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Fixed Point Theory for Study the Controllability of Boundary Control Problems in Reflexive Banach Spaces

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Abstract:
In this paper, we extend the work of our proplem in uniformly convex Banach spaces using Kirk fixed point theorem. Thus the existence and sufficient conditions for the controllability to general formulation of nonlinear boundary control problems in reflexive Banach spaces are introduced. The results are obtained by using fixed point theorem that deals with nonexpanisive mapping defined on a set has normal structure and strongly continuous semigroup theory. An application is given to illustrate the importance of the results.

Keywords: Controllability, Reflexive Banach Space, Opial's condition, Normal Structure, Semigroup Theory.

نظرية النقطة الصامدة لدراسة قابلية السيطرة لمسائل السيطرة في فضاءات بناخ الانعكاسية

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الخلاصه
في هذا البحث تم توسعة العمل لمسألتنا المطروحة ضمن البحث المعرفة على فضاءات بناخ المحدب بأنتظام بأستخدام نظرية النقطة الصامدة لكريك. لذلك تم تقديم الوجود للحل والشروط الكافية لامكانية السيطرة للصيغة العامة لمسائل السيطرة الحدودية غير الخطية في فضاءات بناخ الانعكاسية. تم الحصول على النتائج من خلال استخدام نظرية النقطة الصامدة التي تتعامل مع التطبيق غير الممدد والمعرف على مجموعة تمتلك البنية العمودية وكذلك نظرية شبة الزمرة المستمرة بقوة. ثم تم اعطاء تطبيق يوضح قيمة النتائج التي تم الحصول عليها.

1. Introduction

Many engineering and scientific systems in the control theory in infinite dimensional spaces can be formulated by partial differential equations, integral equations, or fractional differential equations.

We can characterize these systems as differential equations by using semigroup theory,and then

study the solution of these problems. Controllability is one of most significant properties of the control system, it means that the ability to transmit the system from an arbitrary initial

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state to an arbitrary final state of a given set in a finite time by a convenient option of the control function, one can refer to the references [1-3].

In this paper, we introduce the sufficient conditions for controllability of the following boundary control problem in arbitrary reflexive Banach spaces (rBs).

$$\left.\begin{array}{c} \dot{w}(t) = Bw(t) + As(t) + g(t, w(t)) + H\Big(t, N(t, w(t))\Big);\ \textit{almost everywhere in } J = [0, a] \\ \tau w(t) = A_1 s(t), \\ w(0) = w_0 \end{array}\right\} \quad (1.1)$$

where $w(.)$ takes values in (rBs) Z with norm $\|.\|$, the control function $s(.) \in L^2(J, S)$ be a (rBs) of admissible control functions, with S is a Banach space (Bs). Let B a closed linear and densely defined operator, with the domain of B, $D(B) \subseteq Z$, $\|B\| \leq c_1$, where c_1 is a positive constant, and τ be a linear operator such that $D(\tau) \subseteq Z$ and the range of τ, $R(\tau) \subseteq E$, where E is a (Bs), $A_1: S \rightarrow E$ be a linear continuous operator. The nonlinear operators g, N and H are continuous from $J \times Z$ into Z and all of them satisfy Lipschitz condition on the second argument. Here B be a linear operator generates a strongly continuous semigroup (C_0 – semigroup) $Y(t)$, $t \geq 0$, on (rBs) Z and $A: S \rightarrow Z$ be a bounded linear operator with $\|A\| \leq k$, where k is a positive constant.

Fixed point theorems (FPTs) are basic mathematical tools which are used in studying the controllability results of nonlinear equations. Controllability of the system (1.1) with different geometric conditions on the spaces Z and S has been studied by using Banach contraction theorem, Schauder (FPT) and Kirk (FPT), see [4-7].

Nonexpansive mappings on a space has normal structure, these mappings play an important role in fixed point theory, see [8,11].

Since every uniformly convex Banach space (ucBs) is (rBs) ,howeverthe converse is not ture in general, [7], as well as a nonexpansive mapping on a (Bs) has no fixed point (FP) in general. . Then we extend the work of our problem by using Kirk (FPT) [5]. Thus, the aim of this article is to study the controllability of the system (1.1) in arbitrary (rBs) by using (FPT) that deals with nonexpansive mapping defined on a set has normal structure.

2. Preliminaries

In this section some well known definitions, theorems and examples that will be used in the proof of the main results.

Definition 2.1 [8]: Let X be a normed space, a self mapping T is said to be Lipschitz continuous**,** if there is $\varphi \geq 0$, such that $\|T(x_1) - T(x_2)\| \leq \varphi\|x_1 - x_2\|$ for all x_1 , $x_2 \in X$. The smallest φ is the Lipschitz constant of T. If $\varphi < 1$ then T is contraction and if $\varphi \leq 1$ then T is nonexpansive **.**

It is clear that from the previous definition the contraction mapping is nonexpansive and isometry mapping is nonexpansive ,while it is not contraction. Isometry mapping means that T satisfies the following condition $\|T(x_1) - T(x_2)\| = \|x_1 - x_2\|$ for all $x_1, x_2 \in X$.

Definition 2.2 [9]: A (Bs) X is said to be satisfy Opial's condition if for each x in X and each sequence $\{x_n\}$ converges weakly to x, then $\lim_{n\to\infty} inf\|x_n - y\| > \lim_{n\to\infty} inf\|x_n - x\|$ holds for all $y \neq x$. Finite dimensional (Bs), l_p spaces for $1 < p < \infty$ and L_p for $p = 2$ (Hilbert space) satisfy Opial's condition.

Definition 2.3 [7]: Let X^* and X^{**} be the first and second dual spaces of a normed space X. Define a mapping $\mathcal{J}: X \rightarrow X^{**}\ by\ \mathcal{J}(x) = F_x$, where $F_x = f(x)$, and $f \in X^*$. The normed space X is called reflexive if the natural embedding is an onto mapping.It is clear (Bs) is reflexive if the natural embedding is an onto mapping from X into X^{**} .

Example 2.4 [7]: The Euclidean space R^n, all finite dimensional spaces and Hilbert spaces are reflexive, so L_p and l_p spaces for $1 < p < \infty$ and (ucBs) are reflexive, while the space of continuous real value function on [0,1] is not reflexive .

Definition 2.5 [7]: Let X be a normed space, a subset K of X is called weakly compact, if every sequence $\{x_n\}$ $in\ K$ contains a subsequence which converges weakly in .

Remark 2.6: Every nonempty, bounded, closed and convex (bcc) subset of (rBs) is weakly compact [8,9].

Definition 2.7 [8]: Let X be a Banach space, and $K \subseteq X$ be nonempty, (bcc). A point $x \in K$ is said to be diametral if $sup\{\|x-k\| : k \in K\} = diam\, K$. A subset K of X has normal structure, if for each nonempty, convex $D \subseteq K$ with diam > 0 , there exist a point $x \in D$ which is not diametral .

***Example 2.8 [8]*:** In (Bs) compact convex set has normal structure, so nonempty, (bcc) subset of (ucBs) has normal structure, Opial's condition also implies normal structure, see [10].

To have an extension of Kirk (FPT), on (ucBs), we need some geometric conditions on the spaces in the domain of the nonexpansive maps in (rBs).

Theorem 2.9 [8,11]: Let T be nonexpansive mapping from C into C, where C is a nonempty weakly compact convex subset having normal structure in a (Bs) X, then T has a (FP) in C.

Remark 2.10 [11]: In previous theorem the convexity can't be dispense one can see the following simple example

Let $C = [-2,-1] \cup [1,2] \subset R$ and T is a self mapping on C defined by $Tx = -x,\ x \in C$, therefore T is nonexpansive, but T has no (FP) in C.

Note that, the nonexpansive map on a non convex set in (Bs) has no fixed point .

Definition 2.11 [1]: Let X be a (Bs). A one parameter family $Y(t),\ 0 < t < \infty$ of linear bounded operators from a (Bs) X into itself , is called a strongly continuous semigroup (C_0 – semigroup) , if it's satisfied the following conditions:

(i) $(0) = I$, (ii) $Y(t+s) = Y(t)Y(s)$ for every $t,s \geq 0$ (the semigroup property) .

(iii) $\lim_{t\to 0} Y(t)x = x$ for every $x \in X$.

Definition 2.12 [11]: The infinitesimal generator B of the semigroup $Y(t)$ on a (Bs) X is defined by: $Bx = \lim_{t\to 0^+} \frac{1}{t}(Y(t)-I)x$, for $x \in D(B)$ whenever the limit exists .

3. Controllability Of Nonlinear Control Problems

The main objective of this section, is to study the controllability of mild solution to the boundary value control problem (1.1) in (rBs) by using C_0 – semigroup and Theorem 2.9.

Let $B_1 : Z \to Z$, be the linear operator, defined by $B_1 w = Bw,\ w \in D(B_1)$, where $D(B_1) = \{w \in D(B): \tau w = 0\}$. Here, let $Z_0 = \{ w : w \in C(J,Z), w(0) = w_0 , \|w(t)\| \leq r,$ for all $t \in J\}$, where r is a positive constant, be a subset of Z, since Z_0 is closed subset of a (rBs), then Z_0 is a (rBs) [7].

Throughout this paper, we also suppose the basic hypothesis as follows:

(C_1) $D(B) \subset D(\tau)$ and the restriction of τ to $D(B)$ is continuous relative to such graph norm of $D(B_1)$.

(C_2) The infinitesimal generator B_1 generates a C_0-semigroup $Y(t)$ with $\max_{t\geq 0} \| Y(t) \| \leq l$

(C_3) There exists a linear continuous operator $A_2 : S \longrightarrow Z$, with $BA_2 \in L(S,Z), \tau(A_2 s) = A_1 s$, for all $s \in S$. And $A_2 s(t)$ is continuously differentiable, such that $\|A_2 s\| \leq L\|A_1 s\|$ for all $s \in S$, where L is a constant.

(C_4) For all $t \in (0,a]$ and $s \in S$, $Y(t)A_2 s \in D(B_1)$. Further, there exists a positive function $f(.) \in L^1(0,a)$, such that $\|B_1 Y(t) A_2\| \leq f(t)\ a.e., t \in (0,a)$.

(C_5) The nonlinear operators $: J \times Z \to Z$, and $N: J \times Z \to Z$ are continuous and they satisfy Lipschitz condition on the second argument.

Let $w_1, w_2 \in Z_0$, such that $\|g(t,w_1(t)) - g(t,w_2(t))\| \leq l_1\|w_1(t) - w_2(t)\|$, and $\|N(t,w_1(t)) - N(t,w_2(t))\| \leq l_2\|w_1(t) - w_2(t)\|$, where l_1 , l_2 are positive constants.

Also, let $l_3 = \max_{t\in J}\|g(t,0)\|$, and $k_1 = \max_{t\in J}\|w(t)\|$.

Further, the nonlinear operator $H: J \times Z \to Z$ is continuous and satisfy Lipschitz condition, such that for all $w_1, w_2 \in Z_0$, we have :
$\|H(t, N(t, w_1(t))) - H(t, N(t, w_2(t)))\| \leq l_4 \|N(t, w_1(t)) - N(t, w_2(t))\| \leq l_4 l_2 \|w_1(t) - w_2(t)\|$, where l_4 is positive constant. Also, let $l_5 = \max_{t\in J} \|H(t, N(t, 0))\|$.

(C$_6$) The linear operator U from $L^2(J, S)$ into , defined by :

$$Us = \int_0^a Y(t-z)As(z)dz$$

This leads to a bounded inverse operator $\widetilde{U}^{-1}$ defined on $L^2(J, S)$ /ker(U), and hence $\|\widetilde{U}^{-1}\| \leq k_2$, where k_2 is a positive constant. For more details about the existence of bounded inverse operator of U, see [5,12].

3.1 Result of Controllability to Problem (1.1):

Throughout this subsection, we want to define, and to find the mild solution to the problem (1.1).

Suppose that $w(.) \in Z$ be a solution of problem (1.1), then we can define a function :

$$O(t) = w(t) - A_2 s(t) \tag{3.1}$$

From assumptions, we obtain that $O(t) \in D(B_1)$. Therefore the problem (1.1) can be written in term of B_1 and A_2, as follows :

$$\begin{aligned} \dot{w} &= B[O(t) + A_2 s(t)] + As(t) + g(t, w(t)) + H(t, N(t, w(t))) \ \ a.e., in\ J = [0, a] \\ &= BO(t) + BA_2 s(t) + As(t) + g(t, w(t)) + H(t, N(t, w(t))) \end{aligned}$$

Since $O(t) \in D(B_1)$, hence $B_1 O(t) = BO(t)$, thus

$$\left.\begin{aligned} \dot{w}(t) = B_1 O(t) + BA_2 s(t) + As(t) + g(t, w(t)) + H(t, N(t, w(t))); (a.e.)\ in\ J = [0, a] \\ w(t) = O(t) + A_2 s(t) \\ w(0) = w_0 \end{aligned}\right\} \tag{3.2}$$

By condition (C$_3$), we have $A_2 s(t)$ is continuously differentiable, if w is continuously differentiable on J, then by definition of the mild solution $O(t) = w(t) - A_2 s(t)$, it can be defined as a mild solution to Cauchy problem [1],

$$\frac{d}{dt} O(t) = \frac{d}{dt} w(t) - A_2 \frac{d}{dt} s(t)$$

By equation (3.2), we get that

$$\begin{aligned} \dot{O}(t) = B_1 O(t) + BA_2 s(t) + (As)(t) + g(t, w(t)) + H(t, N(t, w(t))) \\ - A_2 \frac{d}{dt} s(t), a.e\ in\ J = [0, a] \\ O(0) \\ = w_0 - A_2 s(0) \end{aligned} \tag{3.3}$$

From condition (C$_2$), we get that $(t), t \geq 0$, which is the C_0-semigroup generated by the linear operator B_1, and $O(t)$ is a solution of (3.3), hence the function $Q(z) = Y(t-z)O(z)$ is differentiable for $0 < z < t$ for more dailies see [1].

$\frac{d}{dz} Q(z) = Y(t-z) \frac{d}{dz} O(z) + O(z) \frac{d}{dz} Y(t-z)$. Thus by equation (3.3) we have

$$\frac{d}{dz} Q(z) = Y(t-z)[B_1 O(z) + BA_2 s(z) + (As)(z) + g(z, w(z)) + H(z, N(z, w(z))) - A_2 \frac{d}{dz} s(z)] + O(z)[-BY(t-z)]$$

$$\begin{aligned} \frac{d}{dz} Q(z) = Y(t-z) B_1 O(z) + Y(t-z) BA_2 s(z) + Y(t-z)(As)(z) + Y(t-z) g(z, w(z)) \\ + Y(t-z) H(z, N(z, w(z))) - Y(t-z) A_2 \frac{d}{dz} s(z) - Y(t-z) BO(z), \end{aligned}$$

Since $B_1 O(t) = BO(t)$, hence

$$\frac{d}{dz} Q(z) = Y(t-z) BA_2 s(t) + Y(t-z)(As)(t) + Y(t-z) g(z, w(z)) + Y(t-z) H(z, N(z, w(z))) - Y(t-z) A_2 \frac{d}{dz} s(z).$$

On integrating both sides from 0 $to\ t$, yields :
$Q(t) - Q(0) = \int_0^t Y(t-z)BA_2 s(z)dz + \int_0^t Y(t-z)As(z)dz + \int_0^t Y(t-z)\, g(z,w(z))dz + \int_0^t Y(t-z)\, H(z,N(z,w(z)))dz - \int_0^t Y(t-z)\, A_2 \frac{d}{dz}\, s(z)dz$
(3.4)
From the definition of $Q(z) = Y(t-z)O(z)$, we get :
$Q(t) = Y(t-t)O(t) = Y(0)[w(t) - A_2 s(t)] = w(t) - A_2 s(t), Y(0) = I,$
(3.5)
And,
$Q(0) = Y(t-0)O(0) = Y(t)[w(0) - A_2 s(0)] = Y(t)w_0 - Y(t)\, A_2 s(0).$
(3.6)

Now, by integrating the term $\int_0^t Y(t-z)A_2 \frac{d}{dz}\, s(z)dz$, in (3.4) by parts we get that:

$$\int_0^t Y(t-z)A_2 \frac{d}{dz}\, s(z)dz = Y(t-z)\, A_2 s(z)\Big|_0^t + \int_0^t s(z)B_1\, Y(t-z)A_2 dz$$
$$= A_2 s(t) - Y(t)\, A_2 s(0) + \int_0^t s(z)B_1\, Y(t-z)A_2 dz$$

(3.7) Therefore,
By substituting the equations (3.5), (3.6), and (3.7) into equation (3.4), we get that
$w(t) - A_2 s(t) - Y(t)w_0 + Y(t)\, A_2 s(0) = \int_0^t Y(t-z)BA_2 s(z)dz + \int_0^t Y(t-z)As(z)dz + \int_0^t Y(t-z)\, g(z,w(z))dz + \int_0^t Y(t-z)\, H(s,N(z,w(z)))dz - A_2 s(t) + Y(t)\, A_2 s(0) - \int_0^t s(z)B_1\, Y(t-z)A_2 dz$. Therefore
$w(t) =$
$Y(t)w_0 + \int_0^t Y(t-z)BA_2 s(z)dz + \int_0^t Y(t-z)As(z)dz + \int_0^t Y(t-z)\, g(z,w(z))dz + \int_0^t Y(t-z)\, H(z,N(z,w(z)))dz - \int_0^t B_1\, Y(t-z)A_2 s(z)dz\, .$
(3.8)
If the function $w: [0,a] \to Z$ given by (3.8) is continuous on $[0,a]$, and it is continuously differentiable on $(0,a)$, and $w(z) \in$ Z for $0 < z < t$, then we say that w(.) is a mild solution to problem (1.1).

Definition 3.1: The nonlinear boundary value control system (1.1) is said to be controllable on the interval $J = [0,a]$, if $\forall\, w_0, w_1 \in Z$, $\exists\, s(.) \in L^2(J,S)$ such that the mild solution $w(t)$ satisfies $w(a) = w_1$.

3.2 Main Results

In this section, we will prove the theorem that deals with the controllability of the problem (1.1).

Theorem 3.1: Let Z be a (rBs) which satisfying Opial's condition and the hypothesis (C_1)_(C_6) are satisfied for the nonlinear boundary control problem (1.1)

$$\left.\begin{aligned} \dot{w}(t) = Bw(t) + As(t) + g(t,w(t)) + H(t,N(t,w(t))); \quad (a.e.)\ in\ J = [0,a] \\ \tau w(t) = A_1 s(t), \\ w(0) = w_0 \end{aligned}\right\}$$

Further, suppose that
(C_7) There exists a constant $k_3 > 0$, such that $\int_0^a f(t) \le k_3$.
(C_8) Let $b_1 = all_1 k_1$, $b_2 = all_3$, $b_3 = all_4 l_2 k_1$ and $b_4 = all_5$, such that:
$(l\|w_0\| + b_1 + b_2 + b_3 + b_4 + [ak_2 l\|BA_2\| + k_2 k_3 + ak_2 lk][\|w_1\| + l\|w_0\| + b_1 + b_2 + b_3 + b_4]) \le r$

(C$_9$) $\varphi = [[all_1 + all_4 l_2] + [ak_2 l\|BA_2\| + k_2 k_3 + ak_2 lk][all_1 + all_4 l_2]]$, such that $0 \leq \varphi \leq 1$.

Then the system (1.1) is controllable on J **.**

***Proof*:** By using definition (3.1) and mild solution equation (3.8) we obtain that

$$w_1 = w(a) = Y(a)w_0 + \int_0^a Y(a-z)BA_2 s(z)dz + \int_0^a Y(a-z)(As)(z)dz + \int_0^a Y(a-z)\, g(z,w(z))dz + \int_0^a Y(a-z)\, H(z,N(z,w(z)))dz - \int_0^a B_1 Y(t-z)A_2 s(z)dz$$

$w_1 = Y(a)w_0 + Us + \int_0^a Y(a-z)\, g(z,w(z))dz + \int_0^a Y(a-z)\, H(z,N(z,w(z)))dz.$

Therefore,

$Us = w_1 - Y(a)w_0 - \int_0^a Y(a-z)g(z,w(z))dz - \int_0^a Y(a-z)H(z,N(z,w(z)))dz.$

From constriction the operator U in (C$_6$) since $s(t) = \widetilde{U}^{-1}(Us(t))$, thus

$$s(t) = \widetilde{U}^{-1}(w_1 - Y(a)w_0 - \int_0^a Y(a-z)g\big(z,w(z)\big)dz - \int_0^a Y(a-z)H(z,N(z,w(z)))dz)(t) \quad (3.9)$$

We can explain implication when using this control, the operator defined by

$$(\beta w)(t) = \mathrm{Y(t)}w_0 + \int_0^t Y(t-z)\, BA_2 s(z)dz + \int_0^t Y(t-z)(\mathrm{As})(\mathrm{z})dz + \int_0^t Y(t-z)\, g(z,w(z))dz + \int_0^t Y(t-z)\, H(z,N(z,w(z)))dz - \int_0^t B_1 Y(t-z)A_2 s(z)dz,$$

has a (FP), this (FP) is then a solution of (1.1).

Note that, $(\beta w)(a) = w_1$, it know that the control s transmit the control system from the initial w_0 to w_1 in time a, provided we can obtain a (FP) of the nonlinear operator β .

Let Z be a (rBs) that satisfies the Opial's condition and let Z_0={ $w : w \in C(J,Z), w(0) = w_0, \|w(t)\| \leq r$, for $t \in J$}, it's clear that Z_0 is (bcc) subset of Z [7]. We have also Z_0 is weakly compact and has normal structure in (rBs) Z, see Remark 2.6 and Example 2.8.

Here, we will define a mapping $\beta : Z \to Z_0$ by :

$$(\beta w)(t) = \mathrm{Y(t)}w_0 + \int_0^t Y(t-z)\, g(z,w(z))\mathrm{dz} + \int_0^t Y(t-z)\, H(z,N(z,w(z)))dz + \int_0^t [Y(t-\xi)BA_2 - B_1 Y(t-\xi)A_2 + Y(t-\xi)A]\widetilde{U}^{-1}[w_1 - Y(a)w_0 - \int_0^a Y(a-z)g(z,w(z))\, dz - \int_0^a Y(a-z)H(z,N(z,w(z)))\, dz](\xi)d\xi \quad (3.10)$$

Now we want to show that the operator β is continuous and maps Z_0 into itself. Thus, we take the norm of both sides of (3.10)

$$\|(\beta w)(t)\| = \Big\| Y(t)w_0 + \int_0^t Y(t-z)\, g(z,w(z))dz + \int_0^t Y(t-z)\, H(z,N(z,w(z)))dz + \int_0^t [Y(t-\xi)BA_2 - B_1Y(t-\xi)A_2 + Y(t-\xi)A]\widetilde{U}^{-1}[w_1 - Y(a)w_0 - \int_0^a Y(a-z)g(z,w(z))\,dz - \int_0^a Y(a-z)H(z,N(z,w(z)))\,dz](\xi)d\xi \Big\|$$

$$\|(\beta w)(t)\| \le \|Y(t)\|\|w_0\| + \Big\|\int_0^t Y(t-z)[\, g(z,w(z)) - g(z,0) + g(z,0)]dz\Big\| + \Big\|\int_0^t Y(t-z)\,[H(z,N(z,w(z))) - H(z,N(z,0)) + H(z,N(z,0))]dz\Big\| + \Big\|\int_0^t [Y(t-\xi)BA_2 - B_1Y(t-\xi)A_2 + Y(t-\xi)A]\widetilde{U}^{-1}[w_1 - Y(a)w_0 - \int_0^a Y(a-z)[g(z,w(z)) - g(z,0) + g(z,0)]\,dz - \int_0^a Y(a-z)[H(z,N(z,w(z))) - H(z,N(z,0)) + H(z,N(z,0))]\,dz](\xi)d\xi\Big\|$$

$$\|(\beta w)(t)\| \le \|Y(t)\|\|w_0\| + \int_0^t \|Y(t-z)\|[\,\|g(z,w(z)) - g(z,0)\| + \|+g(z,0)\|]dz + \int_0^t \|Y(t-z)\|\,[\|H(z,N(z,w(z))) - H(z,N(z,0))\| + \|H(z,N(z,0))\|]dz + \int_0^t [\|Y(t-\xi)\|\|BA_2\|\|B_1Y(t-\xi)A_2\|\,\|Y(t-\xi)\|\|A\|]\big\|\widetilde{U}^{-1}\big\|[\|w_1\| + \|Y(a)\|\|w_0\| + \int_0^a \|Y(a-z)\|\,[\|g(z,w(z)) - g(z,0)\| + \|g(z,0)\|]\,dz + \int_0^a \|Y(a-z)\|\,[\|H(z,N(z,w(z))) - H(z,N(z,0))\| + \|H(z,N(z,0))\|]dz](\xi)d\xi$$

Under the conditions (C_1) _(C_7), we obtain that:

$$\|(\beta w)(t)\| \le l\|w_0\| + \int_0^t l\,[l_1\|w(z)\| + l_3]dz + \int_0^t l[l_4l_2\|w(z)\| + l_5]dz + \int_0^t [l\|BA_2\| + f(t) + lk]k_2[\|w_1\| + l\|w_0\| + al[l_1\|w(z)\| + l_3] + al[l_4l_2\|w(z)\| + l_5]]\,(\xi)d\xi$$

Since g , N and H are continuous and by condition (C_8) $\|(\beta w)(t)\| \le r$, it follows that β is also continuous mapping from Z_0 into itself .

Now, we must prove that the operator β is nonexpansive mapping from Z_0 into itself :

Let w_1 , $w_2 \in Z_0$, then

$$\begin{aligned}
&\|(\beta w_1)(t)-(\beta w_2)(t)\| \\
&\quad = \Big\| \mathrm{Y(t)}w_0 + \int_0^t Y(t-z)\, g(z,w_1(z))dz + \int_0^t Y(t-z)\, H(z,N(z,w_1(z)))dz \\
&\quad + \int_0^t [Y(t-\xi)BA_2 - B_1Y(t-\xi)A_2 + Y(t-\xi)A]\widetilde{U}^{-1}[w_1 - Y(a)w_0 \\
&\quad - \int_0^a Y(a-z)g(z,w_1(z))\, dz - \int_0^a Y(a-z)H(z,N(z,w_1(z)))\mathrm{dz}](\xi)d\xi \\
&\quad - [\mathrm{Y(t)}w_0 + \int_0^t Y(t-z)\, g(z,w_2(z))dz + \int_0^t Y(t-z)\, H(z,N(z,w_2(z)))\mathrm{dz} \\
&\quad + \int_0^t [Y(t-\xi)BA_2 - B_1Y(t-\xi)A_2 + Y(t-\xi)A]\widetilde{U}^{-1}[w_1 - Y(a)\, w_0 \\
&\quad - \int_0^a Y(a-z)g(z,w_2(z))\, dz - \int_0^a Y(a-z)H(z,N(z,w_2(z)))\, dz](\xi)d\xi]\|
\end{aligned}$$

$$\begin{aligned}
&\|(\beta w_1)(t)-(\beta w_2)(t)\| \\
&\quad = \Big\| \mathrm{Y(t)}w_0 + \int_0^t Y(t-z)\, g(z,w_1(z))dz + \int_0^t Y(t-z)\, H(z,N(z,w_1(z)))dz \\
&\quad + \int_0^t [Y(t-\xi)BA_2 - B_1Y(t-\xi)A_2 + Y(t-\xi)A]\widetilde{U}^{-1}[w_1 - Y(a)w_0 \\
&\quad - \int_0^a Y(a-z)g\big(z,w_1(z)\big)\, dz - \int_0^a Y(a-z)H(z,N(z,w_1(z)))\, dz](\xi)d\xi \\
&\quad - \mathrm{Y(t)}w_0 - \int_0^t Y(t-z)\, g(z,w_2(z))dz - \int_0^t Y(t-z)\, H(z,N(z,w_2(z)))dz \\
&\quad - \int_0^t [Y(t-\xi)BA_2 - B_1Y(t-\xi)A_2 + Y(t-\xi)A]\widetilde{U}^{-1}[w_1 - Y(a)w_0 \\
&\quad - \int_0^a Y(a-z)g(z,w_2(z))\, dz - \int_0^a Y(a-z)H(z,N(z,w_2(z)))\, dz](\xi)d\xi\|
\end{aligned}$$

$$\begin{aligned}
&\|(\beta w_1)(t)-(\beta w_2)(t)\| \\
&\quad = \Big\| \int_0^t Y(t-z)\, [g(z,w_1(z)) - g(z,w_2(z))]dz \\
&\quad + \int_0^t Y(t-z)\, [H(z,N(z,w_1(z))) - H(z,N(z,w_2(z)))] \\
&\quad + \int_0^t [Y(t-\xi)BA_2 - B_1Y(t-\xi)A_2 \\
&\quad + Y(t-\xi)A]\widetilde{U}^{-1}[\int_0^a Y(a-z)\, [g(z,w_1(z)) - g(z,w_2(z))]dz \\
&\quad - \int_0^a Y(a-z)[H(z,N(z,w_1(z))) - H(z,N(z,w_2(z)))]\mathrm{dz}\,](\xi)d\xi\|
\end{aligned}$$

$$\|(\beta w_1)(t)-(\beta w_2)(t)\| \le \int_0^t \|Y(t-z)\|\,\big\|g\big(z,w_1(z)\big)-g\big(z,w_2(z)\big)\big\|dz + \int_0^t \|Y(t-z)\|\|H(z,N(t,w_1(z)))-H(z,N(z,w_2(z)))\|dz + \int_0^t \|Y(t-\xi)\|\|BA_2\|\|B_1Y(t-\xi)A_2\|\,\|Y(t-\xi)\|\|A\|\big\|\widetilde{U}^{-1}\big\|[\int_0^a \|Y(a-z)\|\,\|g(z,w_1(z))-g(z,w_2(z))\|dz + \int_0^a \|Y(a-z)\|\|H(z,N(t,w_1(z)))-H(z,N(z,w_2(z)))\|dz]\,(\xi)d\xi$$

From condition(C_1) _(C_7), we obtain that:

$$\|(\beta w_1)(t)-(\beta w_2)(t)\| \le \int_0^t ll_1\|w_1(z)-w_2(z)\|dz + \int_0^t ll_4l_2\|w_1(z)-w_2(z)\|dz + \int_0^t [l\|BA_2\|+f(t)+lk]\,k_2[all_1\,\|w_1(z)-w_2(z)\| + all_4l_2\|w_1(z)-w_2(z)\|]dz$$

$$\|(\beta w_1)(t)-(\beta w_2)(t)\| \le [all_1+all_4l_2]\|w_1(t)-w_2(t)\| + [alk_2\|BA_2\|+k_2k_3 + ak_2lk][all_1+all_4l_2]\|w_1(t)-w_2(t)\|$$

$$\|(\beta w_1)(t)-(\beta w_2)(t)\| \le [all_1+all_4l_2][alk_2\|BA_2\|+k_2k_3 + ak_2lk][all_1+all_4l_2]]\|w_1(t)-w_2(t)\|$$

By condition (C_9), we get that:

$\|(\beta w_1)(t)-(\beta w_2)(t)\| \le \varphi\|w_1(t)-w_2(t)\|$

Consequentially, β is nonexpansive mapping .Thus from Theorem (2.9) , there exists a (FP) $w\in Z_0$, such that $(\beta w)(t)=w(t)$, and hence this (FP) is a solution of system (1.1) on the interval J, which satisfied $w(a)=w_1$. Therefore the nonlinear control system (1.1) is controllable on J.

4. Applications

Let ψ be a bounded and open subset of R^n, and let M be a boundary control integrodifferential system

$$\frac{\partial y(t,x)}{\partial t} - Ky(t,x) = \omega_1\Big(t, y(t,x), \int_0^t \omega_2\big(t,z,y(z,x)\big)dz\Big) \text{in} L_1 = (0,a)\times\psi \qquad (4.1)$$

$$y(t,0) = s(t,0) \text{ ,on } L_2 = (0,a)\times M,\ t\in[0,a] \qquad (4.2a)$$

$$y(0,x) = y_0(x),\ \text{for } x\in\psi \qquad (4.2b)$$

where $s\in L^2(L_2), y_0\in L^2(\psi), \omega_1 \in L^2(L_1)$ and $\omega_2\in L_1$.

This problem can be characterization as a boundary control problem of the form (1.1) by suitably taking the spaces, *C*, *E*, *S* and the operators *B₁*, ω, and x as follows:

Let $C= L^2(\psi)$, E= $H^{-\frac{1}{2}}(\Gamma)$, $S=L^2\ (M)$, $B_1= I$ (the identity operator) and $D(\omega)=\{y \in L^2(\psi); Ky \in L^2(\psi)\}$, $\omega = K$. The operator x is the "trace" operator with $xy = y|_M$ is well defined and belongs to $H^{-\frac{1}{2}}(\Gamma)$ for each $y \in D(\omega)$ (see[13]) and the operator B is given by $B = K,\ D(B) = H_0^1\ (\psi) \cup H^2(\psi)$ (Here $H^k\ (\psi), H^{\propto}(\Gamma)$ and $H_0^1(\psi)$ are usual Sobolev Spaces on ψ, Γ).

Define the linear operator A: $L^2(M) \rightarrow\ L^2(\psi)$ by $As = Us$ where Us is the unique solution to the Dirichlet boundary value problem,

$$DUs = 0 \text{ in } \psi$$
$$Us = s \text{ in } M$$

In other words, (see [14])

$$\int_{\psi} Us \triangle U dx = \int_{\Gamma} s \frac{\partial U}{\partial n} dx, \quad \text{for} \quad \text{all} \quad U \in H_0^1 \cup H^2(\quad)$$

(4.3)

Where $\frac{\partial U}{\partial n}$ denotes the outward normal derivative of U which is well-defined as an element of $H^{1/2}(\Gamma)$. From (4.3) it follows that,

$$\|Us\|_{L^2(\psi)} \leq C_1 \|s\|_{H^{-1/2}}(\Gamma), \text{ for all } s \in H^{1/2}(\Gamma)$$

and

$$\|Us\|_{H^1(\psi)} \leq C_2 \|s\| H^{1/2}(\Gamma), \text{ for all } s \in H^{1/2}(\Gamma),$$

where C_i , i=1,2 are positive constants independent of u. From the above estimates it follows by an interpolation argument [15] that

$$\|BY(t)A\|_{L(L^2(\Gamma),L^2(\Gamma))} \leq Ct^{-3/4}, \text{ for all } t > 0 \text{ with } f(t) = Ct^{-3/4}$$

Further assume that the bounded invertible operator $\widetilde{U}$ exists. Choose a and other constants, such that satisfying the last condition (C_8). Hence, one can see that all the conditions stated in the theorem are satisfied and so the system (1.1) is controllable on $(0, a)$.

Conclusions

- We have been studied the controllability of different kinds of general formulation of control systems.
- Extended the method using Kirk fixed point theorem which deals with contraction mapping to fixed point theorem of nonexpansive mapping.
- The controllability of the above systems discussed by using the concepts of semigroup theory with fixed point theorem.

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