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On Invariant Approximations in Modular Spaces

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Abstract
This article is devoted to presenting results on invariant approximations over a non-star-shsped weakly compact subset of a complete modular space by introduced a new notion called S-star-shaped with center *f*: if $S: B \to B$ be a mapping and $\forall e \in B$, $\lambda Se + (1-\lambda)(f - Sf) \in B,\ f \in B$. Then the existence of common invariant best approximation is proved for Banach operator pair of mappings by combined the hypotheses with Opial's condition or demi-closeness condition

Keywords: Banach operator pair, best approximation, fixed point, modular space, S-non-expansive mappings.

حول التقريب الثابت في فضاءات الوحدات

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الخلاصة
كُرست هذه المقالة لتقديم نتائج حول التقريب الثابت تحت شرط مجموعة جزئية متراصة بضعف في فضاء الوحدات الكامل من خلال تقديم مفهوم جديد يسمى S-star-shaped مع المركز *f* : إذا كان $S: B \to B$تطبيق و
$\forall e \in B$, $\lambda Se + (1-\lambda)(f - Sf) \in B,\ f \in B$. ثم تم إثبات وجود أفضل تقريب ثابت مشترك لتطبيقين من نوع زوج مؤثر بناخ وذلك من خلال دمج الفرضيات مع شرط أوبل أو شرط شبه القرب.

Introduction and Preliminaries

Recent paper contains applications of fixed point of non-expansive mappings in a modular space which is known as the following:

Definition 1.1 [1] Let B be real linear space over R, a function $\gamma: \text{B} \to (0, \infty)$ is called modular if for $e, h \in \text{B}$ and $\alpha, \beta \in R$:

(i) $\gamma(e) = 0$ if and only if $e = 0$,

(ii) $\gamma(\alpha e) = \gamma(e)$ with $|\alpha| = 1$,

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(iii) $\gamma(\alpha e+\beta h)\leq\gamma(e)+\gamma(h)$ if and only if $\alpha,\beta\geq 0$.

If (iii) replaced by (iii)` $\gamma(\alpha e+\beta h)\leq\alpha\gamma(e)+\beta\gamma(h)$, for $\alpha,\beta\geq 0$, $\alpha+\beta=1$, for all *e, h* $\in$ Ƀ, then γ is called convex modular.

Definition 1.2 [1] If γ is a convex modular on Ƀ. A corresponding set $Ƀ_\gamma=\{e\in Ƀ:\gamma(\alpha e)\to 0 \text{ whenever } \alpha\to 0\}$ is called convex modular space.

The distance between two vectors e,h is $d_\gamma(e,h)=\gamma(e-h)$. The collection of all balls in a modular space $Ƀ_\gamma$ generates a locally convex Hausdorff topological linear space [2]. Many works in this space can be found in [3-6]. Recently, Abed and Abdul Jabbar [7-10] introduced the concept of normalized duality mappings in the real convex modular spaces proved some of its properties which related to uniformly smooth of these spaces. Also, they presented some results concerning the convergence and equivalence iterative sequences for multivalued strongly pseudo-contractions mappings. Now, we recall the concepts required for our results. Let Ƀ be a convex real modular space with dual $Ƀ'_\gamma$ (for details of dual $Ƀ'_\gamma$, see [8])

Definition 1.3 [8]

(1) A sequence $\{e_n\}\subset Ƀ_\gamma$ is said to be

(a) γ-convergent to $e\in Ƀ_\gamma$ and write $e_n\overset{\gamma}{\to}e$ if $\gamma(e_n-e)\to 0$ as n$\to\infty$.

(b) γ- Cauchy whenever $\gamma(e_n-e_m)\to 0$ as $n,m\to\infty$.

(2) If any γ- Cauchy sequence in $Ƀ_\gamma$ is γ- convergent then $Ƀ_\gamma$ is called γ- complete.

(3) If for any sequence $\{e_n\}\subset B\subset Ƀ_\gamma$ is γ- convergent to a point in B then B is called γ- closed. And, if any sequence $\{e_n\}\subset B\subset Ƀ_\gamma$ has a γ- convergent subsequence then B is called γ- compact.

Definition (1.4) [7] A sequence $\{e_n\}$ in $Ƀ_\gamma$ is said to be weakly convergent if there is an $e\in Ƀ_\gamma$ such that for every $P\in Ƀ'_\gamma$, $\lim_{n\to\infty}\gamma(Pe_n-Pe)=0$ This denoted by $e_n\overset{w}{\to}e$.

Definition (1.5) [3] Let B be a subset of a convex real modular space Ƀ. For $e_\circ\in Ƀ$, $BA(e_\circ)$ denotes the set of best B-approximation to $x_\circ$,

$BA(e_\circ)=\{h\in B:\gamma(h-e_\circ)=d(e_\circ,B)\}$

where $d(e_\circ,B)=\inf_{z\in B}\gamma(z-e_\circ)$

Definition (1.6) [3] Let $S,T:B\to B$, then $F(T)$ denotes the set of all fixed points of T and $F(S,T)$ is the set of all common fixed points of S and T.

Definition (1.7) [9] A mapping S is called T- γ -contraction if for all $e,f\in B,\gamma(Se-Sf)\leq\alpha\gamma(Te-Tf)$,

$\alpha\in(0,1)$. And it is called T-γ-non-expansive if for all $e,f\in B$, $\gamma(Se-Sf)\leq\gamma(Te-Tf)$.

Definition (1.8) [7] A modular space Ƀ satisfies Opial's condition if for every sequence $\{e_n\}\subseteq Ƀ$ weakly convergent to $e\in Ƀ$, $\lim_{n\to\infty}\inf\gamma(e_n-e)<\lim_{n\to\infty}\inf\gamma(e_n-h)$ holds, for all $h\neq e$.

Definition (1.9) [7] Let $B\subseteq Ƀ$, a mapping $T:B\to Ƀ$ is called demi-closed on B. If for every sequence $\{e_n\}\subseteq B$, $\{e_n\}$ converges weakly to $e\in B$ and $\{Te_n\}$ converges strongly to $h\in Ƀ$, implies $h=Te$.

Definition (1.10) [2] Let $B\subseteq Ƀ$, B is called

i) Convex if $\lambda e+(1-\lambda)f\in B,\forall e,f\in B,\forall\lambda\in[0,1]$

ii) Star-shaped with center $f\in B$ if $\lambda e+(1-\lambda)f\in B,\forall e\in B$.

Main Results

In the following, a simple modification of star-shaped:

Definition (2.1) Let $\emptyset \neq B \subset Ƀ$ and $S: B \to B$ be a mapping, B is called S-star-shaped with center f if $\forall e \in B$, $\lambda Se + (1-\lambda)(f - Sf) \in B$.

Theorem (2.2) Let $Ƀ$ be a complete space $e_\circ \in Ƀ$ and $S: Ƀ \to Ƀ \ni S(\partial B) \subseteq B$

If B is w-compact and S-star-shaped such that

i) S is γ- non-expansive on $BA(e_\circ)$.

ii) $\gamma((Se - Se_\circ) \leq \gamma(e - e_\circ), \forall e \in BA(e_\circ)$.

iii) $I - S$ is γ -demiclosed on $BA(e_\circ)$.

Then S has a fixed point closed set to $e_\circ$.

Proof: Firstly, we prove S is self-mapping on $BA(e_\circ)$. Let $e \in BA(e_\circ)$, then

$\gamma(Se - e_\circ) \leq \gamma(Se - Se_\circ) \leq \gamma(e - e_\circ)$.

This implies that Se is also closed to $e_\circ$ so, $Se \in BA(e_\circ)$.

Let f be star center of $BA(e_\circ)$.

Define $S_{m_i}e = m_i Se + (1 - mi)(f + Sf)$

Where $\{m_i\}$ real sequence, $0 < mi < 1$ and $\lim_{i\to\infty} m_i = 1, e \in BA(e_\circ)$.

To prove S_n is γ -contraction, follow:

let $e, g \in BA(e_\circ), \gamma(S_n e - S_n g) = \gamma(m_i S_n e - m_i S_n g)$

$$= m_i \gamma(S_n e - S_n g)$$
$$\leq m_i \gamma(e - g)$$

since $m_i < 1$ then S_n is γ-contraction.

By argument in [11] $\forall n, S_n$ has a fixed point e_n is $BA(e_\circ)$, $i.e, S_n e_n = e_n, \forall n$

Since B is w-compact, then the sequence of fixed point $\{e_n\}$ has a convergent subsequence $\{e_{n_i}\}$ w-convergent to $\bar{e}$, say. Now,

$$\gamma\left(e_{n_i} - Se_{n_i}\right) = \gamma\left(S_n e_{n_i} - Se_{n_i}\right)$$
$$= \gamma(m_{n_i} Se_{n_i} + (1 - m_{n_i})(f + Sf) - Se_{n_i}$$
$$= \left(m_{n_i} - 1\right)Se_{n_i} + \left(1 - m_{n_i}\right)(f + Sf)$$
$$\to 0 \quad \text{as} \quad i \to \infty.$$

Since $I - S$ is demi-closed, then, $0 = (I - S)\bar{e}$ and $\bar{e} = S\bar{e}$, thus S has a fixed point $\bar{e}$ closest to $e_\circ$.

Now, we recall the definition of Banach operator pairs, for a mapping $T: Ƀ \to Ƀ$, $\quad F(T) := \{e \in Ƀ : Te = e\}$

Definition (2.3) [4] The ordered pair (S, T) of two self-mappings S and T of $Ƀ$ is called a Banach operator if $S(F(T) \subseteq F(T)$i.e. the set $F(S)$ is T-invariant.

Remark (2.4) If S and T are commute on $Ƀ$ then (S, T) must be a Banach operator. But the converse is not true. For example:

Example (2.5) Consider $Ƀ = R$ and

$S(e,f) = (e^2 + f^2 + e - 1\,,\ e^2 + f^2 + t - 1)$

$T(e,f) = ((e - f)^2 + 2e - f\,, (\ e - f)^2 + e)$

For any $(e, f) \in Ƀ$. Then

$F(S) = \{(e,f) \in Ƀ: e^2 + f^2 - 1 = 0\}$

$F(T) == \{(e,f) \in Ƀ: s - t = 0\ or\ s - t + 1 = 0\}$

So, (i) $S(F(T)) \subseteq F(T) \to (S,T)$ is Banach operator pair on $Ƀ$.

(ii) (T, S) is not Banach operator pair, since for $(1,0) \in F(S)\,, T(1,0) = (3,2) \notin F(S)$.

Remark (2. 6) [4] In general metric spaces the pair (S, T) is Banach operator $\Leftrightarrow S$ and T commute on $F(T)$. In the above Example (2.5-i) T, S are commute on the set $F(T)$, so, (S, T) is Banach operator by Remark (2.4).

To prove a common fixed point theorem, we need the following Lemma in general metric spaces.

Lemma (2. 7) [12] Let B be a closed subset of a metric space $(\mathbb{B}, d)$ and $S, T: B \to B$. If (S,T) is a Banach operator pair and S is T- γ -contraction on B , T is continuous, $F(T) \neq \emptyset$ $\overline{S(B)}$ is complete. Then $F(S,T) =$singleton.

Theorem (2.8) Let B be a w-compact subset of convex real modular space $\mathbb{B}$, B is S-star-shaped respect to $f \in B$. Let (S,T) be a Banach operator on B $f \in F(T)$. If T is both weakly continuous and strongly continuous on B, $F(T)$ is star-shaped with f, $\overline{S(B)}$ is complete and if either

(i) $\mathbb{B}$ satisfies Opial's condition.

or

(ii) $I - S$ is demiclosed on B.

Then $F(S,T) \neq \emptyset$

Proof: Let $\{m_i\}$ be real sequence $\ni 0 < m_i < 1$, $m_i \to 1$ as $i \to \infty$. Define

$S_i(e) = m_i Se + (1 - m_i)(f + Sf), \forall e \in B$

Since B is S-star-shaped respect to $f \in B, e \in B$ then

- $\forall i, S_i: B \to B$, since B is S-star-shaped <u>with</u> f.
- Since S is T-non-expasine on B then $\forall n, \forall e, f \in B$

$$\gamma(S_i e - S_i f) = \gamma\big(m_i S(e) - m_i S(f)\big) \\ \leq m_i \gamma(T(e) - T(f))$$

$\to$ each S_i is g- γ -contraction on B.

- For each $i, (S_i, T)$ is a Banach operator pair on B, since (S,T) is a Banach operator pair on B, since (S,T) is a Banach operator, for $e \in F(T) \to S(e) \in F(T)$.

$\to S_i(e) = m_i Se + (1 - m_i)(f + Sf) \in F(T)$

By the fact $F(T)$ is S-star-shaped with $f \in F(T)$

- The completeness of $\overline{S(B)} \to$ completeness of $\bar{S_1}(B)$. And w-compactness of $B \to B$ is closed.

Next, by Lemma (2. 7), $\forall i, \exists e_i \in B \ni e_i \in F(S_i, T)$

Since w-compactness of B and w-continuity of T implies w-compactness of $F(T) \subseteq B, \exists \{e_{i_k}\}$ subsequence converges weakly to $e_\circ \in F(T)$.

Now, we must prove $e_\circ \in F(S,T)$.

As known w-compactness weak bounded and thus bounded$\to \{Se_{i_k}\} \subseteq B$ is bounded.

Since

$$\gamma\big((S - T)e_{i_k}\big) = \gamma\Big((m_i - 1)Se_{n_i} + (1 - m_i)(f + Sf)\Big) \\ = \gamma\Big((1 - m_i)(f + Sf - Se_{n_i})\Big)$$

$$\gamma\big((T - S)e_{n_i}\big) \leq (m_i - 1)\Big(\gamma(f + Sf) + \gamma\big(Se_{n_i}\big)\Big) \quad \text{.....(**)} \\ \to 0 \;\; as \;\; i \to \infty$$

Now, if the hypothesis (i) holds and $Te_\circ \neq Se_\circ$.

Form (**) and T-contraction and T-non-expansive, we get

$$\lim_{k\to\infty} \inf \gamma\left(Te_{i_k} - Te_\circ\right) < \lim_{k\to\infty} \inf \gamma\left(Te_{i_k} - Se_\circ\right) \\ \leq \lim_{k\to\infty} \inf \gamma\Big((T - S)e_{i_k}\Big) + \lim_{k\to\infty} \inf\Big(Se_{n_{i_k}} - Se_\circ\Big) \\ \leq \lim_{k\to\infty} \inf\big(Te_{i_k} - Te_\circ\big)$$

which is a contraction$\cdot S_\circ, Te_\circ = Se_\circ$ and then $e_\circ \in F(S,T)$.

If the hypothesis (ii) holds, i.e, if $T - S$ is demiclosed on B. Then by (**) and w-convergence of $\{e_{i_k}\}$ to $e_\circ \to (T - S)e_\circ = 0$

i.e. $Te_{\circ} = Se_{\circ} \rightarrow e_{\circ} \in F(S,T)$.

As application of theorem (2.8) to best approximation, we present the following:

Theorem (2.9) Assume that $\mathbb{B}$ satisfies Opial's condition, $S,T: \mathbb{B} \rightarrow \mathbb{B}$ and $B \subseteq \mathbb{B}$ with $S(\partial B \cap B) \subseteq B$ and $\overline{S(B)}$ complete. Let $e_{\circ} \in F(S,T) \ni \emptyset = BA(e_{\circ})$ is w-compact and S-star-shaped w.r.t. $f \in F(T)$. If (S,T) is Banach operator pair on $BA(e_{\circ})$, S is T-non-expansive on $BA(e_{\circ}) \cup \{e_{\circ}\}$, and if T is both weakly and strongly continuous on $BA(e_{\circ})$. $F(T)$ is $S-$star-shaped w.r.t.f and $T(BA(e_{\circ})) \subseteq BA(e_{\circ})$ then $BA(e_{\circ}) \cap F(S,T) \neq \emptyset$.

Proof:

If $e_{\circ} \in B$ then $e_{\circ} \in BA(e_{\circ}) \cap F(S,T)$ and the result holds.

Now, if $e_{\circ} \notin B \rightarrow BA(e_{\circ}) \subseteq \partial B \cap B \rightarrow S: BA(e_{\circ}) \rightarrow C$ .Since $S(e_{\circ}) = T(e_{\circ}) = e_{\circ}$ and S is T-non-expasive. On $BA(e_{\circ}) \cup \{e_{\circ}\} \rightarrow \forall h \in BA(e_{\circ})$,

$\gamma(Sh - e_{\circ}) = \gamma(Sh - Se_{\circ}) \leq \gamma(Th - Te_{\circ}) = d(e_{\circ}, B)$.

Since $T(BA(e_{\circ})) \subseteq BA(e_{\circ})$ implies $Sh \in BA(e_{\circ})$. Hence, $S: BA(e_{\circ}) \rightarrow BA(e_{\circ})$.

The completeness of $\overline{S(B)} \rightarrow$completeness of $\overline{S(BA(e_{\circ}))}$ and $BA(e_{\circ}) \cap F(T)$ is *S*-star-shaped w.r.t.f. By applying Theorem (2.8) to S and T on $BA(e_{\circ})$ we get $BA(e_{\circ}) \cap F(S,T) \neq \emptyset$

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