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Direct sum of π -projective semimodules

Muna M. T. Altaee, Asaad. A. M. Alhossaini

College of Education for Pure Science, University of Babylon

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Abstract:

Let A , and N are a semiring, and a left A - semimodule, respectively. In this work we will discuss two cases:

1. The direct summand of π -projective semi module is π -projective, while the direct sum of two π -projective semimodules in general is not π -projective . The details of the proof will be given.
2. We will give a condition under which the direct sum of two π -projective semi modules is π -projective, as well as we also set conditions under which π -projective semi modules are projective.

Keywords: π -projective semimodule, finitely generated semimodule, finite dimensional semimodule, hereditary semimodule.

الجمع المباشر لشبه المقاسات الإسقاطية من النمط π

اسعد محمد علي الحسيني، منى توفيق احمد الطائي

قسم الرياضيات، كلية التربية للعلوم الصرفة، جامعة بابل، بابل، العراق

الخلاصة

لتكن A شبه حلقة و N شبه مقياس على تلك الحلقة. بواسطة هذا البحث سوف نناقش حالتين:
1- الجمعي المباشر لشبه المقاس الإسقاطي من النمط π يكون إسقاطي من النمط π , لكن الجمع المباشر لشبه مقاسين إسقاطيين من النمط π لا يكون دائما إسقاطي من النمط π . على سبيل المثال $\mathbb{Z}_2, \mathbb{Z}_4$ على \mathbb{Z} - شبه مقياس يكونان شبه مقاسين إسقاطيين من النمط π بينما في حالة $\mathbb{Z}_2 \oplus \mathbb{Z}_4 = N$ فان N ليست إسقاطي من النمط π وبرهان هذه الحقيقة سوف يعطى.
2- الشرط الذي يجعل الجمع المباشر لشبه مقاسين إسقاطيين من النمط π إسقاطي من النمط π .

1.Introduction

We assume that A , and N are a semiring, and a left A - semimodule respectively. The direct sum of semimodules, and π -projective semimodules were studied by several authors [1, 3], that means if for every two subsemimodules M and L with $M+L=N$, then there exist $f, g \in \text{End}(N)$ such that $f+g=1_N, f(N) \subseteq M$ and $g(N) \subseteq L$.

The main goal of this work is to study the direct sum of π -projective semimodules.

The paper is divided into three sections. In section 2, we give preliminaries that are used throughout this work. In section 3, we will give proofs for some facts and results in details.

1- The direct summand of π -projective semimodule is π - projective, however the direct sum of two π -projective semimodules. In general it is not π -projective for instant: \mathbb{Z}_2 and \mathbb{Z}_4 are π -projective semimodule , however, if $N = \mathbb{Z}_2 \oplus \mathbb{Z}_4$, then N is not π -projective.

2- We give a condition under which the direct sum of two π -projective semimodules is π -projective, as well as we also set conditions under which projective semimodules are projective.

2. Preliminaries

In this section, we will give the basic definitions, and concepts that will be used throughout this paper. We will start with Some concepts that will use to complete this research:

1- Hereditary semimodule: An A -semimodule is said to be hereditary if all its subsemimodules are projective.

2- Finite dimensional semimodule: An A -semimodule is called finite dimensional if it contains no infinite independent families of non-zero subsemimodules.

3- Non-singular semimodule: Let N be an A -semimodule, a singular subsemimodule of N is denoted by $\mathcal{Z}(N)$, we say that N is singular if $\mathcal{Z}(N)=N$ and nonsingular if $\mathcal{Z}(N)=0$.

Definition 2.1.[2] Let A be a semiring. A left A -semimodule N is a commutative monoid $(N, +, 0)$, if

we have a function $A \times N \rightarrow N$ defined by $(a, n) \rightarrow an$ ($a \in A$ and $n \in N$) such that for all $a, a' \in A$ and $n, n' \in N$, the following conditions are satisfied :

a) $a(n+n') = a n + a n'$

b) $(a + a')n = an + a'n$

c) $aa'(n) = a(a'n)$

d) $0n = 0$

Throughout this work, we assume that A -semimodule is left unitary A -semimodule ($1n=n$ for all n in N).

Definition 2.2.[2] A set D is said to be subsemimodule of a semimodule N if it is a nonempty subset of N and it is closed under addition and scalar multiplication, which is denoted by $D \leq N$.

Definition 2.3. [3] A subsemimodule C of a semimodule N is essential in N if for each subsemimodule D of N , $C \cap D = 0$, implies $D = 0$.

Definition 2.4.[2] Let N be a semimodule and D be a subsemimodule of N , then D is said to be subtractive if for all d and $c \in N$, $d \in D$ and $(d + c) \in D$ implies that $c \in D$.

Definition 2.5. [2] A semimodule N is said to be semisubtractive, if for any $n, n' \in N$ there is always some $k \in N$ that satisfies the following $n + k = n'$ or $n' + k = n$.

Definition 2.6.[4] An element n of a left S -semimodule N is cancellable if $n + x = n + h$ implies that $x = h$.

Definition 2.7. [2] An S -semimodule N is cancellative if every element of N is cancellable.

Definition 2.8.[2] Let A be a semiring and let K and H be subsemimodules of a semimodule N . N is said to be a direct sum of K and H , denoted by $N = K \oplus H$, if each $x \in N$ uniquely written as $x = k + h$ where $k \in K$ and $h \in H$. In this case, K (similarly H) is called a direct summand of N .

Definition 2.9.[4] Let B and D be A -semimodules. A homomorphism from B to D is a map $h: B \rightarrow D$ such that

1. $h(b + b') = h(b) + h(b')$

2. $h(ab) = ah(b) \quad \forall b, b' \in B$ and $a \in A$.

A homomorphism $h: B \rightarrow D$ is a:

(a) Monomorphism, if it is 1-1.

(b) Epimorphism if it is onto.

(c) Isomorphism if it is monomorphism and epimorphism.

(d) Endomorphism if $B = D$.

(e) $\ker(h) = \{b \in B/h(b) = 0\}$.

For any A -semimodule N , $\text{End}(N)$ means the set of all endomorphisms of N . In fact $\text{End}(N)$ is a semiring with usual addition and composition of maps in N [5].

Definition 2.10.[6, p.4] If D is a subsemimodule of a semimodule N , then N/D is called quotient (factor) semimodule of N by D , which is defined by $N/D = \{[n], n \in N\}$.

Definition 2.11.[6, p.7] A left A -semimodule B is said to be projective relative to N if for every epimorphism $\beta: N \rightarrow L$ and for every homomorphism $\alpha: B \rightarrow L$ there is a homomorphism $g: B \rightarrow N$ such that the diagram :

$$\begin{array}{ccc} & & B \\ & \swarrow g & \downarrow \alpha \\ N & \xrightarrow{\beta} & L \end{array}$$

commutes ($(\beta g) = \alpha$) (where L is any A -semimodule). A semimodule B is projective if it is projective relative to every left A -semimodule.

Definition 2.12.[7] A semimodule N is said to be quasi-projective if it is projective relative to N .

Lemma 2.13.[6, p.14]

Assume that N and P are semimodules, and H is a subsemimodule of N .

. If P is N -projective, then P is H -projective and N/H -projective.

Definition 2.14.[1] An A -semimodule N is said to be π -projective if for every two subsemimodules B and C of N , with $B+C=N$, there exist f and $g \in \text{End}(N)$ such that $f+g=1_N$, $f(N) \subseteq B$, and $g(N) \subseteq C$.

Every projective semimodule is π -projective ([1, 3.10]).

Definition 2.15. [6, p.5] The sequence $L \xrightarrow{\lambda} N \xrightarrow{\phi} B$ is called exact sequence if $\ker \phi = \text{Im} \lambda$.

Definition 2.16. [6, p.5] A short exact sequence of A -semimodules is an exact sequence of the form $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$.

Definition 2.17.[6, p.27] A proper short exact sequence $0 \rightarrow L \xrightarrow{\lambda} N \xrightarrow{\phi} B \rightarrow 0$ is called split or split exact if there is a homomorphism $\chi: B \rightarrow N$ such that $\phi \chi = I_B$.

Definition 2.17.[1] A semimodule W is indecomposable if the direct summands of it are only $\{0\}$ and itself.

Definition 2.19.[2] A set B is said to be generated set of an A -semimodule N , if N is the smallest subsemimodule containing B , we denote this by $N = \langle B \rangle$.

Remark 2.20.[8] $N = \langle B \rangle$ if and only if $\forall n \in N, n = \sum_{finite} a_i b_i, a_i \in A, b_i \in B$.

Definition 2.21.[2] A set B is to be free set if for all $\{b_1, b_2, b_3, \dots, b_n\} \subseteq B$ (n is a positive integer), the combination $\sum_{i=1}^n a_i b_i = 0$ implies $a_i = 0$ for $i=1, 2, \dots, n$.

Definition 2.22.[2] A set B is said to be a basis of the semimodule N if it is a free generating set of N .

Definition 2.23.[2] A semimodule N is said to be a free- semimodule if it has a basis.

Definition 2.24.[5]

Let A be a semiring, and a subset I of A is left (right) ideal of A , if for m and $m' \in I$, and $a \in A$, then $m + m' \in I$ and $am \in I$ ($ma \in I$).

Definition 2.25.[9] Let A -semimodule N is left semimodule, then its left annihilator which is denoted by $\text{ann}_A(N) = \{a \in A: an = 0 \text{ for every } n \in N\}$.

Definition 2.26[10]An ideal I_1 is essential in a semiring A if for each ideal I_2 of A with $I_1 \cap I_2 = 0$, implies $I_2 = 0$.

Definition 2.27.[10] Let N be an A -semimodule, a singular subsemimodule of N , which is denoted by $\mathcal{Z}(N)$, and $\mathcal{Z}(N) = \{n \in N \mid \text{ann}_A n \text{ is essential in } A\}$.

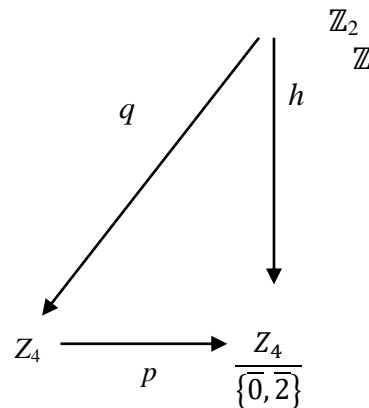
Definition 2.28.[3] Let N be an A -semimodule, we say that N is singular if $\mathcal{Z}(N) = N$ and nonsingular if $\mathcal{Z}(N) = 0$.

Definition 2.29.[1] A semimodule N is said to be hollow if all its proper subsemimodules are small.

In this work we assume that N will be subtractive, semisubtractive and cancellative semimodule

3. Direct sum of π -projective semimodule.

In [11, 4.9], the authors proved that the direct summand of π -projective semimodule is π -projective semimodule. But the direct sum of two π -projective semimodules in general is not π -projective semimodule, for example \mathbb{Z}_2 and \mathbb{Z}_4 are π -projective semimodules because both of them are hollow[1, 3.5], while in the other hand $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ is not π -projective semimodule, In order to prove this case suppose that $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ is π -projective semimodule. In [1,4,3], authors proved that \mathbb{Z}_2 is \mathbb{Z}_4 -projective which is impossible. Now let us consider the next diagram:



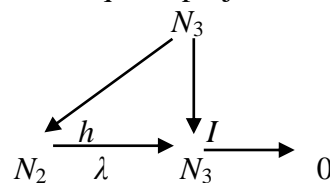
where $h: \mathbb{Z}_2 \rightarrow \frac{\mathbb{Z}_4}{\{0, 2\}}$ is an isomorphism and p is the natural epimorphism. If \mathbb{Z}_2 is \mathbb{Z}_4 -projective, then there exists a non-zero homomorphism $q: \mathbb{Z}_2 \rightarrow \mathbb{Z}_4$ such that $pq = h$. But the homomorphism which defined by $f(\bar{1}) = \bar{2}$ is the only non-zero homomorphism from $\mathbb{Z}_2 \rightarrow \mathbb{Z}_4$, then $h = 0$. Therefore we get contradiction.

proposition[1,4.3] Let $N = K \oplus D$ be a π -projective semimodule, then D is K -projective (and K is D -projective).

Now a condition under which the direct sum of two π -projective semimodules is π -projective, as well as conditions under which π -projective semimodules are projective will be given. The following result for semimodules is converted from a module version in [12, 2.2.5].

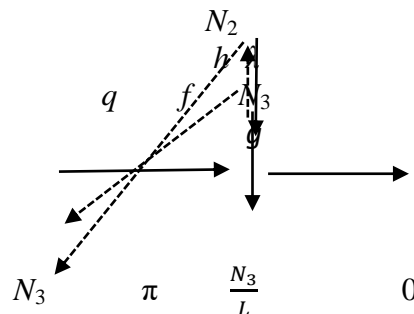
Proposition 3.1. If $0 \rightarrow N_1 \xrightarrow{\phi} N_2 \xrightarrow{\lambda} N_3 \rightarrow 0$ be a short exact sequence, such that $N_2 \oplus N_3$ is π -projective, then the sequence splits and N_3 is quasi-projective.

Proof: See the next diagram:



Since $N_2 \oplus N_3$ is π -projective, then by [1, 4.3] N_3 is N_2 -projective, hence there exists a homomorphism $h: N_3 \rightarrow N_2$ such that $\lambda h = I$. Thus the sequence splits. In order to prove that N_3

is quasi-projective. Let L be a subsemimodule of N_3 , $g: N_3 \rightarrow \frac{N_3}{L}$ be any homomorphism and $\pi: N_3 \rightarrow \frac{N_3}{L}$ be the natural epimorphism. See the next diagram:

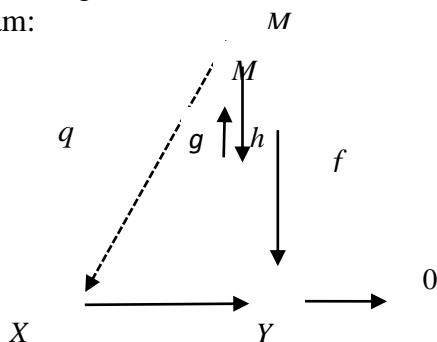


a gain by [1, 4.3] N_2 is N_3 -projective, Thus there exists a homomorphism $q: N_2 \rightarrow N_3$ such that $\pi q = g\lambda$ and $qh = f$. Now to prove that $\pi f = g$. $\pi f = \pi qh = g\lambda h = gI = g$. Then N_3 is quasi-injective semimodule.

Proposition 3.2. Let $M \rightarrow N \rightarrow 0$ be a short exact sequence such that M is projective semimodule. Then we have N is projective if and only if $M \oplus N$ is a π -projective semimodule.

Proof: (\Rightarrow) Since M and N are projective semimodules then $M \oplus N$ is a projective semimodule [1, 4.2], thus $M \oplus N$ by [1, 3.11] is π -projective semimodule.

(\Leftarrow) Assume that $M \oplus N$ is π -projective semimodule, by [1, 4.3] M is N -projective, consider the following diagram:



where X and Y are two semimodules, h is the epimorphism from:

M onto N and g is a right inverse of h (since M is projective, hence h splits) $\lambda: X \rightarrow Y$ any epimorphism and $f: N \rightarrow Y$ any homomorphism.

since M is projective, then there exists a homomorphism $q: M \rightarrow X$ with $\lambda q = fh$. Let $\phi = qg$, then $\lambda\phi = (\lambda q)g = fhg = f$, so $\lambda\phi = f$. Therefore N is projective. ■

Proposition 3.3. For a semiring A , the following conditions are equivalent:

- (i) The direct sum of any two π -projective A -semimodules is π -projective.
- (ii) Every π -projective A -semimodule is projective.

Proof: To prove $i \Rightarrow ii$. Assume that N is a π -projective semimodule, then there exists a short exact sequence $M \rightarrow N \rightarrow 0$, where M is a free semimodule, since N is a π -projective, by assumption then M and N are π -projective and so by (i), $M \oplus N$ is π -projective, then by Proposition (3.2) N is projective.

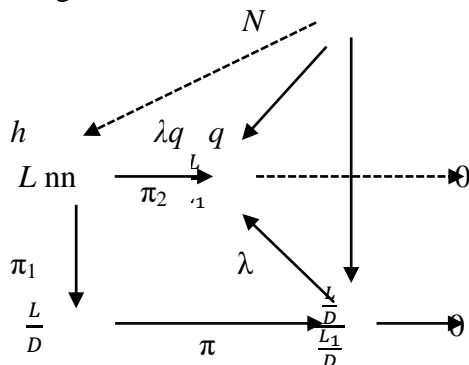
To prove $ii \Rightarrow i$. Suppose that N and L are π -projective semimodules, then N and L are projective by assumption. Hence $N \oplus L$ is a projective semimodule [1,4.2], then $M \oplus L$ is a π -projective [1, 3.11]. ■

The following result for semimodules is converted from a module version in [14], but its proof for modules was given in [13, p.49].

Proposition 3.4. Let $N \oplus L$ be a direct summand of a π -projective semimodule, and \bar{L} a non-zero factor semimodule of the semimodule L , then:

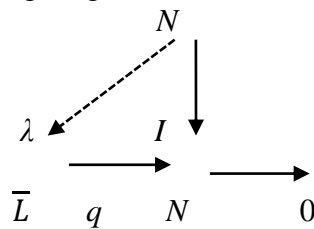
- 1- $N \oplus L$ is a π -projective semimodule and N is a \bar{L} -projective semimodule.
- 2- If there exists an epimorphism $q: \bar{L} \rightarrow N$, then it splits and N is a quasi-projective semimodule isomorphic to a direct summand of \bar{L} .
- 3- If \bar{L} is indecomposable and $q: \bar{L} \rightarrow N$ an epimorphism, then q is an isomorphism.

Proof: 1- $N \oplus L$ is a direct summand of a π -projective semimodule, then by [11, 4.9] $N \oplus L$ is π -projective semimodule and by [1, 4.3] N is L -projective. Let $\bar{L} = \frac{L}{D}$, where D is a subsemimodule of L . Now consider the next diagram:

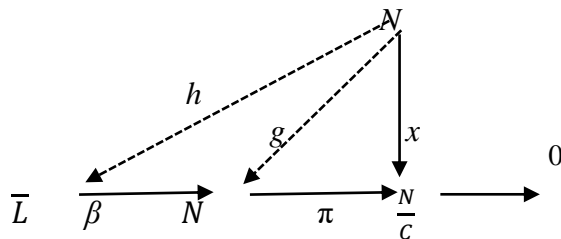


$q: N \rightarrow \frac{L}{D}$ be any homomorphism with $\frac{L_1}{D}$ is subsemimodule of $\frac{L}{D}$, π, π_1, π_2 are the natural epimorphisms. Let $\lambda: \frac{L}{L_1} \rightarrow \frac{L}{L_1}$ an isomorphism (the natural way). In fact, as $\lambda(l+ D) + \frac{L_1}{D} = l + L_1$, that is $\lambda(\pi(\pi_1(l))) = \pi_2(l)$. N is L -projective and there exists a homomorphism $h: N \rightarrow L$ such that $\pi_2 h = \lambda q$. Claim that $\pi_1 h: N \rightarrow \frac{L}{L_1}$ is the required homomorphism. We note that $\lambda \pi \pi_1 = \pi_2$, thus $\lambda \pi \pi_1 h = \pi_2 h = \lambda q$, and since λ is an isomorphism, then $\pi \pi_1 h = q$.

2- Suppose that $q: \bar{L} \rightarrow N$ is an epimorphism , consider the following diagram:



Where I is the identity homomorphism. By(1) N is \bar{L} -projective, hence there exists a homomorphism $\lambda: N \rightarrow \bar{L}$ such that $q\lambda = I$ and that means q has a right inverse, that is, q splits. So, $\bar{L} = \ker q \oplus \bar{L}_1$, where \bar{L}_1 is a suitable subsemimodule of \bar{L} . It is clear that $N \cong \bar{L}/\ker q$ and $\bar{L}_1 \cong \bar{L}/\ker q$, hence $N \cong \bar{L}_1$. To show that N is quasi-projective; let C be a subsemimodule of N and let $\chi: N \rightarrow \frac{N}{C}$ be any homomorphism. We consider the next diagram:



where π is natural epimorphism. Surly that $\pi\beta$ is an epimorphism (where β is an isomorphism). By (1) N is \bar{L} -projective, which implies that there exists a homomorphism $h: N \rightarrow \bar{L}$ such that $\pi\beta h = \chi$. Let $g = \beta h$, then $\pi g = \pi\beta h = \chi$. Thus N is a quasi-projective semimodule.

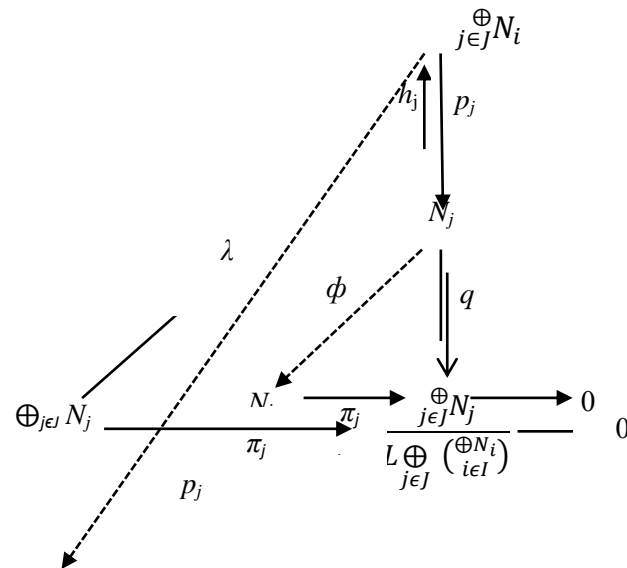
3-Assume that there exists an epimorphism $q: \bar{L} \rightarrow N$, where \bar{L} is indecomposable, by (2) $\ker q$ is a direct summand of \bar{L} , but \bar{L} is indecomposable, which implies that either $\ker q = 0$ or $\ker q = \bar{L}$. If $\ker q = \bar{L}$, then $q = 0$, this means $N = 0$, and this a contradiction, hence $\ker q = 0$, it implies that q is monomorphism, then it is an isomorphism. ■

The condition under which a π -projective module is quasi-projective was given in [14], and the proof was given in [13, p.74-75-76].. Here, this condition will be given for semimodule.

Proposition 3.5. If a semimodule N is a direct sum of a finitely generated semimodules $\{N_j\}_{j \in J}$, then the following cases are equivalent:

- 1- N is a quasi-projective semimodule.
- 2- N is a π -projective semimodule and every N_j is quasi-projective.
- 3- N_j is N_i -projective for every subscripts i and j in J .

Proof: To prove 1 \Rightarrow 2. We have N is quasi-projective, then by [1, 3.10] N is π -projective. See the following diagram



Now for proving 2 \Rightarrow 3. Suppose that $j, k \in J$ and $k \neq j$ (since N_j is quasi-projective, then N_j is N_j -projective for all $j \in J$), but N is a π -projective semimodule so by Proposition(3.4)(1) N_j is

$\frac{\bigoplus_{i \in J, i \neq j} N_i}{\bigoplus_{i \in J, i \neq k} N_i}$ -projective, where $\frac{\bigoplus_{i \in J, i \neq j} N_i}{\bigoplus_{i \in J, i \neq k} N_i} \simeq N_k$. Hence N_j is N_k -projective for all $j \neq k$. Therefore N_j is N_i -projective for all j and $i \in J$.

Finally to prove 3 \Rightarrow 1, we have N_j is N_i -projective for all j and i in J , and since N is finitely generated, then by [1] N_j is $\bigoplus_{j \in J} N_j$ -projective for every $j \in J$, then also by [1] $\bigoplus_{j \in J} N_j$ is $\bigoplus_{j \in J} N_j$ -projective. Hence N is quasi-projective semimodule. ■

The next proposition also gives conditions under which π -projective semimodules are projective. We will compare it with an analogues one in [15, 2.9] for modules. Before this proposition we need to define two concepts hereditary semimodule, and finite dimensional. They are analogous to the concepts in modules for details see [15,16].

Definition 3.6. An A -semimodule N is said to be hereditary if all subsemimodules of it are proujective.

Example: Z_6 is a hereditary semimodule, since all subsemimodules of it are projective.

Definition 3.7. An A -semimodule N is called finite dimensional if N contains no infinite independent families of non-zero subsemimodules. Recall that, a family of subsemimodules is independent if its sum is a direct sum.

Example: Z_6 is a finite dimensional semimodule.

Proposition 3. 8. Let N be a π -projective semimodule such that every cyclic subsemimodule of N is hereditary, N is a finite dimensional and a non-zero direct summand of N has a maximal subsemimodule, then N is projective.

Proof: Let N be a π -projective semimodule such that every cyclic subsemimodule of N is hereditary. By assumption N has a maximal subsemimodule B . let $d \in N$ and $b \notin d$, then the cyclic subsemimodule $\langle d \rangle \neq B$. Since B is maximal, then $\langle d \rangle + B = N$. Since N is π -projective, hence there exist a homomorphisms ψ and $\lambda \in \text{End}(N)$ such that $\psi(N) \leq \langle b \rangle$, $\lambda(N) \leq B$ and $\psi + \lambda = 1_N$. $\psi(N)$ is a subsemimodule of the cyclic subsemimodule $\langle d \rangle$, thus $\psi(N)$ is a projective semimodule and the short exact sequence $N \rightarrow \psi(N) \rightarrow 0$ splits, then $N = C_1 \oplus H_1$, where $C_1 \simeq \psi(N)$, hence C_1 is a projective semimodule. Now let $C_1 \neq 0$, since $N = \psi(N) + \lambda(N) = \psi(N) + B \neq B$, therefore $\psi(N) \neq 0$, consequently $C_1 \neq 0$. If $H_1 = 0$, then $N = C_1$, so that N is projective. Assume that $H_1 \neq 0$, thus hypothesis H_1 has a maximal subsemimodule and H_1 is π -projective by [1]. Similarly, if we take $H_1 = C_2 \oplus H_2$, where H_2 is a subsemimodule of H_1 , then $N = C_1 \oplus C_2 \oplus H_2$, and if $H_2 = 0$, then C_2 is projective, and so on. But N is finite dimensional, then there exists $b \in B$ such that $H_n = 0$ and $N = C_1 \oplus C_2 \oplus C_3 \oplus \dots \oplus C_{n-1}$, and for $i = 1, 2, \dots, n-1$, C_i is projective. Therefore N is projective.

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