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Applications of Affine Systems of Walsh Type to Generate Smooth Basis

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Abstract

The question on affine Riesz basis of Walsh affine systems is considered. An affine Riesz basis is constructed, generated by a continuous periodic function f that belongs to the space L^2 on the real line, which has a derivative almost everywhere; in connection with the construction of this example, we note that the functions of the classical Walsh system suffer a discontinuity and their derivatives almost vanish everywhere. A method of regularization (improvement of differential properties) of the generating function of Walsh affine system is proposed, and a criterion for an affine Riesz basis for a regularized generating function that can be represented as a sum of a series in the Rademacher system is obtained.

Keywords: Affine systems of Walsh type, affine Riesz basis, Rademacher system, Steklov concept.

تطبيقات الأنظمة الأفينية من نوع والش لتوليد أساس سلس

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الخلاصة

تم طرح سؤال عن أساس رايز الخطي للأنظمة الأفينية لوالش. يتم أنشاء وتوليد أساس رايز الأفيني بواسطة دالة f دورية تنتمي الى الفضاء مستمرة على خط الاعداد الحقيقية و التي تمتلك مشتقة على الأغلب عند كل نقاط الأعداد الحقيقية ; وفيما يتعلق ببناء هذا المثال , نلاحظ ان الدوال لنظام والش الكلاسيكية تعاني من عدم الاستمرارية وان مشتقاتها على الأغلب مساويا للصفر في كل نقاط الأعداد الحقيقية. تم اقتراح طريقة تنظيم (تحسين الخصائص التفاضلية) للدالة المتولدة لنظام والش الخطية وتم الحصول على أساس رايز الخطي لدالة التوليد المنتظمة من خلال تمثيلها باستخدام متسلسلة نظام الراديميجر .

الكلمات المفتاحية: الأنظمة الأفينية من نوع والش، أساس الرايز الأفيني، نظام الراديميجر، مفهوم ستيكلوف.

1. Introduction

Let $\mathrm{A}=\bigcup_{k=0}^{\infty}\{0,1\}^k$ – be the set of all finite multi-indices $\alpha=(\alpha_0,\ldots,\alpha_{k-1})$, consisting of zeros and ones, i.e. $\alpha_v=0$ or 1 , $0\le v\le k-1$, for all $\alpha=(\alpha_0,\ldots,\alpha_{k-1})\in\mathrm{A}$ and $\beta=(\beta_0,\ldots,\beta_{l-1})\in\mathrm{A}$, denote by:

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$\alpha\beta = (\alpha_0, \ldots, \alpha_{k-1}, \beta_0, \ldots, \beta_{l-1})$

The concatenation of multi-indices α and β. If $\alpha = (\alpha_0, \ldots, \alpha_{k-1}) \in \mathrm{A}$, then $|\alpha| = k$ is the length of the multi-index α .We will use the standard one-to-one correspondence between the sets of natural numbers $\mathbb{N}$ and the set of multi-indices A , defined by the binary decomposition $n = 2^k + \sum_{v=0}^{k-1} \alpha_v 2^v$ .This standard correspondence allows the replacement of the index n by α .Using the multi-index $\alpha \in \mathrm{A}$, it is convenient to write the product of operators:

$W^{\alpha} = W_{\alpha_0} \ldots W_{\alpha_{k-1}} \qquad , \qquad \alpha = (\alpha_0, \ldots, \alpha_{k-1}),$

The first is the operator $W_{\alpha_{k-1}}$ and the last is W_{α_0} (for $k = 0$, the empty product is equal to the identity operator). Let us consider the contraction-modulation operators $W_0; W_1$, acting in the Hilbert space $H = L_0^2 = L_0^2(0,1)$, which contains all 1-periodic functions $f(t), t \in \mathbb{R}$, such that $f(t) \in L^2(0,1)$, $\int_0^1 f(t)dt = 0$. That is for $f \in L^2(0,1)$, we choose:

$W_0 f(t) = f(2t); \qquad W_1 f(t) = r(t) f(2t),$

where $r(t)$ is the periodic function: Haar –Rademacher -Walsh. Then, the family of all possible products of these operators, applied to the function f , coincides with the Walsh affine system:

$f_n = f_{\alpha} = W^{\alpha} f = W_{\alpha_0} \ldots W_{\alpha_{k-1}} f,$

The operator structure $\{W_0, W_1\}$ has an important property that lies in the fact that given two operators form a multishift in the Hilbert space L_0^2 from the viewpoint of the following definition, which was introduced and studied earlier [1].

Definition(1.1): The family of functions $\{f_{\alpha}\}_{\alpha \in A}$ is called an affine system of functions of the type Walsh generated by function $f \in L_0^2$.

Rademacher system was first introduced by the German mathematician Hans Rademacher in 1922. Rademacher system is an incomplete orthonormal system in $L^2[0,1]$.

Definition (1.2)[2]: Let r be the function defined on [0,1) by :

$$r(t) = \begin{Bmatrix} 1, & 0 < t < 1/2 \\ -1, & 1/2 < t < 1 \\ 0, & t = 0, 1/2, 1 \end{Bmatrix},$$

which can be extended to the whole real numbers by periodicity of period 1. The Rademacher system $\{r_k\}_{k=0}^{\infty}$ is defined by:

$r_k(t) = r(2^k t),$

where, $t \in \mathbb{R}, k \in \mathbb{N}$.

The Walsh system was introduced by the American mathematician R.E. Paley in 1932, as a product of Rademacher functions by the following definition.

Definition(1.3)[2]: The Walsh system $\{w_n\}_{n=0}^{\infty}$ can be defined as:

$w_0(t) \equiv 1, \ \forall t \in [0,1],$

and for each natural number $\mathbb{N}$, using binary expansion of natural number n as :

$n = \sum_{k=0}^{\infty} 2^k n_k$, then $w_n = \prod_{k=0}^{\infty} r_k^{n_k}$.

Remark(1.1): Note that the American mathematician J.L. Walsh (1895-1973), who published a paper entitled "A closed set of normal orthogonal functions" [3] was the first who defined a system of orthogonal functions that is complete over normalized interval [0,1), currently called Walsh functions.

The Haar functions were initially defined by the Hungarian mathematician Alfred Haar in 1910.

Definition(1.4)[4]: The first Haar function is defined by:

$$\chi_0(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}$$

$$\chi_1(t) = \begin{cases} 1, & 0 \le t \le 1/2 \\ -1, & 1/2 \le t < 1 \\ 0, & t \ge 1 \end{cases}$$

In general,

$\chi_n(t) = \chi_1(2^j t - k / 2^j)$,

where $n = 2^j + k$, $j \ge 0, 0 \le k < 2^j$.

The Haar system $\chi = \{\chi_n\}_{n=0}^{\infty}$ forms a complete orthonormal system in $L^2[0,1]$.

Definition(1.5)[5]: Let H be a separable complex Hilbert space with scalar product $\langle f, g \rangle$ and norm $\|f\| = \sqrt{\langle f, f \rangle}$ $\forall f, g \in H$. A system of elements $\{e_n\}_{n=1}^{\infty} \subset H$ is called orthonormal if

$\langle e_i, e_j \rangle = \delta_{ij}, i, j = 1, 2, \ldots,$

where δ_{ij} is the Kronecker symbol.

Definition(1.6)[6]: A sequence $\{\varphi_n\}_{n=1}^{\infty}$ is called a Riesz basis if there exists an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of the space H and a bounded linear operator $T : H \to H$, having a bounded inverse operator $T^{-1} : H \to H$ (that is, the operator T realizes an isomorphism of the space H), such that

$\varphi_n = Te_n$, $n = 1, 2, \ldots.$

Rademacher functions and Walsh functions are fundamental objects of binary harmonic analysis. A considerable amount of studies have been devoted to an overall study of the Rademacher and Walsh systems. In this paper, we will mention the classical monographs of Golubov, Efimov, and Skvortsov [7], as well as those of Shipp, Wade and Simon [8]. The study of the properties of the Rademacher and the Walsh systems greatly influenced both the development of the theory of the function and functional analysis, probability theory, and computational mathematics. These systems found their applications in applied problems of signal processing, coding, and transmission of information.

Apparently, the affine system of Walsh type (in short, the Walsh affine system) generated by the function f was first introduced by Grandos [9] under the term of "Walsh wavelets" and under the additional assumption of the anti-periodicity of the generating function f . For generic functions f of the general form, the term "the affine system of Walsh type" was proposed elsewhere [10]. Terekhin was the first in studying affine systems of Walsh type and proving their orthogonality and completion. Also, he gave the structure of this kind. In another work [11], three sections were given such that an affine system of Walsh type can be classified into three sections; firstly, , the definition of an affine system of functions of the Walsh type was given on the basis of the functional analytic structure of a multishift in a Hilbert space, which is a generalized analogue of the operator(simple, one-side)shift and closely related to the representations of the Cuntz algebra. Secondly, various criteria and signs of the completeness of affine systems of functions were given. Finally, the minimality of the affine system was established and an explicit form of the biorthogonally conjugate system of functions was indicated and its completeness was established. Terekhin [6] provided conditions and, in some particular cases, criteria for the generating function, for the affine system to be Besselian, to form a Riesz basis, or to be an orthonormal system, and separately, to be complete. For this purpose, the concept of the dual function of the generating function of a system was introduced and studied. Mironov, Sarsenbi, and Terekhin [12] studied an affine Bessel sequences in connection with the spectral theory and the multishift structure in Hilbert space. They constructed a non-Besselian affine system generated by a continuous periodic function . Their results were based on Nikishin's example concerning convergence in measure. Also, they showed that affine systems generated by any Lipchitz function are Besselian. A new method for characterization of Bessel system was given later [13]. This method was based on given necessary and sufficiently conditions on the function, which is an affine system of functions of the Walsh type, to be Bessel system in the space. Some examples were also given to explain the representation method. In recently papers [14-15], the basis properties of affine Walsh – type systems were studied in symmetric space. Also, it was shown that such a system can only be an unconditional basis in $L^2[0,1]$.

2. Main results

We consider the simplest method of regularization (i.e., improvement of the differential properties) of the function $f(t)$, namely, Steklov concept:

$$f_h(t)=\frac{1}{2h}\int_{t-h}^{t+h} f(s)ds, \qquad h>0$$

It is well known that, for all $f\in L^2$, the following relation holds:

$$\lim_{h\to 0}\| f_h - f\| = 0.$$

A direct check shows that the Steklov concept $f_h(t)$ has zero mean value:

$$\int_0^1 f_h(t)dt = 0$$

together with the function $f(t)$ and, obviously, inherits the properties of periodicity. In particular, we note that $f_h(t)=0$ for $h=\frac{1}{2}$ for the 1 -periodic function $f(t)$. Hence, $f_h(t)=0$ for all $0<h<\frac{1}{2}$. The Walsh system $\{\omega_n\}_{n=1}^{\infty}$ consists of discontinuous functions. In this connection, a natural question arises: will the affine Walsh systems generated by the functions r_h be the Steklov concept of the function r?

Denote $r_h, h>0$ as the Steklov average of the function r.

Theorem (2.1): For each $h=\frac{1}{2^{n+1}}, n=1,2,\ldots,$ the affine Walsh system, generated by the function $f=r_h$, is a Riesz basis.

Proof. Let us start with the most difficult case of $n=1$. The function $r_{\frac{1}{4}}$ is linear on each interval $[\frac{i}{4},\frac{i+1}{4}], i\in\mathbb{Z}$ and takes the values:

$$r_{\frac{1}{4}}(\frac{i}{4})=\begin{cases} 0, & i=0 \mod 4,\\ 1, & i=1 \mod 4,\\ 0, & i=2 \mod 4,\\ -1, & i=3 \mod 4.\end{cases}$$

It is not difficult to see that if $l(t)=1-2t, 0<t<1,$ then

$$r_{\frac{1}{4}}=\frac{1}{2}(r-W_1^2 l).$$

Since

$$l=\sum_{k=0}^{\infty}\frac{r_k}{2^{k+1}},$$

then the Fourier Walsh series of the function $f=2r_{\frac{1}{4}}$ absolutely converges in blocks:

$$1+\sum_{k=0}^{\infty}\left|\frac{W_1^2 W_0^k r}{2^{k+1}}\right|=2<\infty.$$

We compute the dual function

$$g=T_f^{-1}r=(I-T_{W_1^2 l})^{-1}r=\sum_{d=0}^{\infty}T_{W_1^2 l}^d r.$$

By induction on d, we obtain the equality

$$T_{W_1^2 l}^d r=\sum_{k_1,\ldots,k_d\geq 0}\frac{W_1^2 W_0^{k_1}\ldots W_1^2 W_0^{k_d} r}{2^{k_1+\ldots+k_d+d}}.$$

In the last sum, the multi-indices $\alpha\in\mathrm{A}$, for which $W^{\alpha}=W_1^2 W_0^{k_1}\ldots W_1^2 W_0^{k_d}$. The number of such multi indices of fixed length $k=|\alpha|=k_1+\ldots+k_d+2d$ equals the number of combinations C_{k-d}^d. We denote the set of such multi-indices by $\mathrm{A}(k,d)$. Thus, we have

$$T_{W_1^2 l}^d r=2^d\sum_{k=2d}^{\infty}\frac{1}{2^k}\sum_{\alpha\in\mathrm{A}(k,d)}\omega_{\alpha}.$$

Hence we find that

$$g=r+\sum_{k=2}^{\infty}\frac{1}{2^k}\sum_{d=0}^{[k/2]}2^d\sum_{\alpha\in\mathrm{A}(k,d)}\omega_{\alpha}.$$

Noting that $2^d\leq 2^{k/2}$, we will have

$$1+\sum_{k=2}^{\infty}\frac{1}{2^k}\left\|\sum_{d=0}^{[k/2]}2^d\sum_{\alpha\in\mathrm{A}(k,d)}\omega_{\alpha}\right\|\leq 1+\sum_{k=2}^{\infty}(\frac{1}{2^k}\sum_{d=0}^{[k/2]}C_{k-d}^d)^{1/2}=1+\sum_{k=2}^{\infty}(\frac{F_{k+1}}{2^k})^{1/2}<\infty,$$

where

$$F_0=0, F_1=1, F_k=F_{k-1}+F_{k-2}, k=2,3,\ldots,$$

are Fibonacci numbers, for which $F_k=q^k$ with the constant of the golden ratio $q=\frac{1+\sqrt{5}}{2}$. Therefore, the convergence of the last numerical series follows from the fact that $q<2$. We see that the dual function g also has a Fourier-Walsh series which is absolutely convergent in blocks. Consequently, the function $r_{\frac{1}{4}}$ generates an affine Riesz basis.

Now, consider the case $n \geq 2$. The function r_h is linear on the intervals $[0,h],[\frac{1}{2}-h,\frac{1}{2}+h],[1-h,1]$, and $r_h(t)=1$ for $t\in[h,\frac{1}{2}-h], r_h(t)=-1$ for $t\in[\frac{1}{2}+h,1-h]$. Consider the auxiliary function

$$f(t)=\begin{cases} 2^{n-1}t, & 0\leq t\leq 1/2^{n-1}, \\ 1, & 1/2^{n-1}\leq t\leq 1. \end{cases}$$

As is known, the partial sums $S_n f$ of the Fourier-Walsh series of the function f have the form

$$S_{2^k} f(t)= f_I, \qquad t\in I=(\frac{j}{2^k},\frac{j+1}{2^k}),$$

where

$$f_I = 2^k \int_{j/2^k}^{(j+1)/2^k} f(t)dt.$$

It is clear that

$$S_1 f(t)=\int_0^1 f(t)dt = 1-\frac{1}{2^n}.$$

If $1\leq k\leq n-1$, then when $j=0$, we have

$$f_I = 2^k(\frac{1}{2^k}-\frac{1}{2^n}) = 1-\frac{1}{2^{n-k}}.$$

and it is obvious that $f_I=1$ for $j=1,\ldots,2^k-1$. If $k\geq n$ Then, for $j=0,\ldots,2^{k-n+1}-1$, we have

$$f_I = f(c_I)=2^{n-1}\frac{j+1/2}{2^k}=\frac{j+1/2}{2^{k-n+1}},$$

where c_I is the middle of the interval I, by the linearity of $f(t)$ on I, and it is obvious that $f_I=1$ for $j=2^{k-n+1},\ldots,2^k-1$.

We denote by $I(0)$ and $I(1)$, respectively, the left and right halves of the interval I, which is separated by point c_I. If $0\leq k\leq n-2$, then, when $j=0$

$$f_{I(0)}=1-\frac{1}{2^{n-(k+1)}}, \qquad f_{I(1)}=1,$$

so that

$$\Delta_I=\frac{f_{I(1)}-f_{I(0)}}{2}=\frac{1}{2^{n-k}},$$

Also, it is obvious that $\Delta_I=0$ for $j=1,\ldots,2^k-1$. If $k\geq n-1$, then $j=0,\ldots,2^{k-n+2}-1$

$$\Delta_I=\frac{f(c_{I(1)})-f(c_{I(0)})}{2}=\frac{1}{2}(\frac{j+3/4}{2^{k-n+2}}-\frac{j+1/4}{2^{k-n+2}})=\frac{1}{2^{k-n+4}}$$

and it is obvious that $\Delta_I=0$ for $j=2^{k-n+2},\ldots,2^k-1$. Because

$$f_I=\frac{f_{I(0)}+f_{I(1)}}{2},$$

then

$$\left|S_{2^k} f(t) - S_{2^{k+1}} f(t)\right| = \left|f_I - f_{I(0)}\right| = \left|f_I - f_{I(1)}\right| = \Delta_I ,$$

where $t \in I = (\frac{j}{2^k}, \frac{j+1}{2^k})$. Thus, when $0 \le k \le n-2$, we have

$$\left\|S_{2^k} f - S_{2^{k+1}} f\right\| = (\int_0^{1/2^k} (\frac{1}{2^{n-k}})^2 dt)^{1/2} = \frac{1}{2^{n-k/2}}$$

and for $k \ge n-1$, we have

$$\left\|S_{2^k} f - S_{2^{k+1}} f\right\| = (\sum_{j=0}^{2^{k-n+2}-1} \int_{j/2^k}^{(j+1)/2^k} (\frac{1}{2^{k-n+4}})^2 dt)^{1/2} = \frac{1}{2^{k-n+4}} (\int_0^{1/2^{n-2}} dt)^{1/2} = \frac{1}{2^{k-n/2+3}} .$$

From this, we see that the value of

$$C_n = C_n(f) = \sum_{k=0}^{\infty} \left\|S_{2^k} f - S_{2^{k+1}} f\right\| = \sum_{k=0}^{n-2} \frac{1}{2^{n-k/2}} + \sum_{k=n-1}^{\infty} \frac{1}{2^{k-n/2+3}} = \frac{1}{2^n} \sum_{k=0}^{n-2} 2^{k/2} + \frac{1}{2^{n/2+1}} = A_n + B_n$$

monotonically decreases as $n \ge 2$, since the inequality $C_2 > C_3$ is verified directly by

$$C_2 = \frac{1}{4} + \frac{1}{4} > \frac{1}{8}(1+\sqrt{2}) + \frac{1}{4\sqrt{2}} = C_3,$$

For $n \ge 3$, the value of A_n decreases, which shows the following chain of equivalent inequalities

$$A_{n+1} < A_n \Leftrightarrow \frac{1}{2^{n+1}} \sum_{k=0}^{n-1} 2^{k/2} < \frac{1}{2^n} \sum_{k=0}^{n-2} 2^{k/2} \Leftrightarrow 1+\sqrt{2} < 2^{n/2},$$

where the decrease of B_n is obvious. Thus, for all $n \ge 2$, we have $C_n(f) \le \frac{1}{2}$. Further, note that all the arguments carried out remain valid for the function $f(1-t), 0 \le t \le 1$. Consider the function

$$\overline{f}(t) = \begin{cases} f(2t), & t \in [0,1/2], \\ f(2-2t), & t \in [1/2,1]. \end{cases}.$$

It is clear, as shown before, that

$$S_1 \overline{f}(t) = \int_0^1 \overline{f}(t) dt = 1 - \frac{1}{2^n}$$

And, in view of the symmetry of the graph $\overline{f}$ with respect to $t = \frac{1}{2}$, that

$$S_2 \overline{f} = S_1 \overline{f}.$$

In addition, for all $k \ge 1$, we have $S_{2^k} \overline{f}(t) = S_{2^{k-1}} f(t)$ or $S_{2^{k-1}} f(1-t)$ for $t \in [0,1/2]$ or $t \in [1/2,1]$, respectively. Therefore

$$C_n(\overline{f}) = \sum_{k=1}^{\infty} \left\|S_{2^k} \overline{f} - S_{2^{k+1}} \overline{f}\right\| = C_n(f).$$

Finally, $r_h = W_1 \overline{f}$ — is the anti-periodization of the function $\overline{f}$. Therefore

$$S_2 r_h = S_4 r_h = (1 - \frac{1}{2^n}) r$$

and for all $k \ge 2$, we have $S_{2^k} r_h(t) = \pm S_{2^{k-1}} \overline{f}(t)$. Consequently, we again find that

$$C_n(r_h) = \sum_{k=2}^{\infty} \left\|S_{2^k} r_h - S_{2^{k+1}} r_h\right\| = C_n(\overline{f}) = C_n(f).$$

Then, for the Fourier-Walsh coefficient c_α of the function r_h, we will have

$$\sum_{k=1}^{\infty}(\sum_{|\alpha|=k}|c_\alpha|^2)^{1/2}=C_n(r_h)\le\frac{1}{2}<(1-\frac{1}{2^n})=|c_\phi|.$$

for $n\ge 2$. Then, we have the Riesz basis property of the affine Walsh system generated by the function r_h.

Let $Jf(t)=\int_0^t f(s)ds$ - be an integration operator. It is obvious that, for the 1-periodic function $f(t)$, having a zero mean value on the period, the function $Jf(t)$ is 1-periodic; however, its integral mean, in general, is nonzero. Consider the modified operator

$$J_0=4W_1J,$$

i.e. $J_0f(t)$ is an ant periodization of Jf and therefore the space L_0^2 is invariant under J_0. For each function $f\in L_0^2$, the function J_0f is continuous on the whole axis $(-\infty,\infty)$ and almost everywhere has the derivative L_0^2

$$DJ_0f=8W_1f,\qquad D=\frac{d}{dt}.$$

Theorem (2.2): Suppose that the function $f\in L_0^2$ is representable as a sum of a series in the Rademacher system

$$f=\sum_{k=0}^{\infty}c_kr_k.$$

Then, for an affine Walsh system, generated by the function J_0f, to be a Riesz basis, it is necessary and sufficient that the analytic function

$$f(z)=\sum_{k=0}^{\infty}c_kz^k$$

satisfies the condition

$$\frac{1}{2\pi}\int_0^{2\pi}\left|f(\frac{1}{2}e^{it})\right|^2dt\le 3\left|f(\frac{1}{2})\right|^2.$$

Proof

First, we note that :

$$DJ_0r(t)=8W_1r(t)=8r(t+\frac{1}{4})=4(r(t+\frac{1}{4})-r(t-\frac{1}{4}))=2Dr_{\frac{1}{4}}(t),$$

From this being taken into account, we have

$$J_0r=2r_{\frac{1}{4}}=r-W_1^2l,\Rightarrow 4Jr=1-W_1l.$$

In addition, the relations

$$W_0J=2JW_0,\qquad W_1J=2JW_1,$$

are easily verified by differentiation. Thus, we have

$$J_0f=4W_1J\sum_{k=0}^{\infty}c_kW_0^kr=W_1\sum_{k=0}^{\infty}\frac{c_k}{2^k}W_0^k4Jr=(\sum_{k=0}^{\infty}\frac{c_k}{2^k})r-\sum_{k=0}^{\infty}\frac{c_k}{2^k}W_1W_0^kW_1l.$$

Put

$$\lambda=\sum_{k=0}^{\infty}\frac{c_k}{2^k}$$

and

$$\varphi=\sum_{k=0}^{\infty}\frac{c_k}{2^k}W_1W_0^kW_1l=\sum_{k_1,k_2\geq 0}\frac{c_{k_1}}{2^{k_1+k_2+1}}W_1W_0^{k_1}W_1W_1^{k_2}r.$$

Then, $J_0=\lambda r-\varphi$ and, for the corresponding operators,

$T_{J_0f}=\lambda I-T_\varphi$.

The Walsh affine system generated by the function φ is orthogonal, since for all multi-indices $\alpha\neq\alpha'$, the equality $\alpha(1,0_{k_1},1,0_{k_2})=\alpha'(1,0_{k_1'},1,0_{k_2'})$ entails $k_2=k_2',k_1=k_1'$ and $\alpha=\alpha'$, from which

$$(\varphi_\alpha,\varphi_{\alpha'})=\sum_{k_1,k_2\geq 0}\frac{c_{k_1}}{2^{k_1+k_2+1}}\sum_{k_1',k_2'\geq 0}\frac{c_{k_1'}}{2^{k_1'+k_2'+1}}W^\alpha W_1W_0^{k_1}W_1W_1^{k_2}r,W^{\alpha'}W_1W_0^{k_1'}W_1W_1^{k_2'}r)=0$$

Therefore

$$\|T_\varphi\|=\|\varphi\|=(\sum_{k_1,k_2\geq 0}\frac{|c_{k_1}|^2}{4^{k_1+k_2+1}})^{1/2}=(\frac{1}{3}\sum_{k=0}^{\infty}\frac{|c_k|^2}{4^k})^{1/2}.$$

The resolvent condition $\|T_\varphi\|<\lambda$ obviously ensures the invertibility of the operator $T_{J_0f}=\lambda I-T_\varphi$. Moreover, as argued previously [11], it can be shown that if $\|T_\varphi\|>\lambda$, then the image of the operator $\lambda I-T_\varphi$ is not dense in the space L_0^2 (that is, the corresponding affine system is not complete). Consequently, in view of the closedness of the spectrum of the operator, the condition $\|T_\varphi\|<\lambda$ is equivalent to the invertibility of the operator $\lambda I-T_\varphi$. It remains to note that this resolvent condition can be written in the following form

$$(\frac{1}{3}\sum_{k=0}^{\infty}\frac{|c_k|^2}{4^k})^{1/2}<\left|\sum_{k=0}^{\infty}\frac{c_k}{2^k}\right|,$$

the left-hand side of which is the mean value of the square of the modulus of the analytic function $f(z)$ on the disk $(|z|=\frac{1}{2})$:

$$\frac{1}{\sqrt{3}}(\frac{1}{2\pi}\int_0^{2\pi}\left|f(\frac{1}{2}e^{it})\right|^2dt)^{1/2},$$

and its right-hand side coincides with $\left|f(\frac{1}{2})\right|$.

Conclusions

It was demonstrated that when the Walsh system is regularized by means of Steklov functions, the obtained affine system of Walsh type has the Riesz basis property for certain values of the regularization parameter.

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