3, u_1]_\lambda, u_2]_\lambda + [k_2, u_1, [k_1, u_3, u_2]_\lambda]_\lambda + [[k_1, u_3, k_2]_{\lambda,}, u_1, u_2]_\lambda$$
$$= [k_2, D_{1\lambda}(u_3)(u_1), u_2]_\lambda + [k_2, u_1, D_{1\lambda}(u_3)(u_2)]_\lambda - [[k_1, k_2, u_3]_{\lambda,}, u_1, u_2]_\lambda$$
$$= [D_{1\lambda}(u_3)(u_1), u_2]_{2\lambda} + [u_1, D_{1\lambda}(u_3)(u_2)]_{2\lambda} - \alpha_{u_3}[u_1, u_2]_{1\lambda} - \beta_{u_3}[u_1, u_2]_{2\lambda}$$

Then $for\ all\ u_1, u_2, u_3 \in V$ the equation (8) holds , The same way can be found (9)
Now if $D_\lambda([u_1, u_2]_{1\lambda}) = ad(k_1, k_2)[u_1, u_2]_{1\lambda} = [k_1, k_2, [u_1, u_2]_{1\lambda}]_\lambda$ $=[k_1, k_2, [k_1, u_1, u_2]_\lambda]_\lambda =$
$[k_1, [k_1, k_2, u_1]_\lambda, u_2]_\lambda + [k_1, u_1, [k_1, k_2, u_2]_\lambda]_\lambda + [[k_1, k_2, k_1]_\lambda, u_1, u_2]_\lambda$
$= -[[k_1, k_2, u_1]_\lambda, k_1, u_2]_\lambda + [[k_1, k_2, u_2]_\lambda, k_1, u_1]_\lambda$
$= (-\alpha_{u_1}[k_1, u_2]_\lambda\, k_1 - \beta_{u_1}[k_1, u_2]_\lambda\, k_2) + (\alpha_{u_2}[k_1, u_1]_\lambda\, k_1 - \beta_{u_2}[k_1, u_1]_\lambda\, k_2)$
$= -\alpha_{u_1}D_{\lambda 1}(u_2)(k_1) - \beta_{u_1}D_{\lambda 1}(u_2)(k_2) + \alpha_{u_2}D_{\lambda 1}(u_1)\, k_1\ \beta_{u_2}D_{\lambda 1}(u_1)\,(k_2)$
$= -\alpha_{u_1}[k_1, u_2, k_1]_\lambda - \beta_{u_1}[k_1, u_2, k_2]_\lambda + \alpha_{u_2}[k_1, u_1, k_1]_\lambda + \beta_{u_2}[k_1, u_1, k_2]_\lambda$
$= \beta_{u_1}[k_1, k_2, u_2]_\lambda - \beta_{u_2}[k_1, k_2, u_1]_\lambda = \beta_{u_1}(\alpha_{u_1}k_1 + \beta_{u_2}k_2) - \beta_{u_2}(\alpha_{u_1}k_1 + \beta_{u_1}k_2)$
$= \beta_{u_1}\alpha_{u_1}k_1 + \beta_{u_1}\beta_{u_2}k_2 - \beta_{u_2}\alpha_{u_1}k_1 - \beta_{u_2}\beta_{u_1}k_2 = \beta_{u_2}\alpha_{u_1}k_1 - \beta_{u_2}\alpha_{u_1}k_1$
$D_\lambda([u_1, u_2]_{1\lambda}) = (\beta_{u_1}\alpha_{u_2} - \beta_{u_2}\alpha_{u_1})k_1$

Then $for\ all\ u_1, u_2 \in V, and\ \alpha_{ui}, \beta_{ui} \in F, i = 1,2$ equation (10) holds .The same way can be found (11)

$D_{i\lambda}(u_1)(u_2) = ab(k_i, (u_1)(u_2)) = [k_i, u_1, u_2]_{i\lambda} = -[k_i, u_2, u_1]_{i\lambda}$

$D_{i\lambda}(u_1)(u_2) = -D_{i\lambda}(u_2), (u_1), \forall u_1, u_2 \in V, i = 1,2$ .Then $for\ all\ u_1, u_2 \in V, i = 1,2$ equation (12) hold

Conversely, by equation (5) , for all $u_1, u_2, u_3, u \in V$

$[u_1, u_2, u_3]_\lambda = 0$, $[k_1, u_1, u_2]_\lambda = D_{1\lambda}(u_1)(u_2) = [u_1, u_2]_{1\lambda}$

$[k_2, u_1, u_2]_\lambda = D_{2\lambda}(u_1)(u_2) = [u_1, u_2]_{2\lambda}$, $[k_1, k_2, u]_\lambda = D_\lambda(u) = \alpha_u k_1 + \beta_u k_2$

(13)

Since $D_{i\lambda}(V_i) \subseteq D_\lambda(V_i)$, and $D_{i\lambda}$ are $\lambda - Lie\ homomorphisms$, $i = 1,2$,$U_1 = V_1 + F_{K1}$, $U_2 = V_2 + F_{K2}$ are $3 - \lambda - Lie\ Algebras$ which are $one - dimensional\ extension\ 3 - \lambda - Lie\ Algebras\ of\ \lambda - Lie\ Algebras\ V_i$, $i = 1,2$, respectively.

Next it suffices to prove that the multiplication on U defined by equation (5) satisfies fulfills of the definition 1.6 for all $u_i \in V$ $such\ that$ $1 \le i \le 5$, and the products

$[u_1, u_2, [u_3, u_4, u_5]_\lambda]_\lambda = [[u_1, u_2, u_3]_\lambda, u_4, u_5]_\lambda + [u_3, [u_1, u_2, u_4]_\lambda, u_5]_\lambda + [u_3, u_4[u_1, u_2, u_5]_\lambda]_\lambda$ (14)

and the products $\left[[k_j, u_2, u_3]_\lambda, u_4, u_5\right]_\lambda$, $[[u_1, u_2, u_3]_\lambda, u_4, k_j]_\lambda$ and $\left[[u_1, u_2, k_j]_\lambda, u_4, k_j\right]_\lambda$ with definition 1.6 , $j = 1,2$. Therefore $U_1 = V_1 + F_{K1}$,and $U_2 = V_2 + F_{K2}$ are $one - dimensional\ extension\ 3 - \lambda - Lie\ Algebras$ of V_i, $i = 1,2$ and equation (5) is directly obtained from equation (8), and equation (9). It follows that the products $\left[[k_i, u_1, u_2]_\lambda, k_j, u_3\right]_\lambda$ $1 \le i \ne j \le 2$ fulfill definition 1.6. It follows from equation (10) – (12) that the products $\left[k_1, k_{2,}[k_{i,}u_1, u_2]_\lambda\right]_\lambda$, $[u_1, u_2[k_i, k_2, u_3]_\lambda]_\lambda$,and $\left[k_i u_1, [k_1, k_{2,}, u_2]_\lambda\right]_\lambda$, $i = 1,2$ fulfill the conditions of definition 1.6.

Theorem3.5:- Let (U, [, ,]) be a $3 - \lambda - Lie\ Algebras$. Then U is a $two\ dimensional\ extension\ 3 - \lambda - Lie\ Algebras\ of\ \lambda - Lie\ Algebras$ if and only if there is an $involutive - \lambda - derivation\ D$ on U such that $dimU_1 = 2\ or\ dimU_{-1} = 2$.

Proof . If U is a $two - dimensional\ extension\ 3 - \lambda - Lie\ Algebras\ of\ \lambda - Lie\ Algebras$ then by Theorem 3.2 there are $\lambda - Lie\ Algebras$

$V_1 = (V, [,]_{1\lambda})\ and\ V_2 = (V, [,]_{2\lambda})$

such that $U = V + R$, and the multiplication of U is defined by equation (5) where $R = F_{K1} + F_{K2}$.

Now define the endomorphism D of U by $D(u) = u, D(K_1) = -K_1, D(K_2) = -K_2$ $, or\ D(u) = -u, D(K_1) = K_1, D(K_2) = K_2, \forall u \in V\ then\ D^2 = Id, and\ U_1 = V, U_{-1} = R, or\ U_{-1} = V, U_1 = R$ .Thus by equation (4), and equations (8) - (12) , $involutive - \lambda - derivation\ D$ of U.

Conversely, if there is an $involutive - \lambda - derivation\ D$ on the $3 - \lambda - Lie\ Algebras\ U$ such that $dimU_{-1} = 2\ (or\ dimU_1 = 2)$ then by Theorem 2.8 $we\ have\ [U_1, U_1, U_1] = 0, [U_1, U_1, U_{-1}] \subseteq U_1, [U_1, U_{-1}, U_{-1}] \subseteq U_{-1}. Let\ V = U_1\ and\ U_1 = F_{K1} + F_{K2}$.

$Therefore\ [V, V, K\ 1] \subseteq V, [V, V, K_2] \subseteq V$ and $(V, [,]_{1\lambda})$,and$(V, [,]_{2\lambda})$ are $\lambda - Lie\ Algebras$,where $[u, r]_{1\lambda} = [u, r, k_1]_\lambda$,$[u, r]_{2\lambda} = [u, r, k_2]_\lambda, \forall u, r \in V$. Hence by Theorem 3.4

the $3 - \lambda - Lie\ Algebras\ U$ is a $two - dimensional\ extension\ 3 - \lambda - Lie\ Algebras$ of $\lambda - Lie\ Algebras\ V_1, V_2$.

4 - $Involutive\ \lambda - derivations$ and compatible $3 - \lambda - pre\ Lie\ algebras$

In this section, we study $involutive\ \lambda - derivations$ on compatible $3 - \lambda - pre\ Lie\ algebras$

Definition 4. 1:- A λ −representation of V (or an $V - \lambda - module$) is a pair (U, ρ), where V is a vector space, $\rho^{\lambda} : V \wedge V \to End(U)$ is a linear map such that

$$[\rho^{\lambda}(v_1, v_2)_{\lambda}, \rho^{\lambda}(v_3, v_4)_{\lambda}]_{\lambda} = \rho^{\lambda}(v_1, v_2)_{\lambda}\, \rho^{\lambda}(v_3, v_4)_{\lambda} - \rho^{\lambda}(v_3, v_4)_{\lambda}\, \rho^{\lambda}(v_1, v_2)_{\lambda}$$
$$= \rho^{\lambda}([v_1, v_2, v_3]_{\lambda}, v_4)_{\lambda} - \rho^{\lambda}([v_1, v_2, v_4]_{\lambda}, v_3)_{\lambda}$$
$$\rho^{\lambda}([v_1, v_2, v_3]_{\lambda}, v_4)_{\lambda} = \rho^{\lambda}(v_1, v_2)_{\lambda} \rho(v_3, v_4)_{\lambda} + \rho^{\lambda}(v_2, v_3)_{\lambda} \rho^{\lambda}(v_1, v_4)_{\lambda} + \rho^{\lambda}(v_1, v_3)_{\lambda} \rho^{\lambda}(v_2, v_4)_{\lambda}$$

for all $v_i \in V$, $1 \leq i \leq 4$.

A linear mapping $T^{\lambda} : U \to V$ is called an $\lambda - \wp - operator$ which is associated to an $V - \lambda - module\ (U, \rho)$ if T satisfies

$$[T^{\lambda}u, T^{\lambda}v, T^{\lambda}w]_{\lambda} = T^{\lambda}\left(\rho^{\lambda}(T^{\lambda}u, T^{\lambda}v)w + \rho^{\lambda}(T^{\lambda}v, T^{\lambda}w)u + \rho^{\lambda}(T^{\lambda}w, T^{\lambda}u)v\right)_{\lambda} \quad (15)$$

$for\ all\ u, v, w \in U$, and (V, ad) is called the $adjoint - \lambda - representation\ of\ V$.

Theorem 4.2 : Let $(V, [\cdot,\cdot,\cdot]_{\lambda})$ be a $3 - \lambda - Lie\ algebr$a with an $involutive - \lambda - derivation$ D_{λ}. Then D_{λ} is an $\lambda - \wp - operator$ of V associated to the $adjoint - \lambda - representation\ (V, ad)$, and D satisfies, $\forall\ u_1, u_2, u_3 \in V$

$$[Du_1, Du_2, Du_3]_{\lambda,} = D([Du_1, Du_2, u_3]_{\lambda} + [Du_2, Du_3, u_1]_{\lambda} + [Du_3, Du_1, u_2]_{\lambda}$$

Proof . By defined the a $\lambda - derivation\ D_{\lambda}$, and for all $u_1, u_2, u_3 \in V$,

$D(ad(Du_1, Du_2)_{\lambda} u_3 + ad(Du_2, Du_3)_{\lambda} u_1 + ad(Du_3, Du_1)_{\lambda} u_2)_{\lambda}$

$= D([Du_1, Du_2, u_3]_{\lambda} + [Du_2, Du_3, u_1]_{\lambda} + [Du_3, Du_1, u_2]_{\lambda})$

$= D([Du_1, Du_2, D^2u_3]_{\lambda} + [D^2u_1, Du_2, Du_3]_{\lambda} + [Du_1, D^2u_2, Du_3]_{\lambda})$

$= [Du_1, Du_2, Du_3]_{\lambda,}$. The proof is completed

Definition 4.3 : Let V be an $associative\ \Gamma - algebra$ over a field with a λ-*linear multiplication* $[\ ,\ ,]_{\lambda}: V^{\wedge 3} \to V$, $\forall\, u_1, u_2, u_3, u_4, u_5 \in V$. The pair $(V, \{\ ,,\}_{\lambda})$ is called a $3 - \lambda - preLie\ algebra$ if the next identities are correct

$$\{u_1, u_2, u_3\}_{\lambda} = -\{u_2, u_1, u_3\}_{\lambda} \quad (16)$$

$$\{u_1, u_2, \{u_3, u_4, u_5\}_{\lambda}\}_{\lambda} = \{[u_1, u_2, u_3]_{\lambda c}, u_4, u_5\}_{\lambda} + \{u_3, [u_1, u_2, u_4]_{\lambda c}, u_5\}_{\lambda} + \{u_3, u_4, \{u_1, u_2, u_5\}_{\lambda}\}_{\lambda} \quad (17)$$

$$\{[u_1, u_2, u_3]_{\lambda c}, u_4, u_5\}_{\lambda} = \{u_1, u_2, \{u_3, u_4, u_5\}_{\lambda}\}_{\lambda} + \{u_2, u_3, \{u_1, u_4, u_5\}_{\lambda}\}_{\lambda} + \{u_3, u_1, \{u_2, u_4, u_5\}_{\lambda}\}_{\lambda}$$

(18)

and $[\ ,\ ,]_{\lambda C}$ is defined by $[u_1, u_2, u_3]_{\lambda C} = \{u_1, u_2, u_3\}_{\lambda} + \{u_2, u_3, u_1\}_{\lambda} + \{u_3, u_1, u_2\}_{\lambda}$

(19)

Proposition 4.4 : Let $(V, \{\ ,\ ,\}_{\lambda})$ be a $3 - \lambda - pre\ Lie\ algebra$ .Then the $\{u_1, u_2, u_3\}_{\lambda c}$ defines a $3 - \lambda - Lie\ algebra$

Proof . By previous definition $\{u_1, u_2, u_3\}_{\lambda C}$ is skew-symmetric for all $u_i \in V$,$1 \leq i \leq 5$

$$[u_1, u_2, [u_3, u_4, u_5]_{\lambda c}]_{\lambda c} - [[u_1, u_2, u_3]_{\lambda c}, u_4, u_5]_{\lambda c} - [u_3, [u_1, u_2, u_4]_{\lambda c}, u_5]_{\lambda c} - [u_3, u_4, [\{u_1, u_2, u_5\}]_{\lambda c}]_{\lambda c}$$
$$= \{u_1, u_2, \{u_3, u_4, u_5\}_{\lambda}\}_{\lambda} + \{u_1, u_2, \{u_4, u_5, u_3\}_{\lambda}\}_{\lambda} + \{u_1, u_2, \{u_5, u_3, u_4\}_{\lambda}\}_{\lambda}$$
$$+ \{u_2, [u_3, u_4, u_5]_{\lambda c}, u_1\}_{\lambda} + \{[u_3, u_4, u_5]_{\lambda c}, u_1, u_2\}_{\lambda}$$
$$- \{u_4, u_5, \{u_1, u_2, u_3\}_{\lambda}\}_{\lambda} - \{u_4, u_5, \{u_2, u_3, u_1\}_{\lambda}\}_{\lambda} - \{u_4, u_5, \{u_3, u_1, u_2\}_{\lambda}\}_{\lambda}$$
$$- \{u_5, [u_1, u_2, u_3]_{\lambda c}, u_4\}_{\lambda} - \{[u_1, u_2, u_3]_{\lambda c}, u_4, u_5\}_{\lambda}$$
$$- \{u_3, u_5, \{u_1, u_2, u_4\}_{\lambda}\}_{\lambda} - \{u_3, u_5, \{u_2, u_4, u_1\}_{\lambda}\}_{\lambda} - \{u_3, u_5, \{u_4, u_1, u_2\}_{\lambda}\}_{\lambda}$$
$$- \{u_3, [u_1, u_2, u_4]_{\lambda c}, u_5\}_{\lambda} - \{[u_1, u_2, u_4]_{\lambda c}, u_5, u_3\}_{\lambda}$$
$$- \{u_3, u_4, \{u_1, u_2, u_5\}_{\lambda}\}_{\lambda} - \{u_3, u_4, \{u_2, u_5, u_1\}_{\lambda}\}_{\lambda} - \{u_3, u_4, \{u_5, u_1, u_2\}_{\lambda}\}_{\lambda}$$
$$- \{u_4, [u_1, u_2, u_5]_{\lambda c}, u_3\}_{\lambda} - \{[u_1, u_2, u_5]_{\lambda c}, u_3, u_4\}_{\lambda} = 0$$

This holds because

$\{u_1,u_2,\{u_3,u_4,u_5\}_\lambda\}_\lambda = \{[u_1,u_2,u_3]_{\lambda c},u_4,u_5\}_\lambda + \{u_3,[u_1,u_2,u_4]_{\lambda c},u_5\}_\lambda$
$+\{u_3,u_4,[u_1,u_2,u_5]_{\lambda c}\}_\lambda$

$\{u_1,u_2,\{u_4,u_5,u_3\}_\lambda\}_\lambda = \{[u_1,u_2,u_4]_{\lambda c},u_5,u_3\}_\lambda + \{u_4,[u_1,u_2,u_5]_{\lambda c},u_3\}_\lambda$
$+\{u_4,u_5,[u_1,u_2,u_3]_{\lambda c}\}_\lambda$

$\{u_1,u_2,\{u_5,u_3,u_4\}_\lambda\}_\lambda = \{[u_1,u_2,u_5]_{\lambda c},u_3,u_4\}_\lambda + \{u_5,[u_1,u_2,u_3]_{\lambda c},u_4\}_\lambda$
$+\{u_5,u_3,[u_1,u_2,u_4]_\lambda\}_\lambda$

$\{u_2,[u_3,u_4,u_5]_{\lambda c},u_1\}_\lambda = \{u_4,u_5,\{u_2,u_3,u_1\}_\lambda\}_\lambda + \{u_5,u_3,\{u_2,u_4,u_1\}_\lambda\}_\lambda$
$+\{u_3,u_4,\{u_2,u_5,u_1\}_\lambda\}_\lambda$

$\{[u_3,u_4,u_5]_{\lambda c},u_1,u_2\}_\lambda = \{u_3,u_4,\{u_5,u_1,u_2\}_\lambda\}_\lambda + \{u_4,u_5,\{u_3,u_1,u_2\}_\lambda\}_\lambda$
$+\{u_5,u_3,\{u_4,u_1,u_2\}_\lambda\}_\lambda.$

Therefore the proof is completed.

Theorem 4.5 : Let $(V,[\,,\,,]_\lambda)$ be a $3-\lambda-preLie\,algebra$, $D_\lambda \in D_\lambda(V)$ be an $involutive-\lambda-\,derivation$. Then $(V,\{\,,\,,\}_{\lambda D})$ is a $3-\lambda-preLie\,algebra$ where

$$\{u_1,u_2,u_3\}_{\lambda D} = [Du_1,Du_2,u_3]_\lambda. \qquad (20)$$

Moreover

$$\{u_1,u_2,u_3\}_{\lambda D} = \begin{cases} 0 \quad ,u_1,u_2,u_3 \in v_1 or\ u_1,u_2,u_3 \in v_{-1} \\ [u_1,u_2,u_3]_\lambda \quad u_1,u_2 \in v_1\ ,u_3 \in v_{-1} \\ -[u_1,u_2,u_3]_\lambda \quad u_1 \in v_1,u_2,u_3 \in v_{-1} \\ [u_1,u_2,u_3]_\lambda \quad u_1,u_2 \in v_{-1}\ ,u_3 \in v_1 \\ -[u_1,u_2,u_3]_\lambda \quad u_1 \in v_1\ ,u_2,u_3 \in v_{-1} \end{cases} \qquad (21)$$

And $(V,\{\,,\,,\}_{\lambda D})$ is called the $3-\lambda-preLie\,algebra$ which is associated with the $-\lambda-\,derivation\ D_\lambda$.

Proof . By Theorem 4.2 , D_λ is an $\lambda-\wp-operator$ associate to the $adjoint-\lambda-representation\ (V,ad)$, and for all $v_i \in V, 1 \le i \le 5$,

$[[Du_1,Du_2,\ u_3]_\lambda + [u_1,Du_2,Du_3]_\lambda + [Du_1,u_2,Du_3]_\lambda,\mathrm{D}u_4,u_5]_\lambda$
$= -[u_1,u_2,u_3]_\lambda,\mathrm{D}u_4,u_5]_\lambda$

$[D[Du_1,Du_2,Du_3]_\lambda,\mathrm{D}u_4,u_5]_\lambda = -[D[u_1,u_2,u_3]_\lambda,\mathrm{D}u_4,u_5]_\lambda$

since $\{u_1,u_2,u_3\}_{\lambda D} = [Du_1,Du_2,u_3]_\lambda = -[Du_2,Du_1,u_3]_\lambda = -\{u_2,u_1,u_3\}_{\lambda D}$

we get equation (16) . Since

$[Du_3,Du_4[Du_1,Du_2,u_5]_\lambda]_\lambda = [Du_1,Du_2,[Du_3,Du_4,u_5]_\lambda]_\lambda$
$-[[Du_1,Du_2,Du_3]_\lambda,Du_4,u_5]_\lambda - [Du_3,[Du_1,Du_2,Du_4]_\lambda,u_5]_\lambda.$

Therefore we have

$\{[u_1,u_2,u_3]_{\lambda Dc},u_4,u_5\}_{\lambda D} + \{u_3,[u_1,u_2,u_4]_{\lambda Dc},u_5\}_{\lambda D} + \{u_3,u_4,\{u_1,u_2,u_5\}_{\lambda D}\}_{\lambda D}$
$= [D[u_1,u_2,u_3]_{\lambda Dc},\mathrm{D}u_4,u_5]_\lambda + [Du_3,D[u_1,u_2,u_4]_{\lambda Dc},u_5]_\lambda + [Du_3,Du_4\{u_1,u_2,u_5\}_{\lambda D}]_\lambda$
$= [D([Du_1,Du_2,u_3]_\lambda + [u_1,Du_2,Du_3]_\lambda + [Du_1,u_2,Du_3]_\lambda),\mathrm{D}u_4,u_5]_\lambda$
$+[Du_3,D([Du_1,u_2,u_4]_\lambda + [u_1,Du_2,Du_4]_\lambda + [Du_1,u_2,Du_4]_\lambda),u_5]_\lambda$
$+ [Du_3,Du_4[Du_1,Du_2,u_5]_\lambda]_\lambda$
$= [[Du_1,Du_2,Du_3]_\lambda,Du_4,u_5]_\lambda$
$+ [Du_3,[Du_1,Du_2,Du_4]_\lambda,u_5]_\lambda + [Du_3,Du_4[Du_1,Du_2,u_5]_\lambda]_\lambda$
$= [[Du_1,Du_2,Du_3]_\lambda,Du_4,u_5] + [Du_3,[Du_1,Du_2,Du_4]_\lambda,u_5]_\lambda$
$+[Du_1,Du_2,[Du_3,Du_4,u_5]_\lambda]_\lambda - [[Du_1,Du_2,Du_3]_\lambda,Du_4,u_5]_\lambda$
$- [Du_3,[Du_1,Du_2,Du_4]_\lambda,u_5]_\lambda$
$= [Du_1,Du_2,[Du_3,Du_4,u_5]_\lambda]_\lambda = \{u_1,u_2,\{u_3,u_4,u_5\}_{\lambda D}\}_{\lambda D}$

Then we get equation

(17). By applying the same previous discussion we get equation (18).

Therefore, V is a $3-\lambda-pre\ Lie\ algebra$ in the multiplication (20). The equation (21) follows from equation (1), and equation (23) a direct computation.

Theorem 4.6 : Let $(V,[\ ,\ ,]_\lambda)$be a $3-\lambda-Lie\ algebra$, D_λ be an $involutive-\lambda-derivation$ on V. Then D_λ is an $\lambda-algebra\ isomorphism$ from the $sub-adjacent$ $3-\lambda-Lie\ algebra,(V,\{\ ,\ ,\}_{\lambda Dc})$ of the $3-\lambda-pre\ Lie\ algebra\ (V,\{\ ,\ ,\}_{\lambda D})$ to the $3-\lambda-Lie\ algebra\ (V,[\ ,\ ,]_\lambda)$, and

$$\{u_1,u_2,u_3\}_{\lambda Dc}=\{u_1,u_2,u_3\}_{\lambda D}+\{u_2,u_3,u_1\}_{\lambda D}+\{u_3,u_1,u_2\}_{\lambda D} \quad (22)$$
$$=D[Du_1,Du_2,Du_3]_\lambda, u_1,u_2,u_3\in V$$

Furthermore $$\{u_1,u_2,u_3\}_{\lambda Dc}=\begin{cases}0, u_1,u_2,u_3\in v_1 or\ u_1,u_2,u_3\in v_{-1}\\ -[u_1,u_2,u_3]_\lambda\ \ u_1,u_2\in v_1, u_3\in v_{-1}\\ -[u_1,u_2,u_3]_\lambda\ \ u_1,u_2\in v_{-1}, u_3\in v_1\end{cases} \quad (23)$$

Proof . By equation (20), the $sub-adjacent$ $3-\lambda-Lie\ algebra,(V,\{\ ,\ ,\}_{\lambda Dc})$ with the multiplication

$$\{u_1,u_2,u_3\}_{\lambda Dc}=\{u_1,u_2,u_3\}_{\lambda D}+\{u_2,u_3,u_1\}_{\lambda D}+\{u_3,u_1,u_2\}_{\lambda D}$$

$$=[Du_1,Du_2,u_3]_\lambda+[Du_2,Du_3,u_1]_\lambda+[Du_3,Du_1,u_2]_\lambda=D[Du_1,Du_2,Du_3]_\lambda$$

It follows Equation (22). Since

$$D(\{u_1,u_2,u_3\}_{\lambda Dc})=D(D[Du_1,Du_2,Du_3]_\lambda)=D^2[Du_1,Du_2,Du_3]_\lambda=[Du_1,Du_2,Du_3]_\lambda$$

for all $u_1,u_2,u_3\in V$, $the\ D_\lambda$is an $\lambda-algebra\ isomorphism$. Hence equations(22), and equation (23) hold.

Theorem 4.7 : Let $(V,[\ ,\ ,]_\lambda)$ be a $3-\lambda-Lie\ algebra, and\ D_\lambda$ is an $involutive-\lambda-derivation$ on V. Then there exists a compatible $3-\lambda-pre\ Lie\ algebra\ (V,\{\ ,\ ,\}_{\lambda V})$ where

$$\{u_1,u_2,u_3\}_{\lambda V}=D[u_1,u_2,Du_3]_\lambda \quad (24)$$

Proof. By equation (24), we have $\{u_1,u_2,u_3\}_{\lambda V}=D[u_1,u_2,Du_3]_\lambda=-D[u_2,u_1,Du_3]_\lambda=-\{u_2,u_1,u_3\}_{\lambda V}$ for all $v_i\in V, 1\le i\le 5$, and $\{u_1,u_2,\{u_5,u_3,u_4\}_\lambda\}_\lambda=D[u_1,u_2,D^2[u_3,u_4,Du_5]_\lambda]_\lambda=D[u_1,u_2,[u_3,u_4,Du_5]_\lambda]_\lambda$

we get equation (16) ,and $D[u_3,u_4[u_1,u_2,Du_5]_\lambda]_\lambda=D([u_1,u_2,[u_3,u_4,Du_5]_\lambda]_\lambda -[[u_1,u_2,u_3]_\lambda,u_4,Du_5]_\lambda-[u_3,[u_1,u_2,u_4]_\lambda,Du_5]_\lambda)$

Therefore

$$\{\{u_1,u_2,u_3\}_{\lambda Vc},u_4,u_5\}_{V\lambda}+\{u_3,\{u_1,u_2,u_4\}_{\lambda Vc},u_5\}_{V\lambda}+\{u_3,u_4,\{u_1,u_2,u_5\}_{\lambda V}\}_{V\lambda}$$
$$=D[\{u_1,u_2,u_3\}_{\lambda Vc},u_4,Du_5]_\lambda+D[u_3,\{u_1,u_2,u_4\}_{\lambda Vc},Du_5]_\lambda+D[u_3,u_4\{u_1,u_2,Du_5\}_\lambda]_\lambda$$
$$=D[D([u_1,u_2,Du_3]_\lambda+[u_2,u_3,Du_1]_\lambda+[u_3,u_1,Du_2]_\lambda),u_4,Du_5]_\lambda$$
$$+D[u_3,D([u_1,u_2,Du_4]_\lambda+[u_2,u_4,Du_1]_\lambda+[u_4,u_1,Du_2]_\lambda),Du_5]_\lambda$$
$$+D[u_3,u_4,[u_1,u_2,Du_5]_\lambda]_\lambda$$
$$=D([D(D[u_1,u_2,u_3]_\lambda),u_4,Du_5]_\lambda+[u_3,D(D[u_1,u_2,u_4]_\lambda),Du_5]_\lambda+[u_3,u_4,[u_1,u_2,Du_5]_\lambda]_\lambda)$$
$$=D([D^2[u_1,u_2,u_3]_\lambda),u_4,Du_5]_\lambda+[u_3,D^2[u_1,u_2,u_4]_\lambda),Du_5]_\lambda+[u_3,u_4,[u_1,u_2,Du_5]_\lambda]_\lambda)$$
$$=D([[u_1,u_2,u_3]_\lambda),u_4,Du_5]_\lambda+[u_3,[u_1,u_2,u_4]_\lambda),Du_5]_\lambda+[u_3,u_4[u_1,u_2,Du_5]_\lambda]_\lambda)$$
$$=D([[u_1,u_2,u_3]_\lambda,u_4,Du_5]_\lambda+[u_3,[u_1,u_2,u_4]_\lambda,Du_5]_\lambda+[u_1,u_2,[u_3,u_4,Du_5]_\lambda]_\lambda$$
$$-[[u_1,u_2,u_3]_\lambda,u_4,Du_5]_\lambda-[u_3,[u_1,u_2,u_4]_\lambda,Du_5]_\lambda)$$
$$=D[u_1,u_2,[u_3,u_4,Du_5]_\lambda]_\lambda=\{u_1,u_2,\{u_3,u_4,u_5\}_{\lambda V}\}_{V\lambda}$$

we get equation (17). By the same previous discussion we get equation (18). Hence $\{u_1,u_2,u_3\}_{\lambda Vc}=D([u_1,u_2,Du_3]_\lambda+[u_2,u_3,Du_1]_\lambda+[u_3,u_1,Du_2]_\lambda)$. Hence $(V,\{\ ,,\}_{\lambda V})$ is the compatible a $3-\lambda-pre\ Lie\ algebra$ of $(V,[\ ,\ ,]_\lambda)$.

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