Involutionary Gamma Derivations on n-Gamma Lie Algebra and 3-Pre Gamma-Lie Algebra

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Abstract
In this paper, the structure of \( n - \Gamma - \text{Lie Algebra} \) and \( 3 - \Gamma - \text{Pre Lie Algebra} \) have been introduced and studied. We also obtain that a \( \Gamma - \text{Lie algebra} V \) is one \( \lambda - \text{dimentional extension of a} \Gamma - \text{Lie algebra} \) if and only if there exists an involutive \( \lambda - \text{derivation} D_\lambda \) on \( V \) such that \( \dim V = 1 \) or \( \dim V = 1 \). In addition, we obtain that two \( \lambda - \text{dimentional extension of a} \Gamma - \text{Lie algebras} \) if and only if there is an involutive \( \lambda - \text{derivation} D_\lambda \) on \( U = U_1 \cup U = U_{-1} \) such that \( U_1 = 2 \) or \( \dim U_{-1} = 2 \), where \( U_1 \) and \( U_{-1} \) are subspaces of \( U \) with eigenvalues \( 1 \) and \( -1 \), respectively. We also find that the existence of involutive \( \lambda - \text{derivation} D_\lambda \) on \( 3 - \Gamma - \text{Lie algebra} \) implies that there exists a compatible \( 3 - \Gamma - \text{Pre Lie algebra} \) under appropriate condition.

Keywords: Algebra, Lie Algebra, Derivation, Gamma Lie algebra.

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Introduction

The notion of $n - \text{Lie algebra}$ was introduced by Filippov [1]. Derivation have also a relation with the extensions of $n - \text{Lie algebra}$. The concept of $3 - \text{Lie classical Yang Baxter equations}$ was introduce in [2], as well as $n - \text{Lie algebra}$. In [3] authors investigated the existence of involutive derivations and studied its properties on $n - \text{Lie algebra}$. They also investigated a class of $3 - \text{Lie algebras}$ with involutive derivations which are two -dimensional extension of $\text{Lie algebra}$. A. H. Rezaei and B. Davvaz. in [4] introduced the notion of Construction of $\Gamma - \text{algebra}$ and $\Gamma - \text{lie admissible algebras}$. The concept of compatible with $3 - \text{pre Liealgebra} (A, {\}, \{\})$ such that $A$ is adjacent $3 - \text{Lie algebra}$ in particular is introduced in [5]. For more results on $\text{Gamma - derivations}$ can be found in [6,7].

We study the structure of $n$-$\text{Gamma Lie Algebra}$ and $3$-$\text{Gamma Pre-Algebra}$, and the algebra $D_{\lambda}(V)$ is a $\lambda - \text{subalgebra}$ of $gl_{\lambda}(V)$ has been obtained. We also show that if $n = 2r$ $r \geq 1$ then there is an involutive $\lambda - \text{derivation}$ on $V$ if and only if $V$ is abelian. Furthermore, if $n = 2r + 1$, $r \geq 1$ then there is an involutive $\lambda - \text{derivation}$ on $V$ if and only if $V$ has the decomposition $V = A + B$, so that $A = V_{1}$ and $B = V_{-1}$ as well as if $V$ is a $\lambda - \text{Lie Algebras}$ then $V$ is one dimensional extension of a $\lambda - \text{Lie Algebras}$ $(V, [\ ,\ ])_{\lambda}$ if and only if the exists an involutive $\lambda - \text{derivation}$ $D_{\lambda}$ on $V$ such that $\text{dim} V_{1} = 1$ or $\text{dim} V_{-1} = 1$. Moreover if $(U, [\ ,\ ])_{\lambda}$ is a $3 - \lambda - \text{Lie Algebras}$ then $U$ has a two dimensional extension.

$3 - \lambda - \text{Lie Algebras}$ of $\lambda - \text{Lie Algebras}$ if and only if there is an involutive $\lambda - \text{derivation}$ on $U$ such that $\text{dim} U_{1} = 2$ or $\text{dim} U_{-1} = 2$, where $U_{1}$ and $U_{-1}$ are subspaces of $U$ with eigenvalues $1$ and $-1$, respectively. The existence of involutive $\lambda - \text{derivation}$ $D_{\lambda}$ on $3 - \Gamma - \text{Lie algebra}$ is obtained, it implies that there exists a compatible $3 - \text{pre Lie algebra}$ $(V, [\ ,\ ]_{\lambda})$, where $\{u_{1}, u_{2}, u_{3}\}_{\lambda} = [Du_{1}, Du_{2}, u_{3}]_{\lambda}, \forall u_{1}, u_{2}, u_{3} \in V$. This is done under appropriate condition.

1-Preliminary

In this section, we introduce the basic definitions and examples which are used throughout this paper.

**Definition 1.1 -** [4] Let $\Gamma$ be a groupoid and $V$ be a vector space over a field $F$. Then, $V$ is called a $\Gamma - \text{algebra}$ over the field $F$ if there exists a mapping $V \times \Gamma \times V \rightarrow V$ (the image is denoted by $u_{1} \lambda u_{2}$, for $u_{1}, u_{2}, u_{3} \in V$ and $\lambda \in \Gamma$) such that the following conditions hold:

1. $(u_{1} + u_{2})\lambda u_{3} = u_{1} \lambda u_{3} + u_{2} \lambda u_{3}$, $u_{1} \lambda (u_{2} + u_{3}) = u_{1} \lambda u_{2} + u_{1} \lambda u_{3}$
2. $u_{1}(\lambda + \beta)u_{2} = u_{1}\lambda u_{2} + u_{1}\beta u_{2}$
3. $(u_{1}\lambda)u_{2} = c(u_{1}\lambda u_{2}) = u_{1}(\lambda cu_{2})$, for all $u_{1}, u_{2}, u_{3} \in V$, $c \in F$ and $\lambda, \beta \in \Gamma$.

Moreover, $\Gamma - \text{algebra}$ is called associative if

$(4) \ (u_{1}\lambda u_{2})\beta u_{3} = u_{1}\lambda (u_{3}\beta u_{3})$

**Example 1.2 -** Let $V$ be the set of $2 \times 3$ matrices over the field of real numbers $R$ and

$$
\begin{pmatrix}
\alpha & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & 0
\end{pmatrix}, \alpha, \beta \in R.
$$

Then $V$ is an associative $\Gamma - \text{algebra}$.

**Definition 1.3 -** [4] Let $V$ be an associative $\Gamma - \text{algebra}$ over a field $F$. Then, for every $\lambda \in \Gamma$ one can construct an $\lambda - \text{Lie algebra} L_{\lambda}(V)$ as a vector space, $L_{\lambda}(V)$, which is the same as $V$. The Lie bracket of two elements of $L_{\lambda}(V)$ is defined to be their commutator in $V$, $[u, v]_{\lambda} = u\lambda v - v\lambda u$. Note that $[u, v]_{\lambda} = -[v, u]_{\lambda}$ for every $u, v \in V$ and $\lambda \in \Gamma$. Also, $L_{\lambda}(V)$ is abelian if either char $(F) = 2$ or char $(F) \neq 2$ then $[u, v]_{\lambda} = 0$ for every $\lambda \in \Gamma$. Moreover, if $\lambda \in \Gamma$, then there exists a compatible $3 - \text{pre Lie algebra}$ $(V, [\ ,\ ]_{\lambda})$ where $\{u_{1}, u_{2}, u_{3}\}_{\lambda} = [Du_{1}, Du_{2}, u_{3}]_{\lambda}, \forall u_{1}, u_{2}, u_{3} \in V$. This is done under appropriate condition.
Example 1.4:- Let $V$ be the set of all real $3 \times 5$ matrices of the form

$$
\begin{pmatrix}
0 & a & 0 & 0 & 0 \\
0 & b & 0 & 0 & 0 \\
0 & c & 0 & 0 & 0
\end{pmatrix}
$$

and $\Gamma b$ is the set of all real $5\times 3$ matrices. Then, $\forall \lambda \in \Gamma$ of the shape

$$
\begin{pmatrix}
\alpha & \beta & \delta \\
0 & 0 & 0 \\
\mu & \rho & \sigma \\
\theta & \vartheta & \tau \\
0 & 0 & 0
\end{pmatrix}
$$

Thus for every $A, B \in V$, we have $[A, B]_{\lambda} = 0$, so that $L_{\lambda}(V)$ is abelian, and the $\lambda$-dimension of $V$ is zero.

Definition 1.5:- [4] Let $V$ and $U$ be two associative $\Gamma$-algebras over a field $F$ and $\lambda \in \Gamma$. A linear transformation $\phi^\lambda : V \rightarrow U$ is called an $\lambda$-homomorphism if $\phi^\lambda ([v, u]) = [\phi^\lambda (v), \phi^\lambda (u)]_{\lambda}$ for all $v, u \in V$, and if Ker$\phi^\lambda = 0$, then $\phi^\lambda$ is called an $\lambda$-monomorphism, while it is called $\lambda$-epimorphism if Im$\phi^\lambda = U$. $\phi^\lambda$ is called an $\lambda$-isomorphism if both $\lambda$-monomorphism and $\lambda$-epimorphism are satisfied. If $\phi^\lambda (v) = 0$, then Ker$\phi^\lambda$ is an $\lambda$-ideal of $L_{\lambda}(V)$, certainly, and if $u \in V$ is arbitrary, then $\phi^\lambda ([v, u]) = [\phi^\lambda (v), \phi^\lambda (u)]_{\lambda} = 0$. It is also apparent that Im$\phi^\lambda$ is an $\lambda$-Lie subalgebra of $L_{\lambda}(U)$.

Definition 1.6:- [1] An $n - \lambda$-Lie algebra is a vector space $V$ over a field $F$ endowed with a linear multiplication $\lambda : V \times V \rightarrow V$ satisfying for all $v_1, ..., v_n, u_1, ..., u_n \in V \{[v_1, ..., v_n], u_1, ..., u_n\} = \sum_{i=1}^{n} [v_1, ..., [v_i, u_2, ..., u_n], ..., v_n]$. This equation is usually called the generalized Jacobi identity, or Filippov identity. The Lie subalgebra generated by the vectors $[v_1, ..., v_n]$ for any $v_1, ..., v_n \in V$ is called the derived algebra of $V$, which is denoted by $V'$. If $V' = 0$, $V$ is called an abelian algebra.

Definition 1.7:- [1] The derived algebra of an $n - \lambda$-Lie algebra $V$ is a subalgebra of $V$ generated by $[v_1, ..., v_n]$ for all $v_1, ..., v_n \in V$ and is a linear transformation $D : V \rightarrow V$. Satisfying $D([v_1, ..., v_n]) = \sum_{i=1}^{n} [v_1, ..., D(v_i), ..., v_n]$ for all $v_1, ..., v_n \in V$ and the set of all derivation is denoted by Der$V$ for all $v_1, ..., v_n \in V$. The map $ad(v_1, ..., v_{n-1}) : V \rightarrow V$ is given by $ad(v_1, ..., v_{n-1})(u) = [v_1, ..., v_{n-1}, u]$ for all $u \in V$.

2-Involutive Gamma Derivation on $n - \lambda$-Lie Algebra

In this section, we study involutive $\lambda$-derivations on $n - \lambda$-Lie algebras.

Definition 2.1:- Let $V$ be an associative $\Gamma$-algebra over a field $F$, then for all $\lambda \in \Gamma$, $n - \lambda$-Lie algebra $L_{\lambda}(V)$ can be defined with a linear multiplication $\lambda : \Lambda^n V \rightarrow V$ satisfies for all $v_1, ..., v_n, u_1, ..., u_n \in V$, $[v_1, ..., v_n]_{\lambda}, [v_1, u_2, ..., u_n]_{\lambda} = \sum_{i=1}^{n} [v_1, ..., [v_i, u_2, ..., u_n]]_{\lambda}, ..., v_n]_{\lambda}$, then $A$ is an $n - \lambda$-Lie subalgebra of $(V, [], ...)$ if it is closed under the bracket, that means if $[A, A, ..., A, A]_{\lambda} \subseteq A$, and subspace $J$ of $V$ is called an ideal if $[J, V, V, ..., V, V]_{\lambda} \subseteq J$, and the center of $(V, [], ...)$ is denoted by $Z(V) = \{v \in V : [v, v_1, ..., v_n]_{\lambda} = 0 \}$ for all $v_1, ..., v_n \in V$, $Z(V)$ is an abelian ideal of $V$.

Definition 2.2:- Let $V$ be an $n - \lambda$-Lie algebra over $F$, a transformation linear $D : V \rightarrow V$ satisfies $D([v_1, ..., v_n]) = \sum_{i=1}^{n} [v_1, ..., D(v_i), ..., v_n]_{\lambda}$ is $\lambda$-derivation of $V$ for all $v_1, ..., v_n \in V$. The set of all $\lambda$-derivation $D$ is defined by Der$_{\lambda}(V)$, and if a $\lambda$-derivation $D$ satisfies $D^2 = I_d$, then $D$ is called an involutive $\lambda$-derivation on $V$, and if $V$ is a finite dimensional vector space over $F$, and $D$ is an $\lambda$-endomorphism of $V$ with $D^2 = I_d$, then $V$ can be decomposed into the direct sum of
subspaces $V = V_1 + V_{-1}$ where
$V_1 = \{ v \in V | Dv = v \}$ and $V_{-1} = \{ v \in V | Dv = -v \}$. And if $D$ is an involutive $\lambda$-derivation on $V$.

Then $D([v_1, \ldots, v_n]_{\lambda}) = \sum_{i=1}^{n} [v_1, \ldots, D(v_i), \ldots, v_n]_{\lambda} = n! [v_1, \ldots, v_n]_{\lambda}, \forall v_1, \ldots, v_n \in V$.

**Example 2.3**: Let $V$ be a 3-dimensional $3 - \lambda - \text{Lie algebra}$ with the multiplication of $V$ in the basis $\{e_1, e_2, e_3\}$ as follows, $[e_1, e_2, e_3]_{\lambda} = e_1$. A linear mapping $D : V \rightarrow V$ defined by $D(e_i) = e_i$ for $1 \leq i \leq 2$ and $D(e_3) = -e_3$ is an involutive $\lambda$-derivation on $V$, and it satisfies $e_1, e_2 \in V_1$ and $e_3 \in V_{-1}$.

**Theorem 2.4**: For any $n - \lambda - \text{Lie algebra} V$ the algebra $D_{\lambda}(V)$ is a $\lambda$-Lie subalgebra of $gl(V)$.

**Proof**: Since $D([v_1, \ldots, v_n]_{\lambda}) = \sum_{i=1}^{n} [v_1, \ldots, D(v_i), \ldots, v_n]_{\lambda}$, then for all $D_1, D_2 \in D_{\lambda}(V)$ and $v_1, \ldots, v_n \in V$ we have

$D_1 D_2 ([v_1, \ldots, v_n]_{\lambda}) = D_1 \sum_{i=1}^{n} [v_1, \ldots, D_2(v_i), \ldots, v_n]_{\lambda}$

$= \sum_{i=1}^{n} [v_1, \ldots, D_1 D_2(v_i), \ldots, v_n]_{\lambda} + \sum_{1 \leq i < j \leq n} [v_1, \ldots, D_1(v_i), D_2(v_j), \ldots, v_n]_{\lambda}$

Similarly, we get

$D_2 D_1 ([v_1, \ldots, v_n]_{\lambda}) = D_2 \sum_{i=1}^{n} [v_1, \ldots, D_1(v_i), \ldots, v_n]_{\lambda}$

$= \sum_{i=1}^{n} [v_1, \ldots, D_1 D_2(v_i), \ldots, v_n]_{\lambda} + \sum_{1 \leq i < j \leq n} [v_1, \ldots, D_2(v_i), D_1(v_j), \ldots, v_n]_{\lambda}$

Hence it implies

$(D_1 D_2 - D_2 D_1)([v_1, \ldots, v_n]_{\lambda}) = \sum_{i=1}^{n} [v_1, \ldots, (D_1 D_2 - D_2 D_1)(v_i), \ldots, v_n]_{\lambda}$

$= \sum_{i=1}^{n} [v_1, \ldots, D_1 D_2(v_i), \ldots, v_n]_{\lambda} - \sum_{i=1}^{n} [v_1, \ldots, D_2 D_1(v_i), \ldots, v_n]_{\lambda}$

Therefore, the result is obtained.

**Lemma 2.5**: Let $V$ be an $n - \lambda - \text{Lie algebra}$ over $F$. If $D \in D_{\lambda}(V)$ is an involutive $\lambda$-derivation then for all $v_1, \ldots, v_n \in V$

$[v_1, \ldots, v_n]_{\lambda} = \frac{-2}{n-1} \sum_{i=1}^{n} [v_1, \ldots, v_{i-1}, D(v_i), v_{i+1}, \ldots, D(v_j), v_{j+1}, \ldots, v_n]_{\lambda}$

And

$[D(v_1), \ldots, D(v_n)]_{\lambda}$

$= \frac{-2}{n-1} \sum_{i=1}^{n} [D(v_1), \ldots, D(v_{i-1}), v_i, D(v_{i+1}), \ldots, D(v_j), v_j, D(v_{j+1}), \ldots, D(v_n)]_{\lambda}$

**Proof**: If $D$ is an involutive $\lambda$-derivation on $V$ then for all $v_1, \ldots, v_n \in V$ we have

$[v_1, \ldots, v_n]_{\lambda} = D^2([v_1, \ldots, v_n]_{\lambda}) = D(D([v_1, \ldots, v_n]_{\lambda}))$

$= D(\sum_{i=1}^{n} [v_1, \ldots, D(v_i), \ldots, v_n]_{\lambda}) = \sum_{i=1}^{n} [v_1, \ldots, D(D(v_i)), \ldots, v_n]_{\lambda}$

$+ \sum_{i < j} [v_1, \ldots, D(v_i), \ldots, D(v_j), \ldots, v_n]_{\lambda} + \sum_{j < i} [v_1, \ldots, D(v_i), \ldots, D(v_j), \ldots, v_n]_{\lambda}$

$= \sum_{i=1}^{n} [v_1, \ldots, v_i, \ldots, v_n]_{\lambda} + 2 n \sum_{1 \leq i < j} [v_1, \ldots, D(v_i), \ldots, D(v_j), \ldots, v_n]_{\lambda}$

Then

$(n-1)[v_1, \ldots, v_n]_{\lambda} = -2n \sum_{1 \leq i < j} [v_1, \ldots, D(v_i), \ldots, D(v_j), \ldots, v_n]_{\lambda}$

$[v_1, \ldots, v_n]_{\lambda} = \frac{-2}{n-1} \sum_{i=1}^{n} [v_1, \ldots, v_{i-1}, D(v_i), \ldots, v_{j+1}, \ldots, v_n]_{\lambda}$

And

$[D(v_1), \ldots, D(v_n)]_{\lambda}$

$= \frac{-2}{n-1} \sum_{i=1}^{n} [D(v_1), \ldots, D(v_{i-1}), v_i, D(v_{i+1}), \ldots, D(v_j), v_j, D(v_{j+1}), \ldots, D(v_n)]_{\lambda}$

Because $^2 = Id$.

**Theorem 2.6**: Let $V$ be a finite dimensional $n - \lambda - \text{Lie algebra}$ with $n = 2r$, $r \geq 1$. Then there is an involutive $\lambda$-derivation $D$ on $V$ if and only if $V$ is abelian.
Proof: If $V$ is abelian then $\{u_1, \ldots, u_i, v_1, \ldots, v_{n-i}\}_\lambda = 0$, hence $D$ is an involutive $\lambda$-derivation $D$ on $V$. Conversely, let $D$ be an involutive $\lambda$-derivation on $V$, then $V$ can be decomposed into the direct sum of subspaces $V = V_1 + V_{-1}$.

Hence, for any $i \in \mathbb{Z}, 1 \leq i \leq n$, $u_1, \ldots, u_n \in V_1$, and $v_1, \ldots, v_n \in V_{-1}$

$$D([u_1, \ldots, u_i, v_1, \ldots, v_{n-i}]_\lambda) = i[u_1, \ldots, u_i, v_1, \ldots, v_{n-i}]_\lambda(n-i)[u_1, \ldots, u_i, v_1, \ldots, v_{n-i}]_\lambda = (2i - 2r)[u_1, \ldots, u_i, v_1, \ldots, v_{n-i}]_\lambda \in V_{2i-2r}.$$

Then $D([u_1, \ldots, u_n]_\lambda) = 2r[u_1, \ldots, u_n]_\lambda$, and $D([v_1, \ldots, v_n]_\lambda) = -2r[v_1, \ldots, v_n]_\lambda$.

Then $\pm 2r \neq 1$ and $2i - 2r \neq \pm 1$, $V_{2i-2r}, V_{\pm 2r} = 0$. Therefore $V$ is.

**Theorem 2.7:** Let $V$ be a finite dimensional $n - \lambda$-Lie algebra with $n = 2r + 1$, $r \geq 1$, and $D$ be an involutive $\lambda$-derivation on $V$, then $V_1$ and $V_{-1}$ are abelian subalgebras, and

$$\left[\begin{array}{ccc} V_1, & \ldots, & V_1, V_{-1}, \ldots, V_{-1} \\ \lambda & \lambda & \lambda & \lambda \end{array}\right] = 0, \forall 1 \leq j \leq 2r, j \neq r, r + 1$$

$$\left[\begin{array}{ccc} V_1, & \ldots, & V_1, V_{-1}, \ldots, V_{-1} \\ \lambda & \lambda & \lambda & \lambda \end{array}\right] \subseteq V_1, \quad \left[\begin{array}{ccc} V_1, & \ldots, & V_1, V_{-1}, \ldots, V_{-1} \\ \lambda & \lambda & \lambda & \lambda \end{array}\right] \subseteq V_{-1}$$

**proof.** Since $D \in D_\lambda(V)$

$$\left[\begin{array}{ccc} V_1, & \ldots, & V_1, V_{-1}, \ldots, V_{-1} \\ \lambda & \lambda & \lambda & \lambda \end{array}\right] \subseteq V_{2i-2r} = 0, \forall 1 \leq j \leq 2r + 1$$

If $\left[\begin{array}{ccc} V_1, & \ldots, & V_1, V_{-1}, \ldots, V_{-1} \\ \lambda & \lambda & \lambda & \lambda \end{array}\right] \neq 0$ then $2r + 1 - j = 0$ that is $r + 1 = j$. Therefore

$$[V_1, \ldots, V_1]_\lambda = [V_{-1}, \ldots, V_{-1}]_\lambda = 0$$

**Theorem 2.8:** Let $V$ be an $m$-dimensional $n - \lambda$-Lie algebra with $n = 2r + 1$, $r \geq 1$. Then there is an involutive $\lambda$-derivation on $V$ if and only if $V$ has the decomposition $V = A + B$ such that

$$\left[\begin{array}{ccc} A, & \ldots, & A, B, \ldots, B \\ \lambda & \lambda & \lambda \end{array}\right] = 0 \forall 1 \leq i \leq 2r, i \neq r, r + 1$$

(2)

$$\left[\begin{array}{ccc} A, & \ldots, & A, B, \ldots, B \\ \lambda & \lambda & \lambda \end{array}\right] \subseteq B, \quad \left[\begin{array}{ccc} A, & \ldots, & A, B, \ldots, B \\ \lambda & \lambda & \lambda \end{array}\right] \subseteq A$$

(3)

**Proof:** If $D$ is an involutive $\lambda$-derivation on $V$, then by Theorem 2.7 we have $A = V_1$, and $B = V_{-1}$ satisfy

$$\left[\begin{array}{ccc} A, & \ldots, & A, B, \ldots, B \\ \lambda & \lambda & \lambda \end{array}\right] = 0 \forall 1 \leq i \leq 2r, i \neq r, r + 1$$

Now, let $D$ be an endomorphism of $V$ defined by $D(u) = u, D(v) = -v$, for all $u \in A, v \in B$. Then $D^2 = Id$, $A = V_1$ and $B = V_{-1}$ satisfy (2) and (3). Therefore $D$ is an involutive $\lambda$-derivation on $V$.  

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Corollary 2.9: Let \( A \) be a \((2r + 1)\)-dimensional, \((2r + 1) - \lambda - \text{Lie algebra}\) with the multiplication \([e_1, ..., e_{2r+1}]_\lambda = e_1\), where \( \{e_1, ..., e_{2r+1}\} \) is a basis of \( V \). Then the linear mapping \( D: V \to V \). Now by \( D(e_i) = e_i, 1 \leq i \leq r + 1 \), \( D(e_j) = -e_j \) \(, (r + 1) \leq j \leq (2r + 1)\) is an involutive \( \lambda - \text{derivation} \) on \( V \).

Proof. Since an endomorphism \( D \) of \( V \) defined by \( D(e_i) = e_i, 1 \leq i \leq r + 1 \), \( D(e_j) = -e_j \) \(, (r + 1) \leq j \leq (2r + 1)\), however by Theorem 2.7 we get \( D^2 = Id \), so that there is an involutive \( \lambda - \text{derivation} \) on \( V \).

3- Involutive Gamma Derivations with \( 3 - \lambda - \text{Lie Algebras} \)

In this section, we study involutive \( \lambda - \text{derivations} \) on \( 3 - \lambda - \text{Lie Algebras} \).

Definition 3.1:- Let \((V, [\cdot, \cdot]_\lambda)\) be an associative \( \lambda - \text{Lie algebra} \) over \( F \), such that \( \lambda \in \Gamma \) and \( k \in \text{ker} \) which is not contained in \( V \) then \( U = V + F_k \) is a \( 3 - \lambda - \text{Lie Algebras} \) in the multiplication.

\[
[u, r, h]_\lambda = 0 \hspace{1cm} (4) \]

\[
[k, u, r]_\lambda = [u, r]_\lambda \hspace{1cm} \text{for all } u, r, h \in V. \text{ And the } 3 - \lambda - \text{Lie Algebras} \]

\((U, [\cdot, \cdot]_\lambda)\) is called one-dimensional extension of \( V \). For example let \( V \) be an abelian \( \lambda - \text{Lie algebra} \) with the basis \( \{e_1, e_2, e_3\} \), and let \( U = V + F_k \), \( F_k \subseteq Z(U) \), then \([e_1, e_2, e_3]_\lambda = 0\) and for all \( k \in F_k, [k, e_i, e_j]_\lambda = [e_i, e_j]_\lambda, 1 \leq i, j \leq 3, i \leq j \). Therefore \((U, [\cdot, \cdot]_\lambda)\) is one-dimensional extension of \( V \).

Theorem 3.2 :- Let \( V \) be \( 3 - \lambda - \text{Lie Algebras} \) then \( U \) is one dimensional extension of a \( \lambda - \text{Lie Algebra} \) \((V, [\cdot, \cdot]_\lambda)\) if and only if the exists an involutive \( \lambda - \text{derivation} \) \( D_\lambda \) on \( V \) such that either \( \dim V_1 = 1 \), or \( \dim V_{-1} = 1 \).

Proof:- If \( U \) is one-dimensional extension of a \( \lambda - \text{Lie algebra} \) \( V \) then \( U_\lambda = V_\lambda + F_k \).

Since \( D_\lambda: U \to U \) is endomorphism which is defined by \( D_\lambda(k) = k \), \((or(-k))\) with \( D_\lambda(r) = r \) \((or(-r)) \ r \in V \). \( D_\lambda^2(k) = D_\lambda(D_\lambda(k)) = D_\lambda(k) = k \), and \( D_\lambda^2(-k) = -k \) then \( D_\lambda^2 = Id \)

\( D_\lambda([u, r, h]_\lambda) = [D_\lambda(u), r, h]_\lambda + [u, D_\lambda(r), h]_\lambda + [u, r, D_\lambda(h)]_\lambda = 0 \)

\( D_\lambda([k, u, r]_\lambda) = [D_\lambda(k), u, r]_\lambda + [k, D_\lambda(u), r]_\lambda + [k, u, D_\lambda(r)]_\lambda = [k, u, r]_\lambda \). for all \( u, r \in V \).

Therefore \( D_\lambda \) is an involutive \( \lambda - \text{derivation} \) on \( V \) such that \( \dim V_1 = 1 \), or \( \dim V_{-1} = 1 \). Conversely, let \( D_\lambda \) be an involutive \( \lambda - \text{derivation} \) on \( V \) such that \( \dim V_1 = 1 \), or \( \dim V_{-1} = 1 \). Let \( U_{-1} = F_k \), and \( U_1 = V \) \((or U_{-1} = V, \text{ and } U_1 = F_k)\), where \( k \in U - V \). Then by Theorem 2.6 , \( V \) is an \( \lambda - \text{Lie algebra} \) with the multiplication \([u, r]_\lambda = [k, u, r]_\lambda \) for all \( u, r \in V \), and \( U \) is one-dimensional extension of \( V \).

Let \((V, [\cdot, \cdot]_{1\lambda})\) and \((V, [\cdot, \cdot]_{2\lambda})\) be \( \lambda - \text{Lie algebra} \), \( \{v_1, ..., v_n\} \) is a basis of \( V \). It is easy to define \( \lambda - \text{Lie algebras} \) \((V, [\cdot, \cdot]_{1\lambda})\) be \( V_m, m = 1,2 \), and let \( k_1, k_2 \) are two distinct elements which are not contained in \( V \), and \( 3 - \lambda - \text{Lie Algebras} \) \((U_1, [\cdot, \cdot]_{1\lambda})\) and \((U_2, [\cdot, \cdot]_{2\lambda})\) are one-dimensional extension of \( \lambda - \text{Lie algebras} V_1 \), and \( V_2 \), respectively such that \( U_1 = V_1 + F_{K_1} \), \( U_2 = V_2 + F_{K_2} \), then \( D_\lambda(U_1) \) and \( D_\lambda(U_2) \) are subalgebras of \( g_\lambda(V) \).

Definition 3.3 :- Let \( U_1 = (V, [\cdot, \cdot]_{1\lambda}) \), and \( U_2 = (V, [\cdot, \cdot]_{2\lambda}) \) be two \( \lambda - \text{Lie algebras} \), and \( k_1, k_2 \) are two special elements that are not present in \( V \) such that \( U = V + F_{K_1} + F_{K_2} \). Then \( 3 - \lambda - \text{Lie Algebras} \) \((U, [\cdot, \cdot]_{1\lambda})\) is called a two-dimensional extension of \( \lambda - \text{Lie Algebras} V_m, m = 1,2 \) such that \([\cdot, \cdot]_{1\lambda} : U \wedge U \wedge U \to U \) defined by

\[
[u, r, k_1]_{1\lambda} = [u, r]_{1\lambda} \hspace{1cm} [u, r, k_2]_{1\lambda} = [u, r]_{2\lambda} \hspace{1cm} [u, r, h]_{1\lambda} = 0 \hspace{1cm} (5) \]
If $U$ is an $3 - \lambda$ - Lie Algebras then $U$ is called a two-dimensional extension $3 - \lambda$ - Lie Algebras of $\lambda$ - Lie Algebras $V_m, m = 1,2$

Let $R = V_m + R$ be a two-dimensional extension of $\lambda$ - Lie Algebras $V_m, m = 1,2$

And $R = F_{K1} + F_{K2}$. Define linear mappings $3 - \lambda$ - Lie Algebras as follows

$D_{\lambda}(u) = ad(k_1, u), D_{2\lambda}(u) = ad(k_2, u)$, \hspace{1cm} (6)

$D_{\lambda}(u) = ad(k_1, k_2)(u) \forall u \in V$ that is, for all $\forall r \in V$

$D_{\lambda}(u)(r) = [u, r, k_1]_{\lambda} = [u, r]_{\lambda}, \text{ and } D_{\lambda}(u) = [k_1, k_2, u]_{\lambda}$ \hspace{1cm} (7)

Theorem 3.4: Let $3 - \lambda$ - Algebras $U$ be a two-dimensional extension of $\lambda$ - Lie Algebras $V_m, m = 1,2$ then $U$ is a $3 - \lambda$ - Lie Algebras if and only if linear mappings $D_{\lambda_1}, D_{2\lambda}$ and $D_{\lambda}$ where $D_{\lambda_1} : V_1 \rightarrow Der_{\lambda}(V_1), D_{2\lambda} : V_2 \rightarrow Der_{\lambda}(V_2)$ are $\lambda$ - Lie homomorphisms, and

$D_{\lambda_1}(u_3)(u_1, u_2)_{2\lambda} = [D_{\lambda_1}(u_3)(u_1), u_2]_{2\lambda} + ([u_1, D_{\lambda_1}(u_3)u_2]_{2\lambda}$

$D_{2\lambda}(u_3)(u_1, u_2)_{2\lambda} = [D_{2\lambda}(u_3)(u_1), u_2]_{2\lambda} + ([u_1, D_{2\lambda}(u_3)u_2]_{2\lambda}$

$D_{\lambda}(u_3) = [D_{\lambda}(u_3)(u_1), u_2]_{2\lambda} + ([u_1, D_{\lambda}(u_3)u_2]_{2\lambda}$ \hspace{1cm} (8)

$-\alpha_{u_3}[u_1, u_2]_{1\lambda} - \beta_{u_3}[u_1, u_2]_{2\lambda}$

$D_{2\lambda}(u_3)[u_1, u_2]_{1\lambda} = [D_{2\lambda}(u_3)(u_1), u_2]_{1\lambda} + ([u_1, D_{2\lambda}(u_3)u_2]_{1\lambda}$

$D_{\lambda}(u_3)[u_1, u_2]_{2\lambda} = [D_{\lambda}(u_3)(u_1), u_2]_{2\lambda} + ([u_1, D_{\lambda}(u_3)u_2]_{2\lambda}$

$\alpha_{u_3}[u_1, u_2]_{1\lambda} + \beta_{u_3}[u_1, u_2]_{2\lambda}$

$D_{\lambda}(u_1, u_2) = -D_{\lambda_1}(u_2) = -D_{\lambda_2}(u_1)$

$D_{\lambda}(u_1, u_2) = \alpha_{u_1}k_1 + \beta_{u_1}k_2 \hspace{1cm} i = 1,2,3$

Proof: If $U$ is two-dimensional extension $3 - \lambda$ - Lie Algebras then, by definition 3.3 linear mappings $D_{\lambda}$ satisfy $D_{\lambda}(V_i) \subseteq Der_{\lambda}(V_i)$, and $D_{\lambda}$ are $\lambda$ - Lie homomorphisms $i = 1,2$ by (5) we have

$D_{\lambda_1}(u_3) = [D_{\lambda_1}(u_3)(u_1), u_2]_{2\lambda} + ([u_1, D_{\lambda_1}(u_3)u_2]_{2\lambda}$

$D_{2\lambda}(u_3)[u_1, u_2]_{1\lambda} = [D_{2\lambda}(u_3)(u_1), u_2]_{1\lambda} + ([u_1, D_{2\lambda}(u_3)u_2]_{1\lambda}$

$D_{\lambda}(u_3) = [D_{\lambda}(u_3)(u_1), u_2]_{2\lambda} + ([u_1, D_{\lambda}(u_3)u_2]_{2\lambda}$

Then for all $u_1, u_2, u_3 \in V$ the equation (8) holds. The same way can be found (9)

Now if $D_{\lambda}(u_1, u_2) = ad(k_1, k_2)[u_1, u_2]_{1\lambda} = [k_1, k_2, [u_1, u_2]_{1\lambda}]$

$= k_1[1, k_2, u_1, u_2]_{1\lambda} + k_2[1, u_1, k_2, u_2]_{1\lambda} + [1, k_2, u_1, k_2]_{1\lambda}$

$= -\alpha_{u_1}[k_1, u_2]_{1\lambda} + \beta_{u_1}[k_1, u_2]_{1\lambda} + [\alpha_{u_1}[k_1, u_2]_{1\lambda} + \beta_{u_1}[k_1, u_2]_{1\lambda}$

$= -\alpha_{u_1}[D_{\lambda_1}(u_2)](1) - \beta_{u_1}[D_{\lambda_1}(u_2)](2) + \alpha_{u_2}[D_{\lambda_1}(u_1)] k_1 - \beta_{u_2}[D_{\lambda_2}(u_1)] k_2$

$= -\alpha_{u_1}[k_1, u_2]_{1\lambda} - \beta_{u_1}[k_1, u_2]_{1\lambda} + [\alpha_{u_1}[k_1, u_2]_{1\lambda} + \beta_{u_1}[k_1, u_2]_{1\lambda}$

$= \beta_{u_1}[k_1, u_2]_{1\lambda} - \beta_{u_2}[k_1, u_2]_{1\lambda} = \beta_{u_1}[k_1, u_2]_{1\lambda} + \beta_{u_2}[k_1, u_2]_{1\lambda}$

$= \beta_{u_1}[k_1, u_2]_{1\lambda} - \beta_{u_2}[k_1, u_2]_{1\lambda} = \beta_{u_1}[k_1, u_2]_{1\lambda} + \beta_{u_2}[k_1, u_2]_{1\lambda}$

$D_{\lambda}(u_1, u_2) = \beta_{u_1}[\alpha_{u_2} - \beta_{u_2} \alpha_{u_1}] k_1$
Then for all $u_1, u_2 \in V$, and $\alpha_{u_1}, \beta_{u_2} \in F, i = 1, 2$ equation (10) holds. The same way can be found (11)

$$D_{i\lambda}(u_1)(u_2) = ab(k_i, (u_1)(u_2)) = [k_i, u_1, u_2]_{i\lambda} = -[k_i, u_2, u_1]_{i\lambda}$$

$$D_{i\lambda}(u_2)(u_1) = -D_{i\lambda}(u_2), (u_1), \forall u_1, u_2 \in V, i = 1, 2$$

Then for all $u_1, u_2 \in V, i = 1, 2$ equation (12) hold

Conversely, by equation (5), for all $u_1, u_2, u_3, u \in V$

$$[u_1, u_2, u_3]_{i\lambda} = 0, [k_1, u_1, u_2]_{i\lambda} = D_{i\lambda}(u_1)(u_2) = [u_1, u_2]_{1\lambda}$$

$$[k_1, k_2, u]_{i\lambda} = D_{i\lambda}(u) = [u_1, u_2]_{2\lambda}$$

(13)

Since $D_{i\lambda}(V_i) \subseteq D_{i\lambda}(V_i)$, and $D_{i\lambda}$ are $\lambda-Lie$ homomorphisms, $i = 1, 2$, $U_1 = V_1 + F_{K1}$, $U_2 = V_2 + F_{K2}$ are $3-\lambda-Lie$ Algebras which are one-dimensional extension $3-\lambda-Lie$ Algebras of $\lambda-Lie$ Algebras $V_i$, $i = 1, 2$, respectively.

Next it suffices to prove that the multiplication on $U$ defined by equation (5) satisfies fulfills of the definition 1.6 for all $u_i \in V$ such that $1 \leq i \leq 5$, and the products

$$[u_1, u_2, [u_3, u_4, u_5]]_{i\lambda} = [u_1, u_2, u_3, u_4, u_5]_{i\lambda} + [u_3, [u_1, u_2, u_4, u_5]]_{i\lambda} + [u_3, u_4 [u_1, u_2, u_5]]_{i\lambda} + [u_3, u_4 [u_1, u_2, u_5]]_{i\lambda}$$

and the products

$$[[k_1, k_2, k_3, k_4, k_5]_{i\lambda}, [u_1, u_2, u_3, u_4, u_5]]_{i\lambda}$$

(14)

with definition 1.6, $j = 1, 2$. Therefore $U_1 = V_1 + F_{K1}$ and $U_2 = V_2 + F_{K2}$ are one-dimensional extension $3-\lambda-Lie$ Algebras of $V_i$, $i = 1, 2$ and equation (5) is directly obtained from equation (8), and equation (9). It follows that the products

$$[[k_1, k_2, k_3, k_4, k_5]]_{1\lambda}, [u_3, u_4 [u_1, u_2, u_5]]_{i\lambda}$$

(12)

fulfill definition 1.6. It follows from equation (10)–(12) that the products

$$[[k_1, k_2, k_3, k_4, k_5]_{i\lambda}, [u_1, u_2, u_3, u_4, u_5]]_{i\lambda}$$

(i = 1, 2) fulfill the conditions of definition 1.6.

**Theorem 3.5:** Let $(U, [\cdot, \cdot, \cdot])$ be a $3-\lambda-Lie$ Algebras. Then $U$ is a two-dimensional extension $3-\lambda-Lie$ Algebras of $\lambda-Lie$ Algebras if and only if there is an involutive $-\lambda-derivation D$ on $U$ such that $dimU = 2$ or $dimU-1 = 2$.

**Proof.** If $U$ is a two-dimensional extension $3-\lambda-Lie$ Algebras of $\lambda-Lie$ Algebras then by Theorem 3.2 there are $\lambda-Lie$ Algebras $V_1 = (V, [\cdot, \cdot, \cdot])$ such that $U = V + R$, and the multiplication of $U$ is defined by equation (5) where $R = F_{K1} + F_{K2}$.

Now define the endomorphism $D$ of $U$ by $D(u) = u, D(K_1) = -K_1, D(K_2) = -K_2$, or $D(u) = -u, D(K_1) = K_1, D(K_2) = K_2, \forall u \in V$ then $D^2 = Id$, and $U_1 = V, U_{-1} = R, or U_{-1} = V, U_1 = R$. Thus by equation (4), and equations (8)-(12), involutive $-\lambda-derivation D$ of $U$.

Conversely, if there is an involutive $-\lambda-derivation D$ on the $3-\lambda-Lie$ Algebras $U$ such that $dimU_{-1} = 2$ (or $dimU_1 = 2$) then by Theorem 2.8 we have $[U_1, U_1, U_1] = 0, [U_1, U_1, U_{-1}] \subseteq U_1, [U_1, U_{-1}, U_{-1}] \subseteq U_{-1}$. Let $V = U_1$ and $U_1 = F_{K_1} + F_{K_2}$.

Therefore $[V, V, K_1] \subseteq V, [V, V, K_2] \subseteq V$ and $(V, [\cdot, \cdot, \cdot])$ are $\lambda-Lie$ Algebras, where $[u, r]_{1\lambda} = [u, r, k_1]_{1\lambda}, [u, r]_{2\lambda} = [u, r, k_2]_{2\lambda}, \forall u, r \in V$. Hence by Theorem 3.4 the $3-\lambda-Lie$ Algebras $U$ is a two-dimensional extension $3-\lambda-Lie$ Algebras of $\lambda-Lie$ Algebras $V_1, V_2$.
4 - **Involutive \( \lambda \)-derivations and compatible \( 3-\lambda \)-pre Lie algebras**

In this section, we study involutive \( \lambda \)-derivations on compatible \( 3-\lambda \)-pre Lie algebras

**Definition 4.1:** A \( \lambda \)-representation of \( V \) (or an \( V-\lambda \)-module) is a pair \((U, \rho)\), where \( V \) is a vector space, \( \rho^\lambda : V \wedge V \to \text{End}(U) \) is a linear map such that

\[
\rho^\lambda(v_1, v_2) = \rho^\lambda(v_1, v_2) = \rho^\lambda(v_3, v_4) - \rho^\lambda(v_3, v_4) - \rho^\lambda(v_3, v_4) - \rho^\lambda(v_3, v_4) - \rho^\lambda(v_3, v_4) - \rho^\lambda(v_3, v_4).
\]

\[
\rho^\lambda([v_1, v_2, v_3], v_4) = \rho^\lambda(v_1, v_2) = \rho^\lambda(v_3, v_4) + \rho^\lambda(v_3, v_4) = \rho^\lambda(v_3, v_4) + \rho^\lambda(v_3, v_4).
\]

for all \( v_i \in V, \ 1 \leq i \leq 4 \).

A linear mapping \( T^\lambda : U \to V \) is called an \( \lambda - \rho \)-operator which is associated to an \( V-\lambda \)-module \((U, \rho)\) if \( T \) satisfies

\[
[T^\lambda u, T^\lambda v, T^\lambda w]_\lambda = \rho^\lambda(T^\lambda u, T^\lambda v)w + \rho^\lambda(T^\lambda v, T^\lambda w)u + \rho^\lambda(T^\lambda w, T^\lambda u)v,
\]

for all \( u, v, w \in U \), and \((V, ad)\) is called the adjoint \( - \lambda \)-representation of \( V \).

**Theorem 4.2:** Let \((V, \{\cdot, \cdot\}_\lambda)\) be a \( 3-\lambda \)-Lie algebra with an involutive \( \lambda \)-derivation \( D_\lambda \). Then \( D_\lambda \) is an \( \lambda - \rho \)-operator of \( V \) associated to the adjoint \( - \lambda \)-representation \((V, ad)\), and \( D \) satisfies, \( \forall u_1, u_2, u_3 \in V \)

\[
[Du_1, Du_2, Du_3]_\lambda = D(Du_1, Du_2, u_3)_\lambda + [Du_2, Du_3, u_1]_\lambda + [Du_3, Du_1, u_2]_\lambda
\]

**Proof:** By defined the \( \lambda \)-derivation \( D_\lambda \), and for all \( u_1, u_2, u_3 \in V \),

\[
D(ad(Du_1, Du_2)_\lambda u_3) + ad(Du_2, Du_3)_\lambda u_1 + ad(Du_3, Du_1)_\lambda u_2)_\lambda
\]

\[
= D(Du_1, Du_2, u_3)_\lambda + [Du_2, Du_3, u_1]_\lambda + [Du_3, Du_1, u_2]_\lambda
\]

\[
= D(Du_1, Du_2, Du_3)_\lambda + [Du_1, Du_2, Du_3]_\lambda + [Du_1, Du_2, Du_3]_\lambda + [Du_2, Du_3, Du_1]_\lambda
\]

The proof is completed.

**Definition 4.3:** Let \( V \) be an associative \( \Gamma \)-algebra over a field with a \( \lambda \)-linear multiplication \( \{\cdot, \cdot\}_\lambda : V^3 \to V \), \( \forall u_1, u_2, u_3, u_4, u_5 \in V \). The pair \((V, \{\cdot, \cdot\}_\lambda)\) is called a \( 3-\lambda \)-pre Lie algebra if the next identities are correct

\[
\{u_1, u_2, u_3\}_\lambda = -\{u_2, u_3, u_1\}_\lambda \tag{16}
\]

\[
\{u_1, u_2, u_3, u_4, u_5\}_\lambda = \{u_1, u_2, u_3\}_\lambda + \{u_4, u_5\}_\lambda + \{u_3, u_4, u_5\}_\lambda + \{u_3, u_4, u_5\}_\lambda + \{u_3, u_4, u_5\}_\lambda + \{u_3, u_4, u_5\}_\lambda \tag{17}
\]

(18)

and \( \{\cdot, \cdot\}_\lambda \) is defined by \( [u_1, u_2, u_3]_\lambda = \{u_1, u_2, u_3\}_\lambda + \{u_2, u_3, u_1\}_\lambda + \{u_3, u_1, u_2\}_\lambda \)

**Proposition 4.4:** Let \((V, \{\cdot, \cdot\}_\lambda)\) be a \( 3-\lambda \)-pre Lie algebra. Then the \( \{u_1, u_2, u_3\}_\lambda \) defines a \( 3-\lambda \)-Lie algebra

**Proof:** By previous definition \( \{u_1, u_2, u_3\}_\lambda \) is skew-symmetric for all \( u_i \in V, \ 1 \leq i \leq 5 \)

\[
\{u_1, u_2, u_3\}_\lambda = -\{u_2, u_3, u_1\}_\lambda
\]

\[
\{u_1, u_2, u_3, u_4, u_5\}_\lambda = \{u_1, u_2, u_3\}_\lambda + \{u_4, u_5\}_\lambda + \{u_3, u_4, u_5\}_\lambda - \{u_1, u_2, u_3\}_\lambda
\]

(19)
This holds because
\[ \{u_1, u_2, \{u_3, u_4, u_5\}_\lambda\}_\lambda = \{\{u_1, u_2, u_3\}_\lambda, u_4, u_5\}_\lambda + \{u_3, [u_1, u_2, u_4]_\lambda, u_5\}_\lambda + \{u_3, u_4, [u_1, u_2, u_5]_\lambda\}_\lambda \]

Therefore the proof is completed.

**Theorem 4.5:** Let \((V, [\ , \ ]\lambda)\) be a 3-\(\lambda\)-preLie algebra. \(D_\lambda \in D_\lambda(V)\) be an involutive \(-\lambda\)-derivation. Then \((V, \{\ , \ \}_\lambda)\) is a 3-\(\lambda\)-preLie algebra where
\[ \{u_1, u_2, u_3\}_\lambda = [Du_1, Du_2, u_3]_\lambda. \] (20)

Moreover
\[ \{u_1, u_2, u_3\}_\lambda = \begin{cases} 0 & u_1, u_2, u_3 \in v_1 \text{ or } u_1, u_2, u_3 \in \nu_1 -1 \\ \{u_1, u_2, u_3\}_\lambda & u_1, u_2 \in v_1, u_3 \in \nu_1 \\ -[u_1, u_2, u_3]_\lambda & u_1 \in v_1, u_2, u_3 \in \nu_1 \\ [u_1, u_2, u_3]_\lambda & u_1, u_2 \in v_1 -1, u_3 \in \nu_1 \\ -[u_1, u_2, u_3]_\lambda & u_1 \in v_1, u_2, u_3 \in v_1 -1 \end{cases} \] (21)

And \((V, \{\ , \ \}_\lambda)\) is called the 3-\(\lambda\)-preLie algebra which is associated with the \(-\lambda\)-derivation \(D_\lambda\).

**Proof.** By Theorem 4.2, \(D_\lambda\) is an \(\lambda-\varphi\)-operator associate to the adjoint \(-\lambda\)-representation \((V, ad_x)\), and for all \(v_i \in V, 1 \leq i \leq 5,\)
\[ [Du_1, Du_2, u_3]_\lambda + [u_1, Du_2, Du_3]_\lambda + [Du_1, u_2, Du_3]_\lambda, Du_4, u_5]_\lambda \]
\[ = -[u_1, u_2, u_3]_\lambda \]
\[ [D[Du_1, Du_2, u_5]_\lambda, Du_4, u_5]_\lambda = -[D[u_1, u_2, u_3]_\lambda, Du_4, u_5]_\lambda \]
since \(\{u_1, u_2, u_3\}_\lambda = [Du_1, Du_2, u_3]_\lambda = -[Du_1, Du_2, u_3]_\lambda = -[u_2, u_1, u_3]_\lambda\)
due to equation (16). Since
\[ [Du_1, Du_2, u_5]_\lambda = [Du_1, Du_2, Du_3, u_4]_\lambda \]
\[ = -([Du_1, Du_2, Du_3]_\lambda, Du_4, u_5]_\lambda - [Du_1, Du_2, Du_3]_\lambda, Du_4, u_5]_\lambda - [Du_1, Du_2, Du_4]_\lambda, u_5]_\lambda - [Du_1, Du_2, Du_4]_\lambda, u_5]_\lambda \]
Therefore we have
\[ \{u_1, u_2, u_3\}_\lambda + [u_1, u_2, u_4]_\lambda, Du_5]_\lambda + [u_1, u_2, u_5]_\lambda, Du_4]_\lambda \]
\[ = 0 \]
\[ + [Du_1, Du_2, Du_3]_\lambda, Du_4, u_5]_\lambda + [Du_3, Du_4]_\lambda, u_5]_\lambda \]
\[ + [Du_3, Du_4]_\lambda, u_5]_\lambda \]
\[ = [Du_1, Du_2, Du_3]_\lambda, Du_4, u_5]_\lambda + [Du_3, Du_4]_\lambda, u_5]_\lambda \]
\[ = [Du_1, Du_2, Du_3]_\lambda, Du_4, u_5]_\lambda + [Du_3, Du_4]_\lambda, u_5]_\lambda \]

Then we get equation (17). By applying the same previous discussion, we get equation (18).
Therefore, \( V \) is a \( 3 - \lambda - \text{pre Lie algebra} \) in the multiplication (20). The equation (21) follows from equation (1), and equation (23) a direct computation.

**Theorem 4.6**: Let \((V, \{\cdot,\cdot,\cdot\})\) be a \( 3 - \lambda - \text{Lie algebra}, D_\lambda \) be an involutive \(-\lambda-\) derivation on \( V \). Then \( D_\lambda \) is an \(-\lambda-\) algebra isomorphism from the sub \(-\) adjacent \( 3 - \lambda - \text{Lie algebra} \) \((V, \{\cdot,\cdot,\cdot\})_{\lambda DC} \) of the \( 3 - \lambda - \text{pre Lie algebra} \) \((V, \{\cdot,\cdot,\cdot\})_{\lambda AD} \) to the \( 3 - \lambda - \text{Lie algebra} \) \((V, \{\cdot,\cdot,\cdot\})_{\lambda AD} \), and
\[
\{u_1, u_2, u_3\}_{\lambda DC} = \{u_1, u_2, u_3\}_{\lambda AD} + \{u_2, u_3, u_1\}_{\lambda AD} + \{u_3, u_1, u_2\}_{\lambda AD} \quad (22)
\]
\[
D[D_u, D_{u_2}, D_{u_3}]_\lambda, u_1, u_2, u_3 \in V
\]
Furthermore \(\{u_1, u_2, u_3\}_{\lambda DC} = \{u_1, u_2, u_3\}_{\lambda AD} \quad (23)\)

**Proof**: By equation (20), the sub \(-\) adjacent \( 3 - \lambda - \text{Lie algebra} \) \((V, \{\cdot,\cdot,\cdot\})_{\lambda DC} \) with the multiplication
\[
\{u_1, u_2, u_3\}_{\lambda DC} = \{u_1, u_2, u_3\}_{\lambda AD} + \{u_2, u_3, u_1\}_{\lambda AD} + \{u_3, u_1, u_2\}_{\lambda AD}
\]

It follows Equation (22). Since
\[
D([u_1, u_2, u_3]_{\lambda DC}) = D([D[u_1, u_2, D_{u_3}]_\lambda] = D^2[D[u_1, u_2, D_{u_3}]_\lambda = [D[u_1, u_2, D_{u_3}]_\lambda
\]
for all \( u_1, u_2, u_3 \in V \), then \( D_\lambda \) is an \(-\) algebra isomorphism. Hence equations (22), and equation (23) hold.

**Theorem 4.7**: Let \((V, \{\cdot,\cdot,\cdot\})\) be a \( 3 - \lambda - \text{Lie algebra}, D_\lambda \) is an involutive \(-\lambda-\) derivation on \( V \). Then there exists a compatible \( 3 - \lambda - \text{pre Lie algebra} \) \((V, \{\cdot,\cdot,\cdot\})_{\lambda V} \) where
\[
\{u_1, u_2, u_3\}_{\lambda V} = D[\{u_1, u_2, D_{u_3}]_\lambda
\]

**Proof**: By equation (24), we have
\[
\{u_1, u_2, u_3\}_{\lambda V} = D[u_1, u_2, D_{u_3}]_\lambda = - D[u_2, u_1, D_{u_3}]_\lambda = - [u_1, u_2, u_3]_{\lambda V} \quad (24)
\]

Therefore
\[
D[u_1, u_2, u_3]_{\lambda V} = D[u_1, u_2, D_{u_3}]_\lambda + D[u_2, u_3, D_{u_1}]_\lambda + D[u_3, u_1, D_{u_2}]_\lambda + D[u_1, u_2, D_{u_3}]_\lambda
\]
\[
\quad = D[D[u_1, u_2, D_{u_3}]_\lambda + D[u_2, u_3, D_{u_1}]_\lambda + D[u_3, u_1, D_{u_2}]_\lambda + D[u_1, u_2, D_{u_3}]_\lambda]
\]

we get equation (16), and \( D[u_3, u_1, u_2, D_{u_3}]_\lambda \) hold.

Therefore
\[
\{u_1, u_2, u_3\}_{\lambda V} = D[u_1, u_2, D_{u_3}]_\lambda + D[u_2, u_3, D_{u_1}]_\lambda + D[u_3, u_1, D_{u_2}]_\lambda + D[u_1, u_2, D_{u_3}]_\lambda
\]

we get equation (17). By the same previous discussion we get equation (18). Hence \(\{u_1, u_2, u_3\}_{\lambda V} = D[u_1, u_2, D_{u_3}]_\lambda + [u_2, u_3, D_{u_1}]_\lambda + [u_3, u_1, D_{u_2}]_\lambda\). Hence \((V, \{\cdot,\cdot,\cdot\})_{\lambda V}\) is the compatible a \( 3 - \lambda - \text{pre Lie algebra} \) of \((V, \{\cdot,\cdot,\cdot\})_\lambda\).
References


