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On Sandwich Theorems Results for Certain Univalent Functions Defined by Generalized Operators

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Abstract:
In this present paper, we obtain some differential subordination and superordination results, by using generalized operators for certain subclass of analytic functions in the open unit disk. Also, we derive some sandwich results.

Keywords: analytic function, subordinant, differential subordination, dominant, generalized operator, sandwich theorems.

حول نتائج مبرهنات الساندوج للدوال احادية التكافؤ الاكيدة والمعرفة بواسطة مؤثرات معممة

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الخلاصة
في البحث الحالي, حصلنا على بعض نتائج التابعية التفاضلية والتابعية التفاضلية العليا باستخدام مؤثرات معممة لصنف جزئي اكيد من الدوال التحليلية في قرص الوحدة المفتوح.اشتقينا ايضا بعض نتائج الساندوج.

1. Introduction:

Denote by $\mathcal{Y}=\mathcal{Y}(\mathcal{K})$ the class of analytic functions in the open unit disk $\mathcal{K}=\{\mathcal{Z}\in C : |\mathcal{Z}|<1\}$.
For $\hbar$ a positive integer and $a\in\mathbb{C}$,
let $\mathcal{Y}[a,\hbar]$ be the subclass of the function $f\in\mathcal{Y}$ of the form:
$f(\mathcal{Z})=a+a_{\hbar}\mathcal{Z}^{\hbar}+a_{\hbar+1}\mathcal{Z}^{\hbar+1}+\ldots\ (a\in\mathbb{C},\hbar\in N=\{1,2,3\ldots\})$. (1.1)
Also, let $\mathfrak{D}$ be the subclass of $\mathcal{Y}$ consisting of functions of the formula**:**

$$f(\mathcal{Z})=\mathcal{Z}+\sum_{\hbar=2}^{\infty}a_{\hbar}\mathcal{Z}^{\hbar}. \quad (1.2)$$

Several authors studied class of univalent functions for another conditions, like, [1,2].
If $f\in\mathfrak{D}$ is given by (1.2) and $j\in\mathfrak{D}$ given by

$$j(\mathcal{Z})=\mathcal{Z}+\sum_{\hbar=2}^{\infty}b_{\hbar}\mathcal{Z}^{\hbar}.$$

The Hadamard product (or convolution) of f and j is defined by

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$$(f * j)(\mathcal{Z}) = \mathcal{Z} + \sum_{\hbar=2}^{\infty} a_\hbar b_\hbar \mathcal{Z}^\hbar = (j * f)(\mathcal{Z}).$$

For two functions j and f are analytic in $\mathcal{K}$, we say that the function f is subordinate to j in $\mathcal{K}$, written $f \prec j$, if there exists Schwarz function w, analytic in $\mathcal{K}$ with $w(0) = 0$ and $|w(\mathcal{Z})| < 1$ in $\mathcal{K}$ such that $f(\mathcal{Z}) = j(w(\mathcal{Z})), \mathcal{Z} \in \mathcal{K}$.
If j is univalent and $j(0) = f(0)$, then $f(\mathcal{K}) \subset \hbar(\mathcal{K})$.
Let $\psi : \mathbb{C}^2 \times \mathcal{K} \to \mathbb{C}$, and h is univalent in $\mathcal{K}$ with M ∈ Q, where Q is the set of all functions f that are, injective and analytic on $\bar{\mathcal{K}}/E(f)$
such that $f'(\xi) \neq 0$ for $\xi \in \partial\mathcal{K} \setminus E(f)$ and $E(f) = \{\xi \in \partial\mathcal{K} : \lim_{\mathcal{Z}\to\xi} f(\mathcal{Z}) = \infty\}$ (see[3]).
Let $\psi : \mathbb{C}^2 \times \mathcal{K} \to \mathbb{C}$, and h is univalent in $\mathcal{K}$ with M ∈ Q. Miller and Mocanu [4] consider the problem of determining conditions on admissible functions ψ such that

$$\psi(\mathrm{p}(\mathcal{Z}), \mathcal{Z}\mathrm{p}'(\mathcal{Z}); \mathcal{Z}) \prec \mathrm{h}(\mathcal{Z}) \tag{1.3}$$

implies $\mathrm{p}(\mathcal{Z}) \prec \mathrm{M}(\mathcal{Z})$, for all functions $\mathrm{p}(\mathcal{Z}) \in \mathcal{Y}$ [a, $\hbar$]that satisfy the differential subordination (1.3). Moreover, they found conditions so that M is the smallest function with this property, called the best dominant of the subordination (1.3).
Let $\emptyset : \mathbb{C}^2 \times \mathcal{K} \to \mathbb{C}$, and M ∈ $\mathcal{Y}$[a, $\hbar$] with h ∈ $\mathcal{Y}$. Recently Miller and Mocanu [5,3] studied the dual problem and determined conditions on ∅ such that

$$\mathrm{h}(\mathcal{Z}) \prec \emptyset(\mathrm{p}(\mathcal{Z}), \mathcal{Z}\mathrm{p}'(\mathcal{Z}); \mathcal{Z}) \tag{1.4}$$

implies $\mathrm{M}(\mathcal{Z}) \prec \mathrm{p}(\mathcal{Z})$, for all functions p ∈ Q that satisfy the above superaredination. They also found conditions so that the function M is the largest function with this property, called the best subordinant of superordination (1.4).
Using the results, Bulboača [6] considered certain classes of first- order differential superordinations as well as superordination preserving integral operator [6] .Ali et al.[7], have used the results of Bulboača [8] to obtain sufficient conditions for normalized analytic functions to satisfy:

$$\mathrm{M}_1(\mathcal{Z}) \prec \frac{\mathcal{Z}f'(\mathcal{Z})}{f(\mathcal{Z})} \prec \mathrm{M}_2(\mathcal{Z}),$$

where M_1 and M_2 are given univalent functions in $\mathcal{K}$ with $\mathrm{M}_1(0) = \mathrm{M}_2(0) = 1$.Recently, Al-Ameedee et al. [9, 10] and Atshan with several authors (see[11 − 17]) studied sandwich theorems results for another classes of analytic functions .
Swamy [18] defined a new generalized operator $\mathrm{F}^{\mathrm{m}}_{v,\delta}$ on $\mathfrak{D}$ as follows: For $\delta \geq 0$,m ∈ N_0=N∪{0}and v a real number with $v+\delta > 0$. Then for f ∈ $\mathfrak{D}$, the operator $\mathrm{F}^{\mathrm{m}}_{v,\delta}$ is defined by

$\mathrm{F}^0_{v,\delta} f(\mathcal{Z}) = f(\mathcal{Z})$,

$\mathrm{F}^1_{v,\delta} f(\mathcal{Z}) = \dfrac{vf(\mathcal{Z}) + \delta\mathcal{Z}f'(\mathcal{Z})}{v + \delta}$,

⋮

$\mathrm{F}^{\mathrm{m}}_{v,\delta} f(\mathcal{Z})=\mathrm{F}_{v,\delta}\left(\mathrm{F}^{\mathrm{m}-1}_{v,\delta} f(\mathcal{Z})\right)$.

We observe that $\mathrm{F}^{\mathrm{m}}_{v,\delta} : \mathfrak{D} \longrightarrow \mathfrak{D}$ is a linear operator and for f given by (1.2) , we have

$$\mathrm{F}^{\mathrm{m}}_{v,\delta} f(\mathcal{Z}) = \mathcal{Z} + \sum_{\hbar=2}^{\infty} \left(\frac{v + \hbar\delta}{v + \delta}\right)^{\mathrm{m}} a_\hbar \mathcal{Z}^\hbar. \tag{1.5}$$

It follows from (1,5) that

$$\mathrm{F}^{\mathrm{m}}_{v,0} f(\mathcal{Z}) = f(\mathcal{Z}), \tag{1.6}$$

$(v + \delta)\mathrm{F}^{\mathrm{m}+1}_{v,\delta} f(\mathcal{Z}) = v\mathrm{F}^{\mathrm{m}}_{v,\delta} f(\mathcal{Z}) + \delta\mathcal{Z}\left(\mathrm{F}^{\mathrm{m}}_{v,\delta} f(\mathcal{Z})\right)', \delta > 0,$

and

$\mathrm{F}^{\mathrm{m}_1}_{v,\delta}\left(\mathrm{F}^{\mathrm{m}_2}_{v,\delta} f(\mathcal{Z})\right) = \mathrm{F}^{\mathrm{m}_2}_{v,\delta}\left(\mathrm{F}^{\mathrm{m}_1}_{v,\delta} f(\mathcal{Z})\right)$, for all $\mathrm{m}_1, \mathrm{m}_2 \in \mathrm{N}_0$.

We note that
1) $\mathrm{F}^{\mathrm{m}}_{v,1} f(\mathcal{Z}) = \mathrm{F}^{\mathrm{m}}_{v} f(\mathcal{Z})$, $v > -1$ (see Cho and Srivastava [19] and Cho and Kim[20]).
2) $\mathrm{F}^{\mathrm{m}}_{1-\delta,\delta} f(\mathcal{Z}) = \mathrm{D}^{\mathrm{m}}_{\delta} f(\mathcal{Z}), \delta \geq 0$ (see Al-Oboudi [21]).
3) $\mathrm{F}^{\mathrm{m}}_{l+1-\delta,\delta} f(\mathcal{Z}) = \mathrm{F}^{\mathrm{m}}_{l,\delta} f(\mathcal{Z}), l > -1 , \delta \geq 0$ (see Catas[22]).

4) $F^{m}_{1,\delta}f(\mathcal{Z}) = N^{m}_{\delta}f(\mathcal{Z})$, is an operator defined by (see[18]).

$N^{m}_{\delta}f(\mathcal{Z}) = \mathcal{Z} + \sum_{\hbar=2}^{\infty}\left(\frac{1+\hbar\delta}{1+\delta}\right)^{m} a_{\hbar}\, \mathcal{Z}^{\hbar}, (f \in \mathfrak{D})$,

Patel [23] defined an integral operator $I^{m}_{v,\delta}$ on $\mathfrak{D}$ as follows:

For $m \in N_0 = N \cup \{0\}$, $\delta \geq 0$ with $v + \delta > 0$ and v a real number.

Then for $f \in \mathfrak{D}$, we define the operator $I^{m}_{v,\delta}$ by

$I^{0}_{v,\delta}f(\mathcal{Z}) = f(\mathcal{Z})$

$$I^{1}_{v,\delta}f(\mathcal{Z}) = \left(\frac{v+\delta}{\delta}\right)\mathcal{Z}^{1-\left(\frac{v+\delta}{\delta}\right)}\int_{0}^{\mathcal{Z}} t^{\left(\frac{v+\delta}{\delta}\right)-2} f(t)dt, \quad \mathcal{Z} \in \mathcal{K}.$$

$$I^{2}_{v,\delta}f(\mathcal{Z}) = \left(\frac{v+\delta}{\delta}\right)\mathcal{Z}^{1-\left(\frac{v+\delta}{\delta}\right)}\int_{0}^{\mathcal{Z}} t^{\left(\frac{v+\delta}{\delta}\right)-2} I^{1}_{v,\delta}f(t)dt, \quad \mathcal{Z} \in \mathcal{K}.$$

$\vdots$

$$I^{m}_{v,\delta}f(\mathcal{Z}) = \left(\frac{v+\delta}{\delta}\right)\mathcal{Z}^{1-\left(\frac{v+\delta}{\delta}\right)}\int_{0}^{\mathcal{Z}} t^{\left(\frac{v+\delta}{\delta}\right)-2} I^{m-1}_{v,\delta}f(t)dt, \quad \mathcal{Z} \in \mathcal{K}.$$

$$= \underbrace{I^{1}_{v,\delta}\left(\frac{\mathcal{Z}}{1-\mathcal{Z}}\right) * I^{1}_{v,\delta}\left(\frac{\mathcal{Z}}{1-\mathcal{Z}}\right) * \dots * I^{1}_{v,\delta}\left(\frac{\mathcal{Z}}{1-\mathcal{Z}}\right) * f(\mathcal{Z})}_{m-times}.$$

We observe that $I^{m}_{v,\delta}: \mathfrak{D} \longrightarrow \mathfrak{D}$ is an integral operator and for f given by (1.2), we have

$$I^{m}_{v,\delta}f(\mathcal{Z}) = \mathcal{Z} + \sum_{\hbar=2}^{\infty}\left(\frac{v+\delta}{v+\hbar\,\delta}\right)^{m} a_{\hbar}\, \mathcal{Z}^{\hbar}, (\mathcal{Z} \in \mathcal{K}). \tag{1.7}$$

It follows from (1.7) that

$$I^{m}_{v,0}f(\mathcal{Z}) = f(\mathcal{Z}),$$

$$(v+\delta)I^{m}_{v,\delta}f(\mathcal{Z}) = vI^{m+1}_{v,\delta}f(\mathcal{Z}) + \delta\mathcal{Z}\left(I^{m+1}_{v,\delta}f(\mathcal{Z})\right)'. \tag{1.8}$$

We note that

1) $I^{m}_{1,1}f(\mathcal{Z}) = T^{m}f(\mathcal{Z})$ (see [18,23]).

2) $I^{m}_{1-\delta,1}f(\mathcal{Z}) = T^{m}_{\delta}f(\mathcal{Z}), \delta > 0$ (see [23]).

3) $I^{m}_{v,1}f(\mathcal{Z}) = T^{m}_{v}f(\mathcal{Z}), v > 0$ (see [23]).

In this paper, we study some properties on differential subordination and superordination of univalent functions defined by generalized operators.

2. Preliminaries:

In order to prove our subordinations and superordinations results, we need the following lemmas and definition .

Definition (2.1) [4]: Denote by Q the set of all functions q that are injective and analytic on $\overline{\mathcal{K}} \setminus E(M)$, where $\overline{\mathcal{K}} = \mathcal{K} \cup \{\mathcal{Z} \in \partial\mathcal{K}\}$, and

$$E(M) = \{\xi \in \partial\mathcal{K} : \lim_{\mathcal{Z}\to\xi} M(\mathcal{Z}) = \infty\} \tag{2.1}$$

and are such that $M'(\xi) \neq 0$ for $\xi \in \partial\mathcal{K} \setminus E(M)$. Further, let the subclass of Q for which $M(0) = a$ be denoted by $Q(a)$, $Q(0) \equiv Q_0$ and $Q(1) \equiv Q_1$.

Lemma (2.1) [24]: Let $M(\mathcal{Z})$ be convex univalent function in $\mathcal{K}$, let $\beta \in \mathbb{C}/\{0\}, \alpha \in \mathbb{C}$ and suppose that

$$\operatorname{Re}\left\{1 + \frac{\mathcal{Z}M''(\mathcal{Z})}{M'(\mathcal{Z})}\right\} > \max\left\{-\operatorname{Re}\left(\frac{\alpha}{\beta}\right), 0\right\}.$$

If $p(\mathcal{Z})$ is analytic function in $\mathcal{K}$, and $\alpha p(\mathcal{Z}) + \beta\mathcal{Z}p'(\mathcal{Z}) \prec \alpha M(\mathcal{Z}) + \beta\mathcal{Z}M'(\mathcal{Z})$, then $p(\mathcal{Z}) \prec M(\mathcal{Z})$ and $M(\mathcal{Z})$ is the best dominant.

Lemma (2.2) [4]: Let $M(\mathcal{Z})$ convex univalent function in $\mathcal{K}$ and $M(0) = 1$. Let $\beta \in \mathbb{C}$,that Re $(\beta) > 0$. If $p(\mathcal{Z}) \in \mathcal{Y}[M(0),1] \cap Q$ and $p(\mathcal{Z}) + \beta\mathcal{Z}p'(\mathcal{Z})$ is univalent in $\mathcal{K}$, then $M(\mathcal{Z}) + \beta\mathcal{Z}M'(\mathcal{Z}) \prec p(\mathcal{Z}) + \beta\mathcal{Z}p'(\mathcal{Z})$, which implies that $M(\mathcal{Z}) \prec p(\mathcal{Z})$ and $M(\mathcal{Z})$ is the best subordinant .

3- Main Results :

Unless otherwise mentioned, we shall assume in remainder of the paper that $m \in N_0$, $\mathcal{Z} \in \mathcal{K}$ and the power are understood as principle values.

Theorem (3.1) : Let $\mathrm{M}(\mathcal{Z})$ be convex univalent function in $\mathcal{K}$ with $\lambda \in \mathbb{C}^*, \mathrm{M}(0)=1, \kappa>0, \delta>0$, v real number such that $v+\delta>0$ and suppose that

$$\operatorname{Re}\left\{\frac{\mathcal{Z}\mathrm{M}''(\mathcal{Z})}{\mathrm{M}'(\mathcal{Z})}+1\right\}>\max\left\{\operatorname{Re}\left(\frac{\kappa(v+\delta)}{\lambda\delta}\right), 0\right\}, \tag{3.1}$$

if $f(\mathcal{Z}) \in \mathfrak{D}$ satisfies the subordination

$$\emptyset(\mathrm{m}, \lambda, \kappa, v, \delta) \prec \mathrm{M}(\mathcal{Z})+\frac{\lambda\delta}{\kappa(v+\delta)} \mathcal{Z}\mathrm{M}'(\mathcal{Z}), \tag{3.2}$$

where $\emptyset(\mathrm{m}, \lambda, \kappa, v, \delta)$ is given by

$$\emptyset(\mathrm{m}, \lambda, \kappa, v, \delta)=(1-\lambda)\left(\frac{\mathrm{F}_{v,\delta}^{\mathrm{m}} f(\mathcal{Z})}{\mathcal{Z}}\right)^k+\lambda\left(\frac{\mathrm{F}_{v,\delta}^{\mathrm{m}} f(\mathcal{Z})}{\mathcal{Z}}\right)^k\left(\frac{\mathrm{F}_{v,\delta}^{\mathrm{m}+1} f(\mathcal{Z})}{\mathrm{F}_{v,\delta}^{\mathrm{m}} f(\mathcal{Z})}\right), \tag{3.3}$$

then $\left(\frac{\mathrm{F}_{v,\delta}^{\mathrm{m}} f(\mathcal{Z})}{\mathcal{Z}}\right)^k \prec \mathrm{M}(\mathcal{Z})$, and $\mathrm{M}(\mathcal{Z})$ is the best dominant.

Proof : Let

$$\mathrm{p}(\mathcal{Z})=\left(\frac{\mathrm{F}_{v,\delta}^{\mathrm{m}} f(\mathcal{Z})}{\mathcal{Z}}\right)^k. \tag{3.4}$$

Logarithmic differentiation of (3.4) with respect to $\mathcal{Z}$, and use of identity (1.6) in the resulting equation, yields

$$\frac{\mathcal{Z}\mathrm{p}'(\mathcal{Z})}{\mathrm{p}(\mathcal{Z})}=k\left(\frac{v+\delta}{\delta}\right)\left(\frac{\mathrm{F}_{v,\delta}^{\mathrm{m}+1} f(\mathcal{Z})}{\mathrm{F}_{v,\delta}^{\mathrm{m}} f(\mathcal{Z})}-1\right)$$

and which can be written as

$$\frac{\delta}{k(v+\delta)} \mathcal{Z}\mathrm{p}'(\mathcal{Z})=\left(\frac{\mathrm{F}_{v,\delta}^{\mathrm{m}} f(\mathcal{Z})}{\mathcal{Z}}\right)^k\left(\frac{\mathrm{F}_{v,\delta}^{\mathrm{m}+1} f(\mathcal{Z})}{\mathrm{F}_{v,\delta}^{\mathrm{m}} f(\mathcal{Z})}-1\right).$$

Thus, the subordination (3.2) is equivalent to

$$\mathrm{p}(\mathcal{Z})+\frac{\lambda\delta}{k(v+\delta)} \mathcal{Z}\mathrm{p}'(\mathcal{Z}) \prec \mathrm{M}(\mathcal{Z})+\frac{\lambda\delta}{k(v+\delta)} \mathcal{Z}\mathrm{M}'(\mathcal{Z}).$$

Applying Lemma (2.1),with $\gamma=\frac{\lambda\delta}{k(v+\delta)}$,

the proof of Theorem(3.1) is complete.

Taking the convex function $\mathrm{M}(\mathcal{Z})=\frac{1+\mathrm{A}\mathcal{Z}}{1+\mathrm{B}\mathcal{Z}}$ in Theorem (3.1), we have the following corollary.

Corollary (3.2):Let $\lambda, \mathrm{A}, \mathrm{B} \in \mathbb{C}, \mathrm{B} \neq \mathrm{A},|\mathrm{B}|<1, k>0, \operatorname{Re}(\lambda)>0, \delta>0$ and v real number such that $v+\delta>0$, if $f(\mathcal{Z}) \in \mathfrak{D}$ satisfies the condition:

$$\emptyset(\mathrm{m}, \lambda, \kappa, v, \delta) \prec \frac{1+\mathrm{A}\mathcal{Z}}{1+\mathrm{B}\mathcal{Z}}+\frac{\lambda\delta}{k(v+\delta)} \frac{(\mathrm{A}+\mathrm{B})\mathcal{Z}}{(1+\mathrm{B}\mathcal{Z})^2},$$

where $\emptyset(\mathrm{m}, \lambda, \kappa, v, \delta)$ is given by(3.3), then $\left(\frac{\mathrm{F}_{v,\delta}^{\mathrm{m}} f(\mathcal{Z})}{\mathcal{Z}}\right)^k \prec \frac{1+\mathrm{A}\mathcal{Z}}{1+\mathrm{B}\mathcal{Z}}$,

and $\frac{1+\mathrm{A}\mathcal{Z}}{1+\mathrm{B}\mathcal{Z}}$ is the best dominant.

Taking m=0 in Theorem (3.1), we obtain the following result.

Corollary (3.3): Let $\mathrm{M}(\mathcal{Z})$ be univalent function in $\mathcal{K}$ with $\mathrm{M}(0)=1$, $\lambda \in \mathbb{C}^*, \kappa>0, \delta>0$, v real number such that $v+\delta>0$ and suppose that (3.1)holds. If $f(\mathcal{Z}) \in \mathfrak{D}$ satisfies the subordination condition: $\emptyset_1(0, \lambda, \kappa, v, \delta) \prec \mathrm{M}(\mathcal{Z})+\frac{\lambda\delta}{\kappa(v+\delta)} \mathcal{Z}\mathrm{M}'(\mathcal{Z})$,

where $\emptyset_1(0, \lambda, \kappa, v, \delta)$ is given by

$$\emptyset_1(0, \lambda, \kappa, v, \delta)=(1-\lambda)\left(\frac{f(\mathcal{Z})}{\mathcal{Z}}\right)^k+\lambda\left(\frac{f(\mathcal{Z})}{\mathcal{Z}}\right)^k\left(\frac{\mathrm{F}_{v,\delta}^{1} f(\mathcal{Z})}{f(\mathcal{Z})}\right),$$

then $\left(\frac{f(\mathcal{Z})}{\mathcal{Z}}\right)^k \prec \mathrm{M}(\mathcal{Z})$ and $\mathrm{M}(\mathcal{Z})$ is the best dominant.

Taking $v=\delta=1$ in Theorem (3.1), we obtain the following result.

Corollary (3.4): Let $\mathrm{M}(\mathcal{Z})$ be univalent function in $\mathcal{K}$ with $\mathrm{M}(0)=1$, $\lambda \in \mathbb{C}^*, \kappa>0$ and suppose that (3.1)holds. If $f(\mathcal{Z}) \in \mathfrak{D}$ satisfies the subordination:

$$\emptyset_2(\mathrm{m}, \lambda, \kappa, 1,1)=(1-\lambda)\left(\frac{\mathrm{F}_{1,1}^{\mathrm{m}} f(\mathcal{Z})}{\mathcal{Z}}\right)^k+\lambda\left(\frac{\mathrm{F}_{1,1}^{\mathrm{m}} f(\mathcal{Z})}{\mathcal{Z}}\right)^k\left(\frac{\mathrm{F}_{1,1}^{\mathrm{m}+1} f(\mathcal{Z})}{\mathrm{F}_{1,1}^{\mathrm{m}} f(\mathcal{Z})}\right), \tag{3.5}$$

then $\left(\frac{\mathrm{F}_{1,1}^{\mathrm{m}} f(\mathcal{Z})}{\mathcal{Z}}\right) \prec \mathrm{M}(\mathcal{Z})$ and $\mathrm{M}(\mathcal{Z})$ is the best dominant.

In a manner similar to that of Theorem (3.1) , we can easily prove the following theorems taking the identity (1.8).

Theorem (3.5) : Let $f \in \mathfrak{D}, \lambda \in \mathbb{C}^*, \delta > 0, v \in R, v+\delta > 0$, let the function p be univalent in $\mathcal{K}$ and assume that it satisfies:

$$\operatorname{Re}\left(1+\frac{\mathcal{Z}\mathrm{M}''(\mathcal{Z})}{\mathrm{M}'(\mathcal{Z})}\right) > \max\left\{0, -\operatorname{Re}\left(\frac{\mathrm{k}}{\lambda}\right)\right\}, \mathcal{Z} \in \mathcal{K}. \quad (3.6)$$

If $f(\mathcal{Z}) \in \mathfrak{D}$ satisfies the subordination

$$\Psi(\mathrm{k},\mathrm{m},v,\delta,\lambda) \prec \mathrm{M}(\mathcal{Z}) + \frac{\lambda}{k}\mathcal{Z}\mathrm{M}'(\mathcal{Z}), \quad (3.7)$$

where $\Psi(\mathrm{k},\mathrm{m},v,\delta,\lambda) = (1-\lambda)\left(\frac{v+\delta}{\delta}\right)\left(\frac{\mathrm{I}_{v,\delta}^{\mathrm{m}+1}f(\mathcal{Z})}{\mathcal{Z}}\right)^k + \lambda\left(\frac{v+\delta}{\delta}\right)\left(\frac{\mathrm{I}_{v,\delta}^{\mathrm{m}+1}f(\mathcal{Z})}{\mathcal{Z}}\right)^k\left(\frac{\mathrm{I}_{v,\delta}^{\mathrm{m}}f(\mathcal{Z})}{\mathrm{I}_{v,\delta}^{\mathrm{m}+1}f(\mathcal{Z})}\right)$, (3.8)

then $\left(\frac{\mathrm{I}_{v,\delta}^{\mathrm{m}+1}f(\mathcal{Z})}{\mathcal{Z}}\right)^k \prec \mathrm{M}(\mathcal{Z})$, and $\mathrm{M}(\mathcal{Z})$is best dominant.

Proof: Let

$$\mathrm{p}(\mathcal{Z}) = \left(\frac{\mathrm{I}_{v,\delta}^{\mathrm{m}+1}f(\mathcal{Z})}{\mathcal{Z}}\right)^k. \quad (3.9)$$

Logarithmic differentiation of (3.9) with respect to $\mathcal{Z}$, and use of identity(1.8) in the resulting equation , yields

$$\frac{\mathcal{Z}\mathrm{p}'(\mathcal{Z})}{\mathrm{p}(\mathcal{Z})} = k\left(\frac{v+\delta}{\delta}\right)\left(\frac{\mathrm{I}_{v,\delta}^{\mathrm{m}}f(\mathcal{Z})}{\mathrm{I}_{v,\delta}^{\mathrm{m}+1}f(\mathcal{Z})}-1\right),$$

and which can be written as

$$\frac{\delta}{k(v+\delta)}\mathcal{Z}\mathrm{p}'(\mathcal{Z}) = \left(\frac{\mathrm{I}_{v,\delta}^{\mathrm{m}+1}f(\mathcal{Z})}{\mathcal{Z}}\right)^k\left(\frac{\mathrm{I}_{v,\delta}^{\mathrm{m}}f(\mathcal{Z})}{\mathrm{I}_{v,\delta}^{\mathrm{m}+1}f(\mathcal{Z})}-1\right),$$

thus, the subordination (3.7)is equivalent to

$$\mathrm{p}(\mathcal{Z}) + \frac{\lambda}{k}\mathcal{Z}\mathrm{p}'(\mathcal{Z}) \prec \mathrm{M}(\mathcal{Z}) + \frac{\lambda}{k}\mathcal{Z}\mathrm{M}'(\mathcal{Z}).$$

Applying Lemma (2.1) with $=\frac{\lambda}{k}$,

the proof of Theorem 3.5 is complete.

By using $\delta = 1$ in Theorem 3.5, we obtain the following corollary:

Corollary (3.6): Let $\mathrm{M}(\mathcal{Z})$ be univalent function in $\mathcal{K}$ with $\lambda \in \mathbb{C}^*, \mathrm{M}(0)=1, \kappa > 0$ and suppose that (3.1) holds. If $f(\mathcal{Z}) \in \mathfrak{D}$ satisfies the following subordination:

$\Psi_1(\kappa,\lambda,v,1,\delta) \prec \mathrm{M}(\mathcal{Z}) + \frac{\lambda}{k}\mathcal{Z}\mathrm{M}'(\mathcal{Z})$, where

$$\Psi_1(\kappa,\lambda,v,1,\delta) = (1-\lambda)(v+1)\left(\frac{\mathrm{I}_{v,1}^{\mathrm{m}+1}f(\mathcal{Z})}{\mathcal{Z}}\right)^k + \lambda(v+1)\left(\frac{\mathrm{I}_{v,1}^{\mathrm{m}+1}f(\mathcal{Z})}{\mathcal{Z}}\right)^k\left(\frac{\mathrm{I}_{v,1}^{\mathrm{m}}f(\mathcal{Z})}{\mathrm{I}_{v,1}^{\mathrm{m}+1}f(\mathcal{Z})}\right), \quad (3.10)$$

then $\left(\frac{\mathrm{I}_{v,1}^{\mathrm{m}+1}f(\mathcal{Z})}{\mathcal{Z}}\right)^k \prec \mathrm{M}(\mathcal{Z})$ and $\mathrm{M}(\mathcal{Z})$ is the best dominant.

The next theorem is a result concerning a differential superordination.

Theorem (3.7): Let $\mathrm{M}(\mathcal{Z})$ be convex univalent function in $\mathcal{K}$ with $\operatorname{Re}(\lambda) > 0, \mathrm{M}(0)=1, \lambda \in \mathbb{C}, \mathrm{k} > 0\ \delta > 0, v \in \mathrm{R}$ such that $v+\delta > 0$.

If $f(\mathcal{Z}) \in \mathfrak{D}$ such that $\left(\frac{\mathrm{F}_{v,\delta}^{\mathrm{m}}f(\mathcal{Z})}{\mathcal{Z}}\right)^{\mathrm{k}} \in \mathcal{Y}[\mathrm{M}(0),1] \cap \mathrm{Q}$,

and $\emptyset(\mathrm{m},\lambda,\kappa,v,\delta)$ is univalent function in $\mathcal{K}$ and satisfies the superordination

$$\mathrm{M}(\mathcal{Z}) + \frac{\lambda\delta}{\kappa(v+\delta)}\mathcal{Z}\mathrm{M}'(\mathcal{Z}) \prec \emptyset(\mathrm{m},\lambda,\kappa,v,\delta), \quad (3.11)$$

where $\emptyset(\mathrm{m},\lambda,\kappa,v,\delta)$ is given by(3.3), then

$\mathrm{M}(\mathcal{Z}) \prec \left(\frac{\mathrm{F}_{v,\delta}^{\mathrm{m}}f(\mathcal{Z})}{\mathcal{Z}}\right)^k$ and $\mathrm{M}(\mathcal{Z})$ is the best subordinant.

Proof: Let $\mathrm{p}(\mathcal{Z})$ be given by (3.4) and proceeding as in the proof of Theorem(3.1), the superordination(3.11) becomes

$$\mathrm{M}(\mathcal{Z}) + \frac{\lambda\delta}{\kappa(v+\delta)}\mathcal{Z}\mathrm{M}'(\mathcal{Z}) \prec \mathrm{p}(\mathcal{Z}) + \frac{\lambda\delta}{\kappa(v+\delta)}\mathcal{Z}\mathrm{p}'(\mathcal{Z}).$$

The proof of the theorem follows by an application of Lemma 2.2.
By using m=0 in Theorem(3.7) , we obtain the following corollary .

Corollary (3.8): Let $\mathrm{M}(\mathcal{Z})$ be a convex univalent function in $\mathcal{K}$ with $\operatorname{Re}(\lambda) > 0, \mathrm{M}(0) = 1, \lambda \in \mathbb{C}, \mathrm{k} > 0\ \delta > 0, v \in \mathrm{R}$ such that $v + \delta > 0$.
If $f(\mathcal{Z}) \in \mathfrak{D}$, where $\left(\frac{f(\mathcal{Z})}{\mathcal{Z}}\right)^{\mathrm{k}} \in \mathcal{Y}[\mathrm{M}(0), 1] \cap \mathrm{Q}$ and $\emptyset_1(0, \lambda, \kappa, v, \delta)$is univalent function in $\mathcal{K}$ and satisfies the following superordination
$\mathrm{M}(\mathcal{Z}) + \frac{\lambda\delta}{\kappa(v+\delta)} \mathcal{Z}\mathrm{M}'(\mathcal{Z}) \prec \emptyset_1(0, \lambda, \kappa, v, \delta)$,where $\emptyset(\mathrm{m}, \lambda, \kappa, v, \delta)$ is given by (3.3), then $\mathrm{M}(\mathcal{Z}) \prec \left(\frac{f(\mathcal{Z})}{\mathcal{Z}}\right)^{k}$ and $\mathrm{M}(\mathcal{Z})$ is the best subordinant .
We obtain the following corollary on Taking $\delta = 1$ in Theorem (3.7).

Corollary (3.9): Let $\mathrm{M}(\mathcal{Z})$ be a convex univalent function in $\mathcal{K}$ and $\operatorname{Re}(\lambda) > 0, \mathrm{M}(0) = 1, \lambda \in \mathbb{C}, \mathrm{k} > 0, v > 0$. If $f(\mathcal{Z}) \in \mathfrak{D}$, such that
$\left(\frac{\mathrm{F}_{v,1}^{\mathrm{m}} f(\mathcal{Z})}{\mathcal{Z}}\right)^{\mathrm{k}} \in \mathcal{Y}[\mathrm{M}(0), 1] \cap \mathrm{Q}$ and $\emptyset_3(\mathrm{m}, \lambda, \kappa, v, 1)$is univalent function in $\mathcal{K}$and satisfies the following superordination:
$\mathrm{M}(\mathcal{Z}) + \frac{\lambda}{\kappa(v+1)} \mathcal{Z}\mathrm{M}'(\mathcal{Z}) \prec \emptyset_3(\mathrm{m}, \lambda, \kappa, v, 1)$,
where $\emptyset_3(\mathrm{m}, \lambda, \kappa, v, 1) = (1-\lambda)\left(\frac{\mathrm{F}_{v,1}^{\mathrm{m}} f(\mathcal{Z})}{\mathcal{Z}}\right)^{k} + \lambda\left(\frac{\mathrm{F}_{v,1}^{\mathrm{m}} f(\mathcal{Z})}{\mathcal{Z}}\right)^{k}\left(\frac{\mathrm{F}_{v,1}^{\mathrm{m}+1} f(\mathcal{Z})}{\mathrm{F}_{v,1}^{\mathrm{m}} f(\mathcal{Z})}\right)$,
then
$\mathrm{M}(\mathcal{Z}) \prec \left(\frac{\mathrm{F}_{v,1}^{\mathrm{m}} f(\mathcal{Z})}{\mathcal{Z}}\right)^{k}$ and $\mathrm{M}(\mathcal{Z})$ is the best subordinant.

Theorem (3.10) : Let $\mathrm{M}(\mathcal{Z})$ be a convex univalent function in $\mathcal{K}$ with $\operatorname{Re}(\lambda) > 0, \mathrm{M}(0) = 1, \lambda \in \mathbb{C}, \mathrm{k} > 0\ \delta > 0, v \in \mathrm{R}, v + \delta > 0$.
If $f(\mathcal{Z}) \in \mathfrak{D}$, such that
$\left(\frac{\mathrm{I}_{v,\delta}^{\mathrm{m}+1} f(\mathcal{Z})}{\mathcal{Z}}\right)^{\mathrm{k}} \in \mathcal{Y}[\mathrm{M}(0), 1] \cap \mathrm{Q}$,
and $\Psi(k, \mathrm{m}, v, \delta, \lambda)$is univalent function in $\mathcal{K}$ and satisfies the superordination, $\mathrm{M}(\mathcal{Z}) + \frac{\lambda}{\kappa} \mathcal{Z}\mathrm{M}'(\mathcal{Z}) \prec \Psi(k, \mathrm{m}, v, \delta, \lambda)$ and $\Psi(k, \mathrm{m}, v, \delta, \lambda)$ is given by (3.8). Then
$\mathrm{M}(\mathcal{Z}) \prec \left(\frac{\mathrm{I}_{v,\delta}^{\mathrm{m}+1} f(\mathcal{Z})}{\mathcal{Z}}\right)^{k}$, and $\mathrm{M}(\mathcal{Z})$ is the best subordinant.
Putting $\delta = 1$ in Theorem(3.10), we get the following corollary

Corollary (3.11): Let M be a convex univalent function in $\mathcal{K}$ with $\mathrm{k} > 0, \mathrm{M}(0) = 1, \lambda \in \mathbb{C}$, $\operatorname{Re}(\lambda) > 0, v \in \mathrm{R}$. If $f(\mathcal{Z}) \in \mathfrak{D}$, such that
$\left(\frac{\mathrm{I}_{v,1}^{\mathrm{m}+1} f(\mathcal{Z})}{\mathcal{Z}}\right)^{\mathrm{k}} \in \mathcal{Y}[\mathrm{M}(0), 1] \cap \mathrm{Q}$ and $\Psi_1(\kappa, \mathrm{m}, v, 1, \lambda)$ is univalent function in $\mathcal{K}$ and satisfies the superordination
$\mathrm{M}(\mathcal{Z}) + \frac{\lambda}{\kappa} \mathcal{Z}\mathrm{M}'(\mathcal{Z}) \prec \Psi_1(\kappa, \mathrm{m}, v, 1, \lambda)$, and $\Psi_1(\kappa, \mathrm{m}, v, 1, \lambda)$ is given by (3.10). Then
$\mathrm{M}(\mathcal{Z}) \prec \left(\frac{\mathrm{I}_{v,1}^{\mathrm{m}} f(\mathcal{Z})}{\mathcal{Z}}\right)^{k}$, and $\mathrm{M}(\mathcal{Z})$ is the best subordinan.
Combining the results of Theorem (3.1) and Theorem (3.7), we obtain the following sandwich theorem.

Theorem (3.12): Let M_1 and M_2 be convex function*s* in $\mathcal{K}$ with
$\lambda \in \mathbb{C}, \mathrm{M}_1(0) = \mathrm{M}_2(0) = 1, \operatorname{Re}(\lambda) > 0, \mathrm{k} > 0, \delta > 0$ and $v \in \mathrm{R}$, such that $v + \delta > 0$. If $f(\mathcal{Z}) \in \mathfrak{D}$ such that
$\left(\frac{\mathrm{F}_{v,\delta}^{\mathrm{m}} f(\mathcal{Z})}{\mathcal{Z}}\right)^{\mathrm{k}} \in \mathcal{Y}[\mathrm{M}(0),1] \cap \mathrm{Q}$, and $\emptyset(\mathrm{m}, \lambda, \kappa, v, \delta)$is univalent in $\mathcal{K}$ and satisfies:
$\mathrm{M}_1(\mathcal{Z}) + \frac{\lambda\delta}{\kappa(v+\delta)} \mathcal{Z}\mathrm{M}'_1(\mathcal{Z}) \prec \emptyset(\mathrm{m}, \lambda, \kappa, v, \delta) \prec \mathrm{M}_2(\mathcal{Z}) + \frac{\lambda\delta}{\kappa(v+\delta)} \mathcal{Z}\,\mathrm{M}'_2(\mathcal{Z})$, where $\emptyset(\mathrm{m}, \lambda, \kappa, v, \delta)$is given by(3.3), then

$M_1(\mathcal{Z}) \prec \left(\frac{F^{m}_{v,\delta} f(\mathcal{Z})}{\mathcal{Z}}\right)^{k} \prec M_2(\mathcal{Z}),$

and M_1 and M_2 are the best subordinant and the best dominant respectively .

Combining the results of Theorem (3.5) and Theorem (3.10),we obtain the following sandwich theorem.

Theorem (3.13): Let M_1 and M_2 be a convex univalent functions in $\mathcal{K}$ and $k > 0, M_1(0) = M_2(0) = 1$, $\lambda \in \mathbb{C}$ and $\delta > 0$, $\mathrm{Re}(\lambda) > 0, v \in \mathrm{R}$, such that $v + \delta > 0$. If $f(\mathcal{Z}) \in \mathfrak{D}$ such that

$$\left(\frac{I^{m+1}_{v,\delta} f(\mathcal{Z})}{\mathcal{Z}}\right)^{k} \in \mathcal{Y}\,[M(0),1] \cap Q,$$

and $\Psi(m, \lambda, \kappa, v, \delta)$ is univalent in $\mathcal{K}$ and satisfies $M_1(\mathcal{Z}) + \frac{\lambda}{\kappa}\mathcal{Z}M'_1(\mathcal{Z}) \prec \Psi(k, m, v, \delta, \lambda) \prec M_2(\mathcal{Z}) + \frac{\lambda}{\kappa}\mathcal{Z}M'_2(\mathcal{Z})$, where $\Psi(k, m, v, \delta, \lambda)$ is given by (3.8), then

$$M_1(\mathcal{Z}) \prec \left(\frac{I^{m+1}_{v,\delta} f(\mathcal{Z})}{\mathcal{Z}}\right) \prec M_2(\mathcal{Z}),$$

and M_1 and M_2 are the best subordinant and the best dominant respectively.

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