Jordan Generalized \((\mu, \rho)\)-Reverse Derivation from \(\Gamma\)-Semirings Sinto \(\Gamma S\)-Modules

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Abstract
In this study, we introduce and study the concepts of generalized \((\mu, \rho)\)-reverse derivation, Jordan generalized \((\mu, \rho)\)-reverse derivation, and Jordan generalized triple \((\mu, \rho)\)-reverse derivation from \(\Gamma\)-semiring \(S\) into \(\Gamma S\)-module \(X\). The most important findings of this paper are as follows:
If \(S\) is \(\Gamma\)-semiring and \(X\) is \(\Gamma S\)-module, then every Jordan generalized \((\mu, \rho)\)- reverse derivation from \(S\) into \(X\) associated with Jordan \((\mu, \rho)\)-reverse derivation \(d\) from \(S\) into \(X\) is \((\mu, \rho)\)-reverse derivation from \(S\) into \(X\).

Keywords: generalized \((\mu, \rho)\)-reverse derivation, Jordan generalized \((\mu, \rho)\)-reverse derivation, Jordan generalized triple \((\mu, \rho)\)-reverse derivation, \(\Gamma\)-semiring

1. Introduction
The concept of generalized \((\mu, \rho)\)- reverse derivation from \(\Gamma\)-semiring \(S\) into \(\Gamma S\)-module \(X\) is one of most important topics in non-commutative algebra. The definition of \(\Gamma\)-ring was presented by Nobusawa in 1964 [1] and generalized by Barnes in 1966 [2]. Sen and Saha in 1986 [3] and Saha [4] in 1989 presented the concept of \(\Gamma\)-semiring as a generalization of \(\Gamma\)-ring. The definition of \(\Gamma\)-semiring was introduced in [3].

The definition of prime \(\Gamma\)-semiring and semi-prime \(\Gamma\)-semiring was introduced in [5]. The definition of 2-torsion free \(\Gamma\)-semiring was introduced in [6]. These definitions and identity properties of multiplication of inverse elements, were introduced in [6]. The definitions of additive, identity, inverse abelian elements were introduce in [5].
Let $S$ be a $\Gamma$-semiring and $X$ be an additive abelian group, $X$ is called a left $\Gamma S$-module if there exists a mapping $S \times \Gamma \times X \rightarrow X$ (sending $(a, a, x)$ into $a \alpha x$ where $a \in S$, $a \in \Gamma$, and $x \in X$) satisfying the following:

for all $a, a_1, a_2 \in S$, $a, b \in \Gamma$ and $x, x_1, x_2 \in X$:

i) $(a_1 + a_2) \alpha x = a_1 \alpha x + a_2 \alpha x$

ii) $a(\alpha + \beta) x = a\alpha x + a\beta x$

iii) $a(a_1 + a_2) \alpha x = a_1 a \alpha x_1 + a_2 a \alpha x_2$

iv) $(a_1 a_2) \alpha x = a_1 (a_2 \alpha x)$

$X$ is called a right $\Gamma S$-module if there exists a mapping $X \times \Gamma \times S \rightarrow X$. Also, $X$ is called $\Gamma S$-module if its both left and right $\Gamma S$-module. The definitions of left (respectively right) prime, semi prime, and 2-torsion free $\Gamma S$-modules were introduce in [7].

Paul and Halder [7] defined the left derivation and Jordan left derivation of $\Gamma$-ring $M$ onto $\Gamma M$-module $X$ and studied the relations between them. Salih [8] defined the derivation and Jordan derivation from $\Gamma$-ring $M$ into $\Gamma M$-module, along with their generalization [9]. Mahmood, Nayef, and Salih [10] presented the concepts of generalized higher derivations and Jordan generalized higher derivations on $\Gamma M$-modules and studied the relations between them. For more information see [11,12].

In this paper we present the concepts of $(\mu, \rho)$-reverse derivation, Jordan $(\mu, \rho)$-reverse derivation, and Jordan triple $(\mu, \rho)$-reverse derivation from $\Gamma$-semiring $S$ into $\Gamma S$-module $X$. We prove that every Jordan $(\mu, \rho)$-reverse derivation from $\Gamma$-semiring $S$, with additive inverse identity into $\Gamma S$-module $X$ where $\sigma, \tau$ are automorphisms on $S$, is $(\mu, \rho)$-reverse derivation from $S$ into $X$. In addition, we introduce the concepts of generalized $(\mu, \rho)$-reverse derivation, Jordan generalized $(\mu, \rho)$-reverse derivation, and Jordan generalized triple $(\mu, \rho)$-reverse derivation from $\Gamma$-semiring $S$ into $\Gamma S$-module $X$ and we study the relations among them. In this paper, $\mu$ and $\rho$ are automorphisms on $S$.

2. $(\mu, \rho)$-Reverse Derivations from $\Gamma$-Semirings into $\Gamma S$-Modules

Definitions 2.1: Let $S$ be $\Gamma$-semiring and $X$ be $\Gamma$-module. An additive map $\delta$ from $S$ into $X$ is called $(\mu, \rho)$-reverse derivation if an only if, for all $a, b \in S$, $a \in \Gamma$

$$
\delta(a b) = \delta(b) \mu(a) + \rho(b) \delta(a)
$$

$\delta$ is called Jordan $(\mu, \rho)$-reverse derivation from $S$ into $X$ if, for all $a, b \in S$, $a \in \Gamma$:

$$
\delta(a a) = \delta(a) \mu(a) + \rho(a) \delta(a)
$$

$\delta$ is called Jordan triple $(\mu, \rho)$-reverse derivation from $S$ into $X$ if, for all $a, b \in S$, $a, b \in \Gamma$:

$$
\delta(a b \beta a) = \delta(a) \beta \mu(a) \mu(a) + \mu(a) \delta(b) \mu(a) + \rho(a) \beta \delta(b) \delta(a) + \delta(b) \rho(a) \delta(a).
$$

Example 2.2: Let $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in Z \right\}$, $\Gamma = \left\{ \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} : n \in Z \right\}$. $X$ is a $\Gamma$-semiring, and $Y = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c \in R \right\}$, then $X$ is $\Gamma S$-module.

We define $\mu : S \rightarrow S$ and $\rho : S \rightarrow S$ by $\mu(a, b) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $\rho(a, b) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then $\mu$ and $\rho$ are automorphisms.

Now, we define an additive mapping $\delta : S \rightarrow X$ by $\delta(a, b) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

Then, $\delta$ is $(\mu, \rho)$-reverse derivation on $S$ into $X$.

Example 2.3: Let $R$ be a ring, $\mathcal{A} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in R \right\}$, $\mathcal{B} = \left\{ \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix} : n, m \in Z \right\}$, and $\mathcal{Y} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c \in B \right\}$, then $A$ is $\Gamma$-semiring, $Y$ is $\Gamma S$-module, and $d$ is $(\mu, \rho)$-reverse derivation of $A$ into $Y$. Let $S = A \times A$, $\Gamma = \Gamma \times \Gamma$ and $X = Y \times Y$, then $S$ is $\Gamma S$-semiring and $X$ is $\Gamma S$-module. We use the usual addition and multiplication on matrices of $S$ and $X$. We define $B$ as additive mappings from $S$ into $X$, such that $\Delta(B, C) = (\delta(B), \delta(C))$ for all $B, C \in Y$. Then, $\Delta$ is $(\mu, \rho)$-reverse derivation from $S$ into $X$.

Lemma 2.4: Let $\delta$ be Jordan $(\mu, \rho)$-reverse derivation from $\Gamma$-semiring $S$ into $\Gamma S$-module $X$, then for all $a, b \in S$:

$$
\delta(a b + b a) = \delta(b) \mu(a) + \rho(a) \delta(b) + \delta(a) \mu(b) + \rho(a) \delta(b).
$$

Proof: $\delta(a b + b a) = \delta(a b) \mu(a) + \rho(a b) \delta(b)$.

$$
= (\delta(a) \mu(a) + \rho(a) \delta(b)) + \rho(a) (b) \mu(a) + \delta(a) \mu(b) + \rho(a) \delta(b)
$$

$$
= \delta(a) \mu(b) + \delta(a) \mu(b) + \delta(a) \mu(b) + \rho(a) \delta(b) + \rho(a) \delta(b) + \rho(a) \delta(b)
$$

...$(1)$
On the other hand,
\[ \delta((a+b)\alpha(a+b)) = \delta(\ a\alpha a+a\alpha b+b\alpha a+bab) = \delta(a\alpha a) + \delta(b\alpha b) + \delta(a\alpha b+b\alpha a) \]  
\[ = \delta(a)\alpha \mu(a) + \rho(a)\alpha \delta(a) + \delta(b)\alpha \mu(b) + \rho(b)\alpha \delta(b) + \delta(a\alpha b+b\alpha a) \quad \text{...(2)} \]

By comparing (1) and (2), we have
\[ \delta(a\alpha b+b\alpha a) = \delta(b)\alpha \mu(a) + \rho(b)\alpha \delta(a) + \delta(a)\alpha \mu(b) + \rho(a)\alpha \delta(b). \]

**Definition 2.5:** If \( \delta \) is Jordan \((\mu,\rho)\)-reverse derivation from additive inverse \( \Gamma \)-semiring \( S \) into \( \Gamma \)-semimodule \( X \), then we define \( \psi \) by

\[ \psi(a,b)_{\alpha} = \delta(a\alpha b)-\delta(b)\mu(a)-\rho(b)\alpha \delta(a), \quad \text{for all } a,b \in S, \alpha \in \Gamma. \]

**Example 2.6:**
Let \( S \) be a \( \Gamma \)-semiring, \( X \) be a \( \Gamma S \)-module, and \( a \in S \), such that \( a\Gamma a=0 \) and \( x\Gamma a\Gamma x=0 \), for all \( x \in S \). But \( x\Gamma a\Gamma y=0 \), for some \( x,y \in S \), such that \( x\neq y \). Also, let \( \delta \) be a mapping of \( S \) into \( X \) defined by:
\[ \delta(x)=x\alpha a+a\alpha x, \quad \text{for all } x \in S, \alpha \in \Gamma. \]

It is clear that \( \Gamma \) is a Jordan \((\mu,\rho)\)-reverse derivation on \( S \) into \( X \), which satisfies \( \psi \).

In the following lemma, we present the properties of \( \psi(a,b)_{\alpha} \).

**Lemma 2.7:** If \( S \) is additive inverse \( \Gamma \)-semiring, \( X \) is \( \Gamma S \)-module, and \( \delta \) is Jordan \((\mu,\rho)\)-reverse derivation from \( S \) into \( X \), then for all \( a,b,c \in S, \alpha \in \Gamma \):

i) \( \psi(a,b)_{\alpha} = -\psi(b,a)_{\alpha} \)

ii) \( \psi(a+c,b)_{\alpha} = \psi(a,b)_{\alpha} + \psi(c,b)_{\alpha} \)

iii) \( \psi(a,b+c)_{\alpha} = \psi(a,b)_{\alpha} + \psi(a,c)_{\alpha} \)

**Proof:**

i) \[ \delta(a\alpha b+b\alpha a) = \delta(b)\alpha \mu(a) + \rho(b)\alpha \delta(a) + \delta(a)\alpha \mu(b) + \rho(a)\alpha \delta(b) \]
\[ = \delta(a\alpha b) + \delta(b\alpha a) = \delta(b)b\alpha \mu(a) + \rho(b)\alpha \delta(a) + \delta(a)b\alpha \mu(b) + \rho(a)\alpha \delta(b) \]
\[ = \psi(a,b)_{\alpha} - \psi(b,a)_{\alpha} \]

ii) \[ \psi(a+c,b)_{\alpha} = \psi((a+c)b)_{\alpha} = \delta((a+c)\alpha b) - \delta(b)\alpha \mu(a+c) - \rho(b)\alpha \delta(a+c) \]
\[ = \delta(a\alpha b) + \delta(c\alpha b) - \delta(b)\alpha \mu(a) + \rho(b)\alpha \delta(a) - \delta(b)\alpha \mu(c) + \rho(b)\alpha \delta(c) \]
\[ = \psi(a,b)_{\alpha} + \psi(c,b)_{\alpha} \]

iii) \[ \psi(a,b+c)_{\alpha} = \delta((a+b)\alpha c) + \delta(b)\mu(a+c) - \rho(b)\alpha \delta(a+c) \]
\[ = \delta(a\alpha b + b\alpha c) - \delta(b)\alpha \mu(a) + \rho(b)\alpha \delta(a) - \delta(a)\alpha \mu(b) + \rho(a)\alpha \delta(b) \]
\[ = \psi(a,b)_{\alpha} + \psi(a,c)_{\alpha} \]

**Lemma 2.8:** If \( S \) is \( \Gamma \)-semiring with additive invertible identity and \( X \) is \( \Gamma S \)-module, then \( \delta \) is Jordan \((\mu,\rho)\)-reverse derivation from \( S \) into \( X \) iff \( \psi(a,b)_{\alpha}=0 \), for all \( a,b \in S, \alpha \in \Gamma \).

**Proof:** By Lemma 2.4, we get
\[ \delta(a\alpha b+b\alpha a) = \delta(b)\alpha \mu(a) + \rho(b)\alpha \delta(a) + \delta(a)\alpha \mu(b) + \rho(a)\alpha \delta(b) \quad \text{...(1)} \]

Now, we have the following:
Since \( \delta \) is additive mapping, then we have
\[ \delta(a\alpha b+b\alpha a) = \delta(a\alpha b) + \delta(b\alpha a) \quad \text{...(2)} \]

By comparing (1) and (2), we get
\[ \delta(a\alpha b) = \delta(b)\alpha \mu(a) + \rho(b)\alpha \delta(a) \]
\[ \psi(a,b)_{\alpha} = 0 \]

**Converse:** Obvious.

**Theorem 2.9:** Every Jordan \((\mu,\rho)\)-reverse derivation from \( \Gamma \)-semiring \( S \) with additive inverse identity into \( \Gamma S \)-module \( X \) is \((\mu,\rho)\)-reverse derivation from \( S \) into \( X \).

**Proof:** Let \( \delta \) be Jordan \((\mu,\rho)\)-reverse derivation from \( \Gamma \)-semiring \( S \) into \( \Gamma S \)-module.

Then, by Lemma 2.8, we get
\[ \psi(a,b)_{\alpha}=0 \]

Now, by Lemma 2.8, we get
δ is \((\mu, \rho)\)-reverse derivations from \(S\) into \(X\).

**Proposition 2.10:** Every Jordan \((\mu, \rho)\)-reverse derivation from \(\Gamma\)-semiring \(S\) into 2-torsion free \(\Gamma\)-module is Jordan triple \((\mu, \rho)\)-reverse derivation from \(S\) into \(M\).

**Proof:** Since \(\delta\) is Jordan \((\mu, \rho)\)-reverse derivation from \(S\) into \(X\), then, by replacing \(a\beta b + b\alpha a\) by \(b\) in lemma 2.4, we get

\[
\delta((a\alpha(a\beta b + b\beta a)) + (a\beta b + b\alpha a)\alpha a) = \delta((a\alpha b + aab\beta a + (a\beta b\alpha a + b\alpha a))\alpha a) + \delta(a\beta b + b\alpha a)\alpha a) + \delta(2a\beta b + a\alpha b + a\alpha b + b\alpha a)
\]

By comparing (1) and (2), we get

\[
\delta(a\alpha(a\beta b + b\beta a) + (a\beta b + b\alpha a)\alpha a) = \delta(2a\beta b + a\alpha b + a\alpha b + b\alpha a)
\]

Thus, \(\delta\) is Jordan triple \((\mu, \rho)\)-reverse derivation from \(S\) into \(X\).

### 3. Generalized \((\mu, \rho)\)-Reverse Derivations from \(\Gamma\)-Semirings \(S\) into \(\Gamma\)-module \(X\)

**Definitions 3.1:** Let \(S\) be \(\Gamma\)-semiring and \(X\) be \(\Gamma\)-module. Then, an additive map \(\xi\) from \(S\) into \(X\) is called generalized \((\mu, \rho)\)-reverse derivation from \(S\) into \(X\), associated with \((\mu, \rho)\)-reverse derivation \(\delta\) from \(S\) into \(X\), if and only if, for all \(a, b \in S, \alpha \in \Gamma\):

\[
\xi(a\alpha b) = \xi(b)\alpha \mu(a) + \rho(b)\alpha \delta(a)
\]

\(\xi\) is called Jordan generalized \((\mu, \rho)\)-reverse derivation from \(S\) into \(X\), associated with Jordan \((\mu, \rho)\)-reverse derivation \(\delta\) from \(S\) into \(X\), if for all \(a, b \in S, \alpha \in \Gamma\):

\[
\xi((a + b)\alpha(a + b)) = \xi(a + b)\alpha \mu(a + b) + \rho(a + b)\alpha \delta(a + b)
\]

**Example 3.2:** Let \(X\) be \(\Gamma\)-semiring, as in example 2.2, and \(\delta\) be \((\mu, \rho)\)-reverse derivation, as in example 2.2.

Let \(\xi\) be an additive mapping on \(S\) into \(X\) defined by

\[
\xi\left(\begin{array}{c} a \\ 0 \\ 0 \end{array}\right) = \left(\begin{array}{c} a \\ 0 \\ 0 \end{array}\right)
\]

Then, \(\xi\) is generalized \((\mu, \rho)\)-reverse derivation on \(S\) into \(X\).

**Lemma 3.4:** Let \(\xi\) be Jordan generalized \((\mu, \rho)\)-reverse derivation from \(\Gamma\)-semiring \(S\) into \(\Gamma\)-module \(X\), then for all \(a, b \in S\):

\[
\xi((a + b)\alpha(a + b)) = \xi(a + b)\alpha \mu(a + b) + \rho(a + b)\alpha \delta(a + b)
\]

**Example 3.3:** Let \(R\) be a ring, \(A, \Gamma,\) and \(Y\) are defined as in example 2.3, \(\xi\) be generalized \((\mu, \rho)\)-reverse derivation from \(A\) into \(Y\) with \((\mu, \rho)\)-reverse derivation \(\delta\) from \(A\) into \(Y\), and \(S, \Gamma,\) and \(X\) are as in example 2.3. We define \(\xi\) as an additive mapping from \(S\) into \(X\) such that \(\xi(a, b) = (\xi(a), \xi(b))\). Then, \(\xi\) is generalized \((\mu, \rho)\)-reverse derivation associated with \((\mu, \rho)\)-reverse derivation \(\Delta\) from \(S\) into \(X\), as defined in example 2.3.
\[ \xi(a)\mu(a) + \rho(a)\alpha\delta(a) + \xi(b)\mu(b) + \rho(b)\alpha\delta(b) + \xi(a)\alpha\beta(b) + \xi(a)\alpha\beta(a) \]

By comparing (1) and (2), we have

\[ \xi(a)\alpha\beta(b) = \xi(b)\mu(a) + \rho(b)\alpha\delta(a) + \xi(a)\mu(b) + \rho(a)\alpha\delta(b) \]

**Definition 3.5:** If \( \xi \) is Jordan generalized \((\mu, \rho)\)-reverse derivation from additive inverse \(\Gamma\)-semiring \(S\) into \(\Gamma\)-module, then we define \( \phi \) by

\[ \phi(a, b) = \xi(a)\beta(b) + \xi(b)\mu(a) - \rho(b)\alpha\delta(a), \quad \text{for all } a, b \in S, \alpha \in \Gamma. \]

**Example 3.6:**

Let \( S \) be a \( \Gamma \)-semiring, \( X \) be a \( \Gamma S \)-module, and \( a \in S \), such that \( a\Gamma a = 0 \) and \( x\Gamma x = 0 \), for all \( x \in S \), but \( x\Gamma y = 0 \), for some \( x, y \in S \), such that \( x \neq y \). Also, let \( \delta \) be a mapping of \( S \) into \( X \) defined by:

\[ \delta(x) = x\alpha x \quad \text{for all } x \in S, \alpha \in \Gamma. \]

It is clear that \( \xi \) is a Jordan generalized \((\mu, \rho)\)-reverse derivation on \( S \) into \( X \), which satisfies \( \phi \).

**Lemma 3.7:** If \( S \) is additive inverse \(\Gamma\)-semiring, \( X \) is \(\Gamma S\)-module, and \( \xi \) is Jordan generalized \((\mu, \rho)\)-reverse derivations from \( S \) into \( X \), then for all \( a, b, c \in S, \alpha \in \Gamma \):

i) \[ \phi(a, b) = -\phi(b, a) \]

ii) \[ \phi(a + c, b) = \phi(a, b) + \phi(c, b) \]

iii) \[ \phi(a, b + c) = \phi(a, b) + \phi(a, c) \]

**Proof:**

i) \[ \xi(a\alpha b + b\alpha a) = \xi(b)\mu(a) + \rho(b)\alpha\delta(a) + \xi(a)\mu(b) + \rho(a)\alpha\delta(b) \]

\[ \xi(a\alpha b) + f(b\alpha a) = \xi(b)\mu(a) + \rho(b)\alpha\delta(a) + \xi(a)\mu(b) + \rho(a)\alpha\delta(b) \]

\[ \xi(a\alpha b) - \xi(b)\mu(a) + \rho(b)\alpha\delta(a) = -\xi(b\alpha a) + \xi(a)\mu(b) + \rho(a)\alpha\delta(b) \]

\[ \phi(a, b) = -\phi(b, a) \]

ii) \[ \phi(a + c, b) = \xi((a + c)\alpha b) - (\xi(b)\alpha\mu(a) + \rho(b)\alpha\delta(a)) \]

\[ = \xi(a\alpha b) + \xi(c\alpha b) - \xi(b)\alpha\mu(a) - \xi(b)\alpha\mu(c) + \rho(b)\alpha\delta(a) - \rho(b)\alpha\delta(c) \]

\[ = \xi(a\alpha b) + \xi(c\alpha b) - \xi(b)\mu(a) - \mu(b)\alpha\delta(a) + \xi(c\alpha b) - \xi(b)\mu(c) - \mu(b)\alpha\delta(c) \]

\[ = \phi(a, b) + \phi(c, b) \]

iii) \[ \phi(a, b + c) = \phi(a, b) + \phi(c, b) \]

**Lemma 3.8:** If \( S \) is \( \Gamma \)-semiring with additive invertible identity \( X \) is \(\Gamma S\)-module, then \( \xi \) is Jordan generalized \((\mu, \rho)\)-reverse derivations from \( S \) into \( X \) iff \( \phi(a, b) = 0 \), for all \( a, b \in S, \alpha \in \Gamma \).

**Proof:** By Lemma 3.4, we get

\[ \xi(a\alpha b + b\alpha a) = \xi(b)\mu(a) + \rho(b)\alpha\delta(a) + \xi(a)\mu(b) + \rho(a)\alpha\delta(b) \]

\[ \xi(a\alpha b) + f(b\alpha a) = \xi(b)\mu(a) + \rho(b)\alpha\delta(a) + \xi(a)\mu(b) + \rho(a)\alpha\delta(b) \]

Now, we have

Since \( \xi \) is additive mapping, we have

\[ \xi(a\alpha b + b\alpha a) = \xi(a\alpha b) + \xi(b\alpha a) \]

\[ = \xi(a\alpha b) + \xi(a)\mu(b) + \rho(b)\alpha\delta(b) \]

By comparing (1) and (2), we get

\[ \xi(a\alpha b) = \xi(b)\mu(a) + \rho(b)\alpha\delta(a) = 0 \]

\[ \phi(a, b) = 0. \]

Converse: Obvious.

**Theorem 3.9:** If \( S \) is \( \Gamma \)-semiring and \( X \) is \(\Gamma S\)-module, then every Jordan generalized \((\mu, \rho)\)-reverse derivations from \( S \) into \( X \) is \((\mu, \rho)\)-reverse derivation from \( S \) into \( X \).

**Proof:** Let \( \xi \) be Jordan generalized \((\mu, \rho)\)-reverse derivation from \( \Gamma \)-semiring \( S \) into \( \Gamma S \)-module \( X \), then by Lemma 3.8, we get

\[ \phi(a, b) = 0. \]

Now, by Lemma 3.8, we get

\[ \xi \text{ is } (\mu, \rho) \text{-reverse derivation from } \Gamma \text{-semiring into } X \text{.} \]
Proposition 3.10: Every Jordan generalized \((\mu, \rho)\)-reverse derivation from \(\Gamma\)-semiring \(S\) into 2-torsion free \(\Gamma\)-module \(X\) is Jordan generalized triple \((\mu, \rho)\)-reverse derivation from \(S\) into \(X\).

Proof: Since \(\xi\) is Jordan generalized \((\mu, \rho)\)-reverse derivation from \(S\) into \(X\), then by replacing \(a\beta b + b\beta a\) by \(b\) in lemma 3.4, we get
\[
\xi(a\alpha(a\beta b + b\beta a)\alpha a) = \xi((a\alpha)(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = \xi((a\beta b + b\beta a)\alpha(a\beta b + b\beta a)) + (a\beta b + b\beta a)(\xi(a)\beta \mu(b)\alpha \mu(a)) + (\xi(a)\beta \mu(b)\alpha \mu(a))b.
\]

(1)

On the other hand,
\[
\xi(a\alpha(a\beta b + b\beta a)\alpha a) = \xi(a\alpha\beta b + a\beta b\alpha a) + (a\beta b + b\beta a)\alpha a) = \xi((a\alpha\beta b + a\beta b\alpha a) + (a\beta b + b\beta a)\alpha a) = \xi((a\beta b + b\beta a)\alpha(a\beta b + b\beta a)) + (a\beta b + b\beta a)(\xi(a)\beta \mu(b)\alpha \mu(a)) + (\xi(a)\beta \mu(b)\alpha \mu(a))b.
\]

(2)

By comparing (1) and (2), we get
\[
2\xi(a\alpha\beta b\alpha a) = \xi((a\beta b + b\beta a)\alpha(a\beta b + b\beta a)) + (a\beta b + b\beta a)(\xi(a)\beta \mu(b)\alpha \mu(a)) + (\xi(a)\beta \mu(b)\alpha \mu(a))b + \xi((a\beta b + b\beta a)\alpha(a\beta b + b\beta a)) + (a\beta b + b\beta a)(\xi(a)\beta \mu(b)\alpha \mu(a)) + (\xi(a)\beta \mu(b)\alpha \mu(a))b.
\]

Since \(X\) is 2-torsion free, we get
\[
\xi(a\alpha\beta b\alpha a) = \xi((a\beta b + b\beta a)\alpha(a\beta b + b\beta a)) + (a\beta b + b\beta a)(\xi(a)\beta \mu(b)\alpha \mu(a)) + (\xi(a)\beta \mu(b)\alpha \mu(a))b.
\]

Thus \(\xi\) is Jordan generalized triple \((\mu, \rho)\)-reverse derivation from \(S\) into \(X\).

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References