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Best Approximation in Modular Spaces By Type of Nonexpansive Maps

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Abstract
This paper presents results about the existence of best approximations via nonexpansive type maps defined on modular spaces.

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أفضل تقدير في الفراغات المعيارية حسب نوع الخرائط غير التقريبية

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الخلاصة
تقدم هذه الورقة نتائج عن وجود أفضل التقريبات بواسطة تطبيقات من نوع اللاامتمددة معرفة على فضاءات الوحدات.

1. Introduction and Preliminaries

Modular spaces are extensions of Lebesgue, Riesz, and Orlicz spaces of integrable functions [1]. A general theory of modular linear spaces was founded by Nakano 1950 [2] . Nakano's modulars on real linear spaces are convex functionals. Nonconvex modulars and the corresponding modular linear spaces were constructed by Musielak and Orlicz (we refer to [2]). In 2006, Vyacheslav Chistyakov [3, 4] was introduced the concept of a metric modular on a set, inspired partly by the classical linear modulars on function spaces employed by Nakano and other in the sense of Chistyakov. In the formulation given by Kowzslowski[5], "a modular on a linear space $\mathcal{V}$ over the field $\mathcal{K}(=\mathcal{R}\ or\ \mathcal{C})$ is a function $m:\mathcal{V}\to[0,\infty]$such that

(i)$m(x)=0 \Leftrightarrow x=0$;

(ii)$m(\alpha x)=m(x)$ for $\alpha\in\mathcal{K}\ with\ |\alpha|=1$, for all $x\in\mathcal{V}$;

(iii)$m(\alpha x+\beta y)\leq m(x)+m(y)$ such that $\alpha,\beta\geq 0$, for all $x,y\in\mathcal{V}$.

Moreover, modular m is called convex, if (iii) replaced by

(iii$'$) $m(\alpha x+\beta y)\leq \alpha m(x)+\beta m(y)$ if $\alpha,\beta\geq 0, \alpha+\beta=1$ for all $x,y\in\mathcal{V}$."

"A sequence $\{v_n\}\subset\mathcal{V}$is said to be γ-convergent to $v\in\mathcal{V}$and write$v_n\to v$ if $m(v_n-v)\to 0$ as n→ ∞. A sequence $\{v_n\}$ is called Cauchy whenever $m(v_n-v_m)\to 0$ as ,$m,n\to\infty$. Also, $\mathcal{V}$ is called complete if any Cauchy sequence in $\mathcal{V}$ is convergent. A subset $B\subset\mathcal{V}$ is called closed if for any sequence $\{v_n\}\subset B$, convergent to $\in\mathcal{V}$, we have $v\in B$" [6].

"A closed subset $B\subset\mathcal{V}$ is called compact if any sequence $\{v_n\}\subset B$ has a convergent subsequence" [7].

"A selfmap J on $B\subseteq\mathcal{V}$ is called contraction mapping if ∃ h ∈ (0 ,1) for all v,u in $\mathcal{V}$, $m(J(v)-J(u))\leq h\,m(v-u)$

and if h = 1 then J is called a non –expansive mapping" [7].

"A map J is demi-closed at 0 if $\{v_n\}\subseteq B$, v_n convergrs weakly to $v, w_n\in J(v_n)$ $and\ w_n\to 0\Rightarrow 0\in J(v)$.

$\mathcal{V}$ is said to be Opial if for every sequence $\{v_n\}$ in $\mathcal{V}$ weakly convergent to $v\in\mathcal{V}$the inequality

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$$\lim_{n\to\infty} inf\,\gamma(v_n - v) < \lim_{n\to\infty} inf\gamma(v_n - u)$$

holds for all $u \neq v$" [7].

"Let $\mathcal{V}$ and W be two modular spaces, recall that a set -valued mapping $J: \mathcal{V} \to W$ is a subset of $\mathcal{V} \times W$ with domains $\mathcal{V}$; equivalently, J is a point to set map assigning to each $u \in \mathcal{V}$ a nonempty subset $J(u)$ of W.

let $v \in \mathcal{V}$, v is called a fixed point of S if $v \in J(v)$ (when S is single valued , v is fixed point of S if $v = J(v)$ A set-valued mapping J is upper semi continuous (shortly, $u.s.c.$) if and only if the set $\{u \in M_\gamma : J(x) \cap B \neq \emptyset\}$ is closed for each closed subset B of W." See [8].

"Consider $\emptyset \neq B \subset \mathcal{V}$, the element $y \in B$ is a best approximation for a given $x \in \mathcal{V}$; if

$$m(x-y) = d_m(x, B) = inf_{y\in B}\, m(x-y)$$

and $P_B(x)$ or Px the set of all elements of best approximation of x by B.

A subset B is called Chebysev if $\forall x \in \mathcal{V}, \exists! y \in \mathcal{U}$ such that $m(x-y) = d_m(x, B)$. "[9].

Main Results.

First we start with the following definition:

Definition 1: A multivalued map $J: B \to 2^B$ is called *-nonexpansive if $\forall x, y \in B\ and\ a_x \in J(x) with \quad m(x - a_x) = \sigma(x, J(x)),$

$$\exists a_y \in J(y)\ with\, m(y - a_y) = \sigma(y, J(y)) \ni m(a_x - a_y) \le m(x-y).$$

Remark (2) The concept of *-nonexpansive map coincides with a nonexpansive for a single valued map. Thus we have the result shown in [10].

Define *-nonexpansive map $K: B \to 2^B$ by

$K(x) = \cup\{P(y) : y \in J(x), \sigma(J(x), B) = \sigma(y, B)\}$ (1)

For the first result, fix $C(B)$ as the class of all nonempty compact subsets of B and b-starshaped mean starshaped with starcenter at b. Then we have the following

Theorem 2: let B be a nonempty weakly compact $b-$starshped subset of complete convex modular space $\mathcal{V}$, $K\ as\ in(1)$ and $J: B \to C(B)$ is usc such that $\exists x_0 \in B, a_{x_0} \in J(x_0), m(a_{x_0}) < \infty$. If $\forall x, K(x)$ is compact Chebyshev and $I - K$ is demiclosed at 0 then $\exists z \in B \ni \sigma(z, J(z)) = \sigma(J(z), B)$.

Proof:

The compactness of $J(x), \forall x$ implies that $K(x) \neq \emptyset$. Since $K(x)$ is Chebyshev so by definition of *-nonexpansive, $a_x \in K(x)$ is unique and $\exists! a_y \in K(y), \forall y \in B \ni$

$m(a_x - a_y) \le m(x-y)$ (2)

Let $J_n: B \to B$ such that $J_n(x) = \theta_n a_x + (1-\theta_n)b$, where $0 < \theta_n < 1, \forall n$ and $\theta_n \to 1$ as $n \to \infty$. By convexity of $\mathcal{V}$ and (2), we have $\forall x, y \in B$,

$m(J_n(x) - J_n(y)) \le \theta_n m(x-y)$.

So, $\forall n$, J_n is contraction and hence, by [6], has a fixed point $z_n \in B$. the sequence $\{z_n\}$ has a subsequence, also say $\{z_n\}$, converging weakly to $z \in B$. By definition of $J_n, \exists a_n \in K(z_n) \ni$

$$z_n = J_n(z_n) = \theta_n a_n + (1-\theta_n)\mathrm{b}$$

And then

$y_n = a_n - z_n = (1-\theta_n)(a_n - \mathrm{b}) \to 0\ as\ n \to \infty$ (3)

Since $I - K$ is demi-closed at 0, the sequence $\langle z_n \rangle$ converges weakly to z, $y_n \to 0$ where $y_n = a_n - z_n \in K(z_n) - z_n$. Thus $0 \in (I-K)(z) \Rightarrow z \in K(z)$.

Therefore, for some $w \in J(z)$ with

$$m(J(z)) = \sigma(w, B), z \in P(w).$$

We have

$$\sigma(z, J(z)) \le m(z-w) = \sigma(w, B) = \sigma(J(z), B) \le \sigma(z, J(z))$$
$$\Rightarrow \sigma(z, J(z)) = \sigma(J(z), B)$$

The proof is complete.

Now, we state the definition of weak nonexpansive map (shortly, called $w-$nonexpansive map)

Definition 3: A multivalued mapping $J: B \to 2^B$ is called w- nonexpansive if $\forall x \in B, a_x \in J(x)$ there is $a_y \in J(y), \forall y \in B \ni m(a_x - a_y) \le m(x-y)$.

Theorem 4: The result of Theorem (2) also hold if $\mathcal{V}$ satisfies Opial's condition instead of demi closeness.

Proof: Since the *-nonexpansive mapping K is weakly nonexpansive. So, $\forall n, a_n \in K(x_n), \exists b_n \in K(z) such\ that$

$$m(a_n - b_n) \leq m(x_n - z) \quad (4)$$

As $K(z)$ is compact so $\langle b_n \rangle$ converges to some $u \in K(z)$.

Combination of (4) with $b_n \to 0\ and\ z_n \to u \Rightarrow$

$$\liminf\ m(z_n + x_n - b_n) = \liminf m(x_n - u) \leq \liminf m(x_n - z)$$

By Opial's condition, we have

$$\liminf m(x_n - z) < \liminf\ m(x_n - u).$$

Thus $z = u \in K(z)$.

Therefore, the final step of proof follows from previous argument.

About invariant best approximation we prove the following result

Theorem (5): Let B be a closed subspace of a convex modular space V and $J: B \to V$ be a continuous map. If $P°J: B \to B$ is linear nonexpansive map such that $\exists u_0 \in B$ with $(P°J)^2 (u_0) - 2(P°J)(u_0) + u_0 = 0$ then $m(u_0 - J(u_0)) = \sigma(J(u_0), B)$. Moreover, if $J(u_0) \in B$, then J has a fixed point.

Proof:

let $K = P°J$ then $K: B \to B$is linear nonexpansive $\ni$

$$(K)^2 (u_0) - 2(K)(u_0) + u_0 = 0$$

From linearity of K, we have

$(K - I)(K - I)(u_0) = 0$

Let $(K - I)(u_0) = u$

$\Rightarrow (K - I)(u) = 0 \Rightarrow K(u) = u.$

$\Rightarrow K(u_0) = u_0 + u \Rightarrow K^n(u_0) = nu, \forall n \geq 1.$

Consider $nm(u) = m(K^n(u_0) - u_0)$

$$\leq m\big(K^n(u_0) - K(0)\big) + m(u_0)$$
$$\leq 2m(u_0)$$

Hence, $m(u) \leq \frac{2m(u_0)}{n}, \forall n \geq 1$. As $n \to \infty$, we get $u = 0 \Rightarrow K(u_0) = u_0$.

Therefore, $(P°J)(u_0) = u_0 \Rightarrow m(u_0 - J(u_0)) = \sigma(J(u_0), B)$ done.

Open problem

Consider $J: B \to V$, where B is convex set J is midpoint concave (or convex) map if

$$\frac{1}{2}J(x) + \frac{1}{2}J(y) \subseteq J\left(\frac{x}{2} + \frac{y}{2}\right), \forall x, y \in B.$$

(or, $J\left(\frac{x}{2} + \frac{y}{2}\right) \subseteq \left(\frac{1}{2}J(x) + \frac{1}{2}J(y)\right)$ respectively. Is there $u_0 \in B \ni m(u_0 - J(u_0)) = \sigma(J(u_0), B)$?.

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