Shaheen Iraqi Journal of Science, 2021, Vol. 62, No. 4, pp: 1334-1343
DOI: 10.24996/ijs.2021.62.4.29

ISSN: 0067-2904

On Additivity of Jordan Higher Mappings on Generalized Matrix Algebras

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Received: 20/10/2020 Accepted: 18/1/2021

Abstract
In this article, the additivity of higher multiplicative mappings, i.e., Jordan mappings, on generalized matrix algebras are studied. Also, the definition of Jordan higher triple product homomorphism is introduced and its additivity on generalized matrix algebras is studied.

Keywords: Additive map, Multiplicative Map, Jordan Map, Homomorphism, Generalized Matrix algebra.

حول جمعية تطبيقات جوردان العليا على تعميمات جبور المصفوفات

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الخلاصة
في هذا البحث ،جمعية التطبيقات الضربية العليا ،تطبيقات جوردان على تعميم جبور المصفوفات قد درست وكذلك تعريف تشاكل جوردان الثلاثي الاعلى قد قدم ودرست جمعيته على تعميم جبور المصفوفات.

Introduction

Suppose that a ring $\mathcal{R}$ has identity and commutative, and $\mathcal{A}$ and $\mathcal{B}$ are associative algebras on$\mathcal{R}$. Let $\mathcal{M}$ be $(\mathcal{A},\mathcal{B})$-bi module and $\mathcal{N}$ be $(\mathcal{B},\mathcal{A})$- bi module. $\emptyset: \mathcal{M} \otimes_{\mathcal{B}} \mathcal{N} \rightarrow \mathcal{A}$ and $\psi: \mathcal{N} \otimes_{A} \mathcal{M} \rightarrow \mathcal{B}$ are two bi module homomorphisms, where $\mathcal{M}.\mathcal{N} = \emptyset(\mathcal{M} \otimes \mathcal{N})$ and $\mathcal{N}.\mathcal{M} = \psi(\mathcal{N} \otimes \mathcal{M})$, satisfying $(\mathcal{M}.\mathcal{N}).\mathcal{M}' = \mathcal{M}.(\mathcal{N}.\mathcal{M}')$ and $(\mathcal{N}.\mathcal{M}).\mathcal{N}' = \mathcal{N}.(\mathcal{M}.\mathcal{N}')$ for all $\mathcal{M},\mathcal{M}' \epsilon \mathcal{M}$ and $\mathcal{N},\mathcal{N}' \epsilon \mathcal{N}$.

$\mathrm{Mat}(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B}) = \{\begin{bmatrix} \mathcal{A} & m \\ n & \mathcal{B} \end{bmatrix} : A\epsilon\mathcal{A}, m\epsilon\mathcal{M}, n\epsilon\mathcal{N}, \mathcal{B}\epsilon\mathcal{B}\}$, with a usual matrix like multiplication, where either $\mathcal{M} \neq 0$ or $\mathcal{N} \neq 0$ is a generalized matrix algebra. $\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B}) = \{\begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{bmatrix} : A\epsilon\mathcal{A}, m\epsilon\mathcal{M}, \mathcal{B}\epsilon\mathcal{B}\}$, if $\mathcal{M}$ is faithful as a left $\mathcal{A}$- module (resp.,right $\mathcal{B}$-module) and $\mathcal{N} = \{0\}$ is said to be a Triangular algebra [1]. See Cheng and Jing [2] for examples on nest algebras and block upper Triangular matrix algebras . For more examples, see Xiao and Wei [3].

Let$\sigma: \mathcal{A} \rightarrow B$ and $\mathcal{A}, B \in \mathcal{A}$.

1. σ is called multiplicative if $\sigma(\mathcal{A}\mathcal{B}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$.
2. σ is said to be a Jordan map if $\sigma(\mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A}) = \sigma(\mathcal{A})\sigma(\mathcal{B}) + \sigma(\mathcal{B})\sigma(\mathcal{A})$.
3. σ is said to be a Jordan Triple Product Homomorphism if
$\sigma(\mathcal{A}\mathcal{B}\mathcal{A}) = \sigma(\mathcal{A})\sigma(\mathcal{B})\sigma(\mathcal{A})$[1].

Characterizing the interrelation between the additive structures and the multiplicative of algebra or a ring is an interesting topic. This relation was first studied by Martindale [4] on a prime ring with some condition, and in [2,5,6] on operator algebras. Additivity of maps which are multiplicative

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with respect to product on operator algebras, such that Jordan -triple product homomorphism, were investigated by Lu [7]. Ling and Lu [8] studied Jordan maps on nest algebras. They proved that any bijective Jordan map on standard sub algebra of nest algebra is an additive, which was extended to surjective Jordan pair maps of Triangular algebra by Ji [9] . Cheng and Jing [2] studied the linearity of multiplicative(Jordan)(triple) bijective map and elementary surjective map on triangular algebra. Li and Xiao [1], under some conditions, extended the results of Ji [9] to generalized matrix algebras. Shaheen [10] introduced the definition of higher multiplicative mapping and Jordan higher mapping and studied their additivity property on triangular matrix ring . In this article, we study the additivity property of them on generalized matrix algebras .

Li and Jing [11] studied the additive property of Jordan triple product homomorphism of prime ring . Kuzma [12] described Jordan triple product homomorphisms of matrix algebra . Motivated by the results of Li. and Jing [11] and Li. and Xiao [1], the authors in Kim and Park [13] studied the additivity of Jordan -Triple product homomorphism from generalized matrix algebras. For more results about Jordan triple homomorphism, see [14, 15]. In this article , the definition of Jordan higher triple product homomorphism is introduced and its additivity on generalized matrix algebra is studied. We use techniques similar to those used by Lu [7] and Kim and Park [13].

Throughout this article, let $\mathbb{G}= Mat(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$. We set

$\mathbb{G}_{11} = \{\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} : A\epsilon\mathcal{A}\}$, $\mathbb{G}_{12} = \{\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} : M\epsilon\mathcal{M}\}$,

$\mathbb{G}_{21} = \{\begin{bmatrix} 0 & 0 \\ N & 0 \end{bmatrix} : N\epsilon\mathcal{N}\}$, $\mathbb{G}_{22} = \{\begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} : B\epsilon\mathcal{B}\}$

Then $A = A_{11} + A_{12} + A_{21} + A_{22}\epsilon G$, where $A_{ij}\epsilon\mathbb{G}_{ij}$ for $1 \le i,j \le 2$.

2-Additivity of Higher Multiplicative Mappings on Generalized Matrix Algebra

Definition 2.1 [10]. The family $\sigma = \{\sigma_s\}_{s\epsilon N}$ of mappings on a ring R is said to be

1- A higher multiplicative mappings if $\sigma_t(\mathcal{A}\mathcal{B}) = \sum_{k=1}^{t} \sigma_k(\mathcal{A})\sigma_k(\mathcal{B})$,
such that $\sigma_k(\mathcal{A})\sigma_t(\mathcal{B}) = 0 \ \forall k \neq t$.

2- A Jordan higher mappings if $\sigma_t(\mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A}) = \sum_{k=1}^{t} \sigma_k(\mathcal{A})\sigma_k(\mathcal{B}) + \sigma_k(\mathcal{B})\sigma_k(\mathcal{A})$.

Theorem 2.2

Let $\mathbb{G}$ be the generalized matrix algebra that satisfies the conditions:

1. $\forall\ a\epsilon\mathcal{A}, a\mathcal{A} = 0$ or $= 0 \Longrightarrow a = 0$.
2. $\forall\ b\epsilon\mathcal{B}, b\mathcal{B} = 0$ or $\mathcal{B}b = 0 \Longrightarrow b = 0$.
3. for $m\epsilon\mathcal{M}, \mathcal{A}m = 0$ or $\mathrm{m}\mathcal{B} = 0 \Longrightarrow m = 0$.
4. $\forall\ n\epsilon N, \mathcal{B}n = 0$ or $\mathrm{n}\mathcal{A} = 0 \Longrightarrow n = 0$.

Then any bijective higher multiplicative mapping $\emptyset$ from $\mathbb{G}$ onto arbitrary ring $\mathfrak{R}'$ is additive.

We shall introduce the following Lemmas to prove Theorem 2.2. From now on, $\emptyset$ satisfies the hypothesis of Theorem 2.2.

Lemma 2.3: $\emptyset_n(0) = 0$.

Proof: Since $\emptyset_i$ is surjective. Then $\exists\ A\epsilon\mathbb{G}, \emptyset_i(A) = 0$
$\emptyset_n(0) = \emptyset_n(0A) = \sum_{i=1}^{n} \emptyset_i(0)\emptyset_i(A)=0$

Lemma2.4: For
$\emptyset_n(A_{11} + A_{12} + A_{22}) = \emptyset_n(A_{11}) + \emptyset_n(A_{12} + A_{22}) = \emptyset_n(A_{11} + A_{12}) + \emptyset_n(A_{22})$,
for every $A_{ij}\epsilon\mathbb{G}_{ij}$.

Proof: Let $s\epsilon\mathbb{G}$ be an element such that
$\emptyset_n(s) = \emptyset_n(A_{11}) + \emptyset_n(A_{12} + A_{22})$. Then for every $t_{22} \in \mathbb{G}_{22}$

$$\begin{aligned}
\emptyset_n(t_{22}\, s) &= \sum_{i=1}^{n} \emptyset_i(t_{22})\emptyset_i(s) \\
&= \sum_{i=1}^{n} \emptyset_i(t_{22})\,(\emptyset_i(A_{11}) + \emptyset_i(A_{12} + A_{22})) \\
&= \sum_{i=1}^{n} \emptyset_i(t_{22})\,\emptyset_i(A_{11}) + \emptyset_i(t_{22})\emptyset_i(A_{12} + A_{22}) \\
&= \sum_{i=1}^{n} \emptyset_i(t_{22})\,\emptyset_i(A_{22}) = \emptyset_n(t_{22}\, A_{22})
\end{aligned}$$

Then $t_{22}\, s = t_{22}\, A_{22}$, hence $s_{21} = 0$ and $s_{22} = A_{22}$. For $c_{11} \in \mathbb{G}_{11}$, we have $\emptyset_n(sc_{11}) = \sum_{i=1}^{n} \emptyset_i(s)\emptyset_i(c_{11}) = \sum_{i=1}^{n}\big(\emptyset_i(A_{11}) + \emptyset_i(A_{12} + A_{22})\big)\emptyset_i(c_{11})$
$=\emptyset_n(A_{11})\emptyset_n(c_{11}) + \emptyset_n(A_{12} + A_{22})\emptyset_n(c_{11})$
$=\emptyset_n(A_{11})\emptyset_n(c_{11})$.

Then, $sc_{11}=A_{11}c_{11}$ and hence $c_{11} = A_{11}$. In order to determined s_{12} , we have

$\emptyset_n(c_{11}\ s\ t_{22}) = \sum_{i=1}^{n} \emptyset_i(c_{11})\emptyset_i(s)\ \emptyset_i(\ t_{22})$.
$=\sum_{i=1}^{n} \emptyset_i(c_{11})\big(\emptyset_i(A_{11}) + \emptyset_i(A_{12} + A_{22})\big)\emptyset_i(\ t_{22})$
$=\sum_{i=1}^{n} \emptyset_i(c_{11})\emptyset_i(A_{11})\emptyset_i(\ t_{22}) + \sum_{i=1}^{n} \emptyset_i(c_{11})\emptyset_i(A_{12} + A_{22})\emptyset_i(\ t_{22})$
$=\emptyset_n(c_{11}\ A_{12}\ t_{22})$.
It follows that $c_{11}\ s\ t_{22} = c_{11}\ A_{12}\ t_{22}$, which implies that $s_{12} = A_{12}$.
It follows that $\emptyset_n(A_{11} + A_{12} + A_{22}) = \emptyset_n(A_{11}) + \emptyset_n(A_{12} + A_{22})$.
The second claim is proven similarly.
Corollary 2.5: For $A_{ij} \in \mathbb{G}_{ij}$, we have
1. $\emptyset_n(A_{11} + A_{12}) = \emptyset_n(A_{11}) + \emptyset_n(A_{12})$.
2. $\emptyset_n(A_{12} + A_{22}) = \emptyset_n(A_{12}) + \emptyset_n(A_{22})$.
As in Lemma 2.4, we get
Lemma 2.6: For $A_{11} \in \mathbb{G}_{11}$, $e_{21} \in \mathbb{G}_{21}$, $A_{22} \in \mathbb{G}_{22}$, and we have
$\emptyset_n(A_{11} + e_{21} + A_{22}) = \emptyset_n(A_{11}) + \emptyset_n(e_{21} + A_{22}\ ) = \emptyset_n(A_{11} + e_{21}) + \emptyset_n(A_{22}\ )$.
Corollary 2.7: For $A_{ij} \in \mathbb{G}_{ij}$, $e_{21} \in \mathbb{G}_{21}$, we have
1. $\emptyset_n(A_{11} + e_{21}) = \emptyset_n(A_{11}) + \emptyset_n(e_{21})$.
2. $\emptyset_n(e_{21} + A_{22}) = \emptyset_n(e_{21}) + \emptyset_n(A_{22})$.
Lemma 2.8: For $A_{11} \in \mathbb{G}_{11}, A_{12}, c_{12} \in \mathbb{G}_{12}\ e_{21},\ f_{21} \in \mathbb{G}_{21}$, and $A_{22} \in \mathbb{G}_{22}$.
The following two identities hold:
1.$\emptyset_n(A_{11}A_{12} + c_{12}A_{22}\ ) = \emptyset_n(A_{11}A_{12}\ ) + \emptyset_n(c_{12}A_{22})$
2.$\emptyset_n(A_{22}e_{21} +\ f_{21}A_{11}) = \emptyset_n(A_{22}e_{21}) + \emptyset_n(\ f_{21}A_{11})$
Proof :1- By Corollary 2.5 , we have

$$\emptyset_n(A_{11}A_{12} + c_{12}A_{22}\ ) = \emptyset_n((A_{11}+c_{12})(A_{12}+A_{22}\ ))$$

$= \sum_{i=1}^{n} \emptyset_i(A_{11}+c_{12})\emptyset_i(A_{12}+A_{22})$
$=\emptyset_n(A_{11}A_{12}\ ) + \emptyset_n(c_{12}A_{22})$.
By Corollary 2.7, we get (2).
Proof of Theorem 2.2
1- $\emptyset$ is additive on $\mathbb{G}_{12}, \mathbb{G}_{21}$.
Suppose that s$\in \mathbb{G}$, such that $\emptyset_n(s) = \emptyset_n(A_{12}) + \emptyset_n(B_{12})$.
For $c_{11} \in \mathbb{G}_{11}$and $t_{22} \in \mathbb{G}_{22}$, we have

$$\emptyset_n(c_{11}st_{22}) = \sum_{i=1}^{n} \emptyset_i(c_{11})\emptyset_i(s)\ \emptyset_i(\ t_{22})$$

$=\sum_{i=1}^{n} \emptyset_i(c_{11})(\emptyset_i(A_{12}) + \emptyset_i(B_{12}))\ \emptyset_i(\ t_{22})$
$=\sum_{i=1}^{n} \emptyset_i(c_{11})\emptyset_i(A_{12})\ \emptyset_i(\ t_{22}) + \sum_{i=1}^{n} \emptyset_i(c_{11})\emptyset_i(B_{12})\ \emptyset_i(\ t_{22})$
$=\emptyset_n(c_{11}A_{12}t_{22} + c_{11}B_{12}t_{22})$.
Then $c_{11}st_{22}$=$c_{11}A_{12}t_{22} + c_{11}B_{12}t_{22}$.
It follows that s_{12}=$A_{12} + B_{12}$.
Moreover, $\emptyset_n(sc_{11})$=$\sum_{i=1}^{n} \emptyset_i(s)\ \emptyset_i(c_{11})$=$\sum_{i=1}^{n}(\emptyset_i(A_{12}) + \emptyset_i(B_{12}))\ \emptyset_i(c_{11})$
$=\sum_{i=1}^{n} \emptyset_i(A_{12})\ \emptyset_i(c_{11}) + \sum_{i=1}^{n} \emptyset_i(B_{12})\ \emptyset_i(c_{11})$
$=\emptyset_n(A_{12}c_{11}) + \emptyset_n(B_{12}c_{11})$=0.
Then s_{11}=0 and s_{21}=0.
By considering $\emptyset_n(t_{22}s)$, we obtain s_{22}=0.
Hence, we get that $\emptyset$ is additive on $\mathbb{G}_{12}$. By the same way, we can prove that $\emptyset$ is additive on $\mathbb{G}_{21}$.
2- We must prove that $\emptyset$ is additive on $\mathbb{G}_{11}, \mathbb{G}_{22}$.
Suppose that s$\in \mathbb{G}$ such that $\emptyset_n(s) = \emptyset_n(A_{11}) + \emptyset_n(B_{11})$.
For $c_{11} \in \mathbb{G}_{11}$and $t_{22} \in \mathbb{G}_{22}$, we have $\emptyset_n(t_{22}s) = 0$ and $\emptyset_n(c_{11}st_{22}) = 0$, hence $s_{21} = 0$ and $s_{22} = 0, s_{12} = 0$.
On the other hand, for $u_{12} \in \mathbb{G}_{12}$, it follows that
$\emptyset_n(su_{12}) = \sum_{i=1}^{n} \emptyset_i(s)\ \emptyset_i(u_{12})$
$=\sum_{i=1}^{n}(\emptyset_i(A_{11}) + \emptyset_i(B_{11}))\ \emptyset_i(u_{12})$

$=\sum_{i=1}^{n} \emptyset_i(A_{11})\emptyset_i(u_{12}) + \emptyset_i(B_{11})\, \emptyset_i(u_{12})$
$=\emptyset_n(A_{11}u_{12} + B_{11}u_{12})$.
$su_{12} = A_{11}u_{12} + B_{11}u_{12}$
sos_{11}=A_{11}+B_{11}, because $\mathcal{M}$ is faithful as a left A-module .
Hence, $\emptyset$ is additive on $\mathbb{G}_{11}$. By the same way, we can prove that it is additive on $\mathbb{G}_{22}$.
Now, let $A = A_{11} + A_{12} + A_{21} + A_{22}$ and $B = B_{11} + B_{12} + B_{21} + B_{22}$ be elements of $\mathbb{G}$. Then by assertion lemmas ,we have
$\emptyset_n(A + B) = \emptyset_n(A_{11} + B_{11} + A_{12} + B_{12} + A_{21} + B_{21} + A_{22} + B_{22}$
$= \emptyset_n(A_{11} + B_{11}) + \emptyset_n(A_{12} + B_{12}) + \emptyset_n(A_{21} + B_{21})\emptyset_n(A_{22} + B_{22})$.
$= \emptyset_n(A_{11}) + \emptyset_n(B_{11}) + \emptyset_n(A_{11}) + \emptyset_n(B_{12}) + \emptyset_n(A_{21}) + \emptyset_n(B_{21}) + \emptyset_n(A_{22}) + \emptyset_n(B_{22})$
$= \emptyset_n(A) + \emptyset_n(B)$.
Corollary 2.9: Let $\mathcal{A}$ be an algebra (unital, not necessarily prime) over a commutative ring $\mathcal{R}$. Then, any bijective higher multiplicative mapping from $M_n(\mathcal{A})$, $n \geq 2$, onto the arbitrary ring $\mathcal{R}'$ is additive
Proof: Clearly, we achieve the result when $\mathcal{N}$=0.

3- Additivity of Jordan Higher Maps on Generalized Matrix Algebra

In this section, we study the additivity of Jordan higher maps on generalized matrix algebra.

Theorem 3.1

Let $\mathbb{G}$ be the generalized matrix algebra that satisfies the conditions:
(1) for $a\epsilon\mathcal{A}$, if $ax + xa = 0$, then $\forall x\epsilon\mathcal{A} \Longrightarrow a = 0$.
(2) for $b\epsilon\mathcal{B}, b\mathcal{B} + y\mathcal{B} = 0, \forall y\epsilon\mathcal{B} \Longrightarrow b = 0$.
(3) for $m\epsilon M, Am = 0$, or m$\mathcal{B} = 0 \Longrightarrow m = 0$.
(4) for $n\epsilon N, \mathcal{B}n = 0$, or n$\mathcal{A} = 0 \Longrightarrow n = 0$.
Then any bijective Jordan higher mapping $\emptyset$ from $\mathbb{G}$ onto ring $\mathfrak{R}'$ is additive.
We shall introduce the following Lemmas to prove theorem 3.1. From now on, $\emptyset$ satisfies the hypothesis of Theorem 3.1.
Lemma 3.2: $\emptyset_n(0) = 0$.
Proof: Since $\emptyset_i$ is surjective, then $\exists\ A\epsilon\mathbb{G}, \emptyset_i(A) = 0$,
$\emptyset_n(0) = \emptyset_n(0A + A0) = \sum_{i=1}^{n} \emptyset_i(0)\emptyset_i(A) + \emptyset_i(A)\emptyset_i(0)$=0.
Lemma 3.3: For
$\emptyset_n(A_{11} + A_{12} + A_{22}) = \emptyset_n(A_{11}) + \emptyset_n(A_{12} + A_{22}) = \emptyset_n(A_{11} + A_{12}) + \emptyset_n(A_{22})$,
for every $A_{ij}\epsilon\mathbb{G}_{ij}$.
Proof: Let $s\epsilon\mathbb{G}$ be an element such that
$\emptyset_n(s) = \emptyset_n(A_{11}) + \emptyset_n(A_{12} + A_{22})$. Then for every $t_{22} \in \mathbb{G}_{22}$,
$\emptyset_n(s\, t_{22} +\ t_{22}\, s) = \sum_{i=1}^{n} \emptyset_i(\ t_{22})\emptyset_i(s) + \emptyset_i(s)\ \emptyset_i(\ t_{22})$
$= \sum_{i=1}^{n} \emptyset_i(\ t_{22})\,(\emptyset_i(A_{11}) + \emptyset_i(A_{12} + A_{22}))$+ $(\emptyset_i(A_{11}) + \emptyset_i(A_{12} + A_{22}))\ \emptyset_i(\ t_{22})$
$=\emptyset_n((A_{12} + A_{22})\ t_{22} +\ t_{22}\quad A_{22})$
Then, $s\, t_{22} +\ t_{22}\, s = ((A_{12} + A_{22})\ t_{22} +\ t_{22}\quad A_{22}$, hence
$s_{12}\ t_{22} = A_{12}\ t_{22}$ and $t_{22}\ s_{21} = 0$ and $s\ t_{22} +\ t_{22}\ s =\ A_{22}\ t_{22} +\ t_{22}\ A_{22}$.
So, $s_{12} = A_{12},\ s_{21} = 0$, and $s_{22} = A_{22}$.
By the same way,
$\emptyset_n(s\, t_{11} +\ t_{11}\ s) = \emptyset_n(A_{11}\ t_{11} +\ t_{11}\ A_{11}) + \emptyset_n(\ t_{11}(A_{12} + A_{22}) + (A_{12} + A_{22})\ t_{11})$
$=\ \emptyset_n(A_{11}\ t_{11} +\ t_{11}\ A_{11}) + \emptyset_n(\ t_{11}A_{12})$.
Therefore, we have
$\emptyset_n((s\, t_{11} +\ t_{11}\ s)\quad u_{12}\quad +u_{12}(s\, t_{11} +\ t_{11}\ s))$= $\emptyset_n((A_{11}\ t_{11} +\ t_{11}\ A_{11})\quad u_{12}\quad +u_{12}(A_{11}\ t_{11} + t_{11}\ A_{11}))$+ $\emptyset_n((\ t_{11}A_{12}u_{12} + u_{12}\ t_{11}\ A_{12})$
$=\ \ \emptyset_n\big((A_{11}\ t_{11} +\ t_{11}\ A_{11})u_{12}\big)$.
Thus, $(s\ t_{11} +\ t_{11}\ s)\ u_{12}$ +$u_{12}(s\ t_{11} +\ t_{11}\ s)$=$(A_{11}\ t_{11} +\ t_{11}\ A_{11})u_{12}$.
It follows that s_{21}=0 and that $s\ t_{11}\ u_{12}$+ $t_{11}\ su_{12}$=$(A_{11}\ t_{11} +\ t_{11}\ A_{11})u_{12}$.
By the faithfully $\mathcal{M}$, we have
$s\ t_{11}$ + $t_{11}\ s$=$A_{11}\ t_{11} +\ t_{11}\ A_{11}$ and hence s_{11}=A_{11}.
Corollary 3.4 : For $A_{ij} \in \mathbb{G}_{ij}$, we have
1. $\emptyset_n(A_{11} + A_{12}) = \emptyset_n(A_{11}) + \emptyset_n(A_{12})$.
2. $\emptyset_n(A_{12} + A_{22}) = \emptyset_n(A_{12}) + \emptyset_n(A_{22})$.

By the same way in Lemma 3.3, we get

Lemma 3.5: For $A_{11} \in \mathbb{G}_{11}, e_{21} \in \mathbb{G}_{21}$, $A_{22} \in \mathbb{G}_{22}$, and we have

$$\emptyset_n(A_{11} + e_{21} + A_{22}) = \emptyset_n(A_{11}) + \emptyset_n(e_{21} + A_{22}) = \emptyset_n(A_{11} + e_{21}) + \emptyset_n(A_{22}).$$

Corollary 3.6: For $A_{ij} \in \mathbb{G}_{ij}, e_{21} \in \mathbb{G}_{21}$, we have

1- $\emptyset_n(A_{11} + e_{21}) = \emptyset_n(A_{11}) + \emptyset_n(e_{21})$.

2- $\emptyset_n(e_{21} + A_{22}) = \emptyset_n(e_{21}) + \emptyset_n(A_{22})$.

Lemma 3.7: For $A_{11} \in \mathbb{G}_{11}, A_{12}, c_{12} \in \mathbb{G}_{12}\ e_{21},\ f_{21} \in \mathbb{G}_{21}$, and $A_{22} \in \mathbb{G}_{22}$.
The following two identities hold:

1- $\emptyset_n(A_{11}A_{12} + c_{12}A_{22}) = \emptyset_n(A_{11}A_{12}) + \emptyset_n(c_{12}A_{22})$.

2- $\emptyset_n(A_{22}e_{21} + f_{21}A_{11}) = \emptyset_n(A_{22}e_{21}) + \emptyset_n(f_{21}A_{11})$.

Proof : 2- we have

$\emptyset_n\big((A_{22}e_{21} + f_{21}A_{11})\big) = \emptyset_n((A_{22} + f_{21})(e_{21} + A_{11}) + (e_{21} + A_{11})(A_{22} + f_{21}))$
$=\emptyset_n(A_{22} + f_{21})\emptyset_n(e_{21} + A_{11}) + \emptyset_n(e_{21} + A_{11})\emptyset_n(A_{22} + f_{21})$
$=(\emptyset_n(A_{22}) + \emptyset_n(f_{21}))(\emptyset_n(e_{21}) + \emptyset_n(A_{11})) + (\emptyset_n(e_{21}) + \emptyset_n(A_{11}))(\emptyset_n(A_{22}) + \emptyset_n(f_{21}))$
$=\emptyset_n(A_{22}e_{21} + e_{21}A_{22}) + \emptyset_n(A_{22}A_{11} + A_{11}A_{22}) + \emptyset_n(f_{21}e_{21} + e_{21}f_{21}) + \emptyset_n(f_{21}A_{11} + A_{11}f_{21})$
$=\emptyset_n(A_{22}e_{21}) + \emptyset_n(f_{21}A_{11})$.

(1) is proven similarly .

Lemma 3.8:$\emptyset$ is additive on $\mathbb{G}_{12}, \mathbb{G}_{21}$.

Proof: Suppose that s$\in \mathbb{G}$ such that $\emptyset_n(s) = \emptyset_n(A_{21}) + \emptyset_n(B_{21})$.
For $c_{11} \in \mathbb{G}_{11}$and $t_{22} \in \mathbb{G}_{22}$, we have

$$\emptyset_n(sc_{11} + c_{11}s) = \sum_{i=1}^{n} \emptyset_i(s)\emptyset_i(c_{11}) + \emptyset_i(c_{11})\emptyset_i(s)$$

$=\sum_{i=1}^{n}\big(\emptyset_i(A_{21}) + \emptyset_i(B_{21})\big)\emptyset_i(c_{11}) + \emptyset_i(c_{11})\big(\emptyset_i(A_{21}) + \emptyset_i(B_{21})\big)$
$=\sum_{i=1}^{n} \emptyset_i(A_{21}c_{11} + c_{11}A_{21}) + \emptyset_i(B_{21}c_{11} + c_{11}B_{21})$
$=\emptyset_n(A_{21}c_{11}) + \emptyset_n(B_{21}c_{11})$

And from Lemma 3.7 that

$\emptyset_n(t_{22}(sc_{11} + c_{11}s) + (sc_{11} + c_{11}s)t_{22})= \emptyset_n(t_{22}A_{21}c_{11} + A_{21}c_{11}t_{22})+ \emptyset_n(t_{22}B_{21}c_{11} + B_{21}c_{11}t_{22})$
$= \emptyset_n(t_{22}A_{21}c_{11}) + \emptyset_n(t_{22}B_{21}c_{11})= \emptyset_n(t_{22}A_{21}c_{11} + t_{22}B_{21}c_{11})$

It follows that $t_{22}s_{21}c_{11} + c_{11}s_{12}t_{22} = t_{22}A_{21}c_{11} + t_{22}B_{21}c_{11}$.
Then, we get $s_{12} = 0$ and s_{21}=$A_{21} + B_{21}$. Moreover,
for $c_{11} \in \mathbb{G}_{11}$ and $u_{21} \in \mathbb{G}_{21}$, we have

$\emptyset_n(u_{21}(sc_{11} + c_{11}s) + (sc_{11} + c_{11}s)u_{21}) = \emptyset_n(u_{21}c_{11}A_{21} + c_{11}A_{21}u_{21})+ \quad \emptyset_n(u_{21}\ c_{11}B_{21} + c_{11}B_{21}u_{21})$ = 0

Then, $u_{21}sc_{11} + u_{21}c_{11}s + c_{11}su_{21} = 0$
It follows from $s_{12} = 0$ that $u_{21}sc_{11} + u_{21}c_{11}s = 0$.
By the faithful $\mathcal{N}$, then $s_{11}c_{11} + c_{11}s_{11}$= 0,
and hence s_{11}= 0.
Note that

$\emptyset_n(st_{22} + t_{22}s)= \emptyset_n(A_{21}t_{22} + t_{22}A_{21}) + \emptyset_n(B_{21}t_{22} + t_{22}B_{21})= \emptyset_n(t_{22}A_{21}) + \emptyset_n(t_{22}B_{21})$.

Therefore,

$\emptyset_n(u_{21}(st_{22} + t_{22}s) + (st_{22} + t_{22}s)u_{21})= \emptyset_n(t_{22}A_{21} + t_{22}A_{21}u_{21}) + \emptyset_n(u_{21}B_{21} + t_{22}B_{21})$ = 0

Thus $u_{21}st_{22} + st_{22}u_{21} + t_{22}su_{21}$=0. Then, we have from $s_{12} = 0$ that
$s_{22}t_{22}u_{21} + t_{22}s_{22}u_{21} = 0$.
By the faithful $\mathcal{N}$, we get $s_{22}t_{22} + t_{22}s_{22} = 0$,
and hence $s_{22} = 0$. Then, $\emptyset$ is additive on $\mathbb{G}_{21}$. The additivity of $\emptyset$ on $\mathbb{G}_{12}$is proved similarly.

Lemma 3.9: $\emptyset$ is additive on $\mathbb{G}_{11}$ and $\mathbb{G}_{22}$.
For $A_{11}, B_{11} \in \mathbb{G}_{11}$, let $\emptyset_n(s) = \emptyset_n(A_{11}) + \emptyset_n(B_{11})$ with s$\in \mathbb{G}$.
Then, for all $t_{22} \in \mathbb{G}_{22}$,

$\emptyset_n(st_{22} + t_{22}s) = \emptyset_n(A_{11}t_{22} + t_{22}A_{11}) + \emptyset_n(B_{11}t_{22} + t_{22}B_{11})$ = 0.

This implies that $st_{22} + t_{22}s = 0$. Then we have $s_{12} = 0, s_{21}$=0 and $s_{22} = 0$. At the same time , for $u_{12} \in \mathbb{G}_{12}$,
we have $\emptyset_n(u_{12}s + su_{12}) = \emptyset_n(u_{12}A_{11} + A_{11}u_{12}) + \emptyset_n(u_{12}B_{11} + B_{11}u_{12})$
$= \emptyset_n(A_{11}u_{12}) + \emptyset_n(B_{11}u_{12}) = \emptyset_n(A_{11}u_{12} + B_{11}u_{12})$.

Hence, $u_{12}s + su_{12} = A_{11}u_{12} + B_{11}u_{12}$.
Since $s_{12} = 0, s_{21} = 0$ and $s_{22} = 0$,
we have $s_{11}u_{12} = A_{11}u_{12} + B_{11}u_{12}$.
Then $s_{11} = A_{11} + B_{11}$
So, $\emptyset$ is additive on $\mathbb{G}_{11}$, and $\emptyset$ is additive on $\mathbb{G}_{22}$ is proved similarly .
Lemma 3.10: For x$\in \mathbb{G}$, $A_{11} \in \mathbb{G}_{11}$ and $B_{22} \in \mathbb{G}_{22}$, we have
1- $\emptyset_n(x + A_{11}) = \emptyset_n(x) + \emptyset_n(A_{11})$.
2- $\emptyset_n(x + B_{22}) = \emptyset_n(x) + \emptyset_n(B_{22})$.
Proof :1- Let $\emptyset_n(s) = \emptyset_n(x) + \emptyset_n(A_{11})$ for some s$\in \mathbb{G}$.
Then for all $t_{22} \in \mathbb{G}_{22}$
$\emptyset_n(st_{22} + t_{22}s) = \emptyset_n(xt_{22} + t_{22}x) + \emptyset_n(A_{11}t_{22} + t_{22}A_{11}) = \emptyset_n(xt_{22} + t_{22}x)$
So, $st_{22} + t_{22}s = xt_{22} + t_{22}x$.
This implies that $s_{12} = x_{12}$, $s_{21} = x_{21}$ and $s_{22} = x_{22}$.
Also, for all $t_{11} \in \mathbb{G}_{11}$,
$\emptyset_n(st_{11} + t_{11}s) = \emptyset_n(xt_{11} + t_{11}x) + \emptyset_n(A_{11}t_{22} + t_{22}A_{11})$.
Therefore, for $u_{12} \in \mathbb{G}_{12}$, we have
$\emptyset_n(u_{12}(st_{11} + t_{11}s) + (st_{11} + t_{11}s)u_{12}) = \emptyset_n(u_{12}xt_{11} + xt_{11}u_{12} + t_{11}xu_{12}) +$
$\emptyset_n(A_{11}t_{11}u_{12} + t_{11}A_{11}u_{12})$
$= \emptyset_n(u_{12}xt_{11} + x_{11}t_{11}u_{12} + x_{21}t_{11}u_{12} + t_{11}xu_{12}) + \emptyset_n(t_{11}u_{12} + t_{11}A_{11}u_{12})$
$= \emptyset_n(u_{12}xt_{11} + x_{11}t_{11}u_{12} + x_{21}t_{11}u_{12} + t_{11}xu_{12} + A_{11}t_{11}u_{12} + t_{11}A_{11}u_{12})$.
Then $s_{11}t_{11}u_{12} + t_{11}s_{11}u_{12} = x_{11}t_{11}u_{12} + t_{11}x_{11}u_{12} + A_{11}t_{11}u_{12} + t_{11}A_{11}u_{12}$.
By the faithful $\mathcal{M}$, then
$s_{11}t_{11} + t_{11}s_{11} = x_{11}t_{11} + t_{11}x_{11} + A_{11}t_{11} + t_{11}A_{11}.$
This implies that $s_{11} = x_{11} + A_{11}$.
Lemma 3.11: $\emptyset_n(A_{11}B_{12} + s_{21}d_{11}) = \emptyset_n(A_{11}B_{12}) + \emptyset_n(s_{21}d_{11})$ holds for all $A_{11}, d_{11} \in \mathbb{G}_{11}, B_{12} \in \mathbb{G}_{12}$ and $s_{21} \in \mathbb{G}_{21}$.
Proof : $\emptyset_n((A_{11}B_{12} + s_{21}d_{11}) + A_{11}d_{11} + d_{11}A_{11} + B_{12}s_{21}) + s_{21}B_{12})$
$= \emptyset_n((A_{11}B_{12} + s_{21}d_{11}) + A_{11}d_{11} + d_{11}A_{11} + B_{12}s_{21}) + \emptyset_n(s_{21}B_{12})$
$= \emptyset_n((A_{11}B_{12} + s_{21}d_{11}) + \emptyset_n(A_{11}d_{11} + d_{11}A_{11} + B_{12}s_{21}) + \emptyset_n(s_{21}B_{12})$
$= \emptyset_n((A_{11}B_{12} + s_{21}d_{11}) + \emptyset_n(A_{11}d_{11} + d_{11}A_{11}) + \emptyset_n(B_{12}s_{21}) + \emptyset_n(s_{21}B_{12})$
$= \emptyset_n\big((A_{11}B_{12} + s_{21}d_{11}) + \emptyset_n(A_{11}d_{11}\big) + \emptyset_n(d_{11}A_{11}) + \emptyset_n(B_{12}s_{21}) + \emptyset_n(s_{21}B_{12})$.
On the other hand,
$\emptyset_n(A_{11}B_{12} + s_{21}d_{11} + A_{11}d_{11} + d_{11}A_{11} + B_{12}s_{21} + s_{21}B_{12})$
$= \emptyset_n\big((A_{11} + s_{21})(B_{12} + d_{11}) + (B_{12} + d_{11})(A_{11} + s_{21})\big)$
$= \emptyset_n(A_{11} + s_{21})\emptyset_n(B_{12} + d_{11}) + \emptyset_n(B_{12} + d_{11})\emptyset_n(A_{11} + s_{21})$
$= (\emptyset_n(A_{11}) + \emptyset_n(s_{21}))(\emptyset_n(B_{12}) + \emptyset_n(d_{11})) + (\emptyset_n(B_{12}) + \emptyset_n(d_{11}))(\emptyset_n(A_{11}) + \emptyset_n(s_{21}))$
$= \emptyset_n(A_{11})\emptyset_n(B_{12}) + \emptyset_n(A_{11})\emptyset_n(d_{11}) + \emptyset_n(s_{21})\emptyset_n(B_{12}) + \emptyset_n(s_{21})\emptyset_n(d_{11}) + \emptyset_n(B_{12})\emptyset_n(A_{11}) +$
$\emptyset_n(d_{11})\emptyset_n(A_{11}) + \emptyset_n(B_{12})\emptyset_n(s_{21}) + \emptyset_n(d_{11})\emptyset_n(s_{21})$
$= \emptyset_n(A_{11}B_{12} + B_{12}A_{11}) + \emptyset_n(s_{21}d_{11} + d_{11}s_{21}) + \emptyset_n(A_{11}d_{11}) + \emptyset_n(d_{11}A_{11}) + \emptyset_n(B_{12}s_{21}) +$
$\emptyset_n(s_{21}B_{12})$
$= \emptyset_n(A_{11}B_{12}) + \emptyset_n(s_{21}d_{11}) + \emptyset_n(A_{11}d_{11}) + \emptyset_n(d_{11}A_{11}) + \emptyset_n(B_{12}s_{21}) + \emptyset_n(s_{21}B_{12})$.
Therefore, we get the result.
Proof of Theorem 3.1
Step 1-

$$\emptyset_n(A_{11} + B_{12} + C_{21} + d_{22}) = \emptyset_n(A_{11}) + \emptyset_n(B_{12}) + \emptyset_n(C_{21}) + \emptyset_n(d_{22})$$

For $A_{11} \in \mathbb{G}_{11}, B_{12} \in \mathbb{G}_{12}, C_{21} \in \mathbb{G}_{21}$, and $d_{22} \in \mathbb{G}_{22}$,
let $\emptyset_n(s) = \emptyset_n(A_{11}) + \emptyset_n(B_{12}) + \emptyset_n(C_{21}) + \emptyset_n(d_{22})$,
for some s$\in \mathbb{G}$.
By corollaries 3.4 and 3.6,
$\emptyset_n(s) = \emptyset_n(A_{11} + B_{12}) + \emptyset_n(C_{21} + d_{22})$.
Then, for $w_{11} \in \mathbb{G}_{11}$, we have
$\emptyset_n(sw_{11} + w_{11}s) = \emptyset_n(w_{11}(A_{11} + B_{12}) + (A_{11} + B_{12})w_{11} + \emptyset_n(w_{11}(C_{21} + d_{22}) + (C_{21} +$
$d_{22})w_{11})$

$= \emptyset_n(w_{11}A_{11} + w_{11}B_{12} + A_{11}w_{11}) + \emptyset_n(C_{21}w_{11})$
Hence, for $t_{22} \in \mathbb{G}_{22}$, we obtain

$$\emptyset_n(t_{22} \ \ (sw_{11} + w_{11}s) + (sw_{11} + w_{11}s)t_{22})$$
$$= \emptyset_n(w_{11}B_{12}t_{22}) + \emptyset_n(t_{22}C_{21}w_{11}) + \emptyset_n(w_{11}B_{12}t_{22} + t_{22}C_{21}w_{11})$$

It follows that

$$t_{22}sw_{11} + w_{11}st_{22} = w_{11}B_{12}t_{22} + t_{22}C_{21}w_{11}.$$

This implies that $t_{22}sw_{11} = t_{22}C_{21}w_{11}$ and $w_{11}st_{22} = w_{11}B_{12}t_{22}$.
This implies that $s_{21} = C_{21}$ and s_{12}=B_{12}.
Furthermore, we have
$\emptyset_n(u_{12}(\mathrm{s}w_{11} + w_{11}s) + (sw_{11} + w_{11}s)u_{12})$
$= \emptyset_n(w_{11}A_{11}u_{12} + A_{11}w_{11}u_{12}) + \emptyset_n(C_{21}w_{11}u_{12} + u_{12}C_{21}w_{11})$
$= \emptyset_n(w_{11}A_{11}u_{12} + A_{11}w_{11}u_{12} + C_{21}w_{11}u_{12} + u_{12}C_{21}w_{11})$.
It follows that

$$u_{12}sw_{11} + sw_{11}u_{12} + w_{11}su_{12} = w_{11}A_{11}u_{12} + A_{11}w_{11}u_{12} + C_{21}w_{11}u_{12} + u_{12}C_{21}w_{11}$$

Hence

$$s_{11}w_{11}u_{12} + w_{11}s_{11}u_{12} = w_{11}A_{11}u_{12} + A_{11}w_{11}u_{12}$$

By the faithful $\mathcal{M}$, we get

$$s_{11}w_{11} + w_{11}s_{11} = w_{11}A_{11} + A_{11}w_{11}.$$

And hence, $s_{11} = A_{11}$. Similarly , $s_{22} = d_{22}$.
Step- 2
$\emptyset_n(A + B) = \emptyset_n(A) + \emptyset_n(B)$ for all A,B in $\mathbb{G}$,
from Lemmas 3.8 and 3.9 and step 1.

4- Additivity of Jordan Triple Product Homomorphism

Definition 4.1: A family $\sigma \ = \{\sigma_{\ s}\}_{s\epsilon N}$ of mappings on $\mathbb{G}$ is said to be Jordan higher triple product homomorphism if $\sigma_{\ t}(\mathcal{A}\mathcal{B}\mathcal{A}) = \sum_{k=1}^{t} \sigma_{\ k}(\mathcal{A})\sigma_{\ k}(\mathcal{B})\sigma_{\ k}(\mathcal{A})$.
S.t. $\sigma_{\ k}(\mathcal{A})\sigma_{\ t}(\mathcal{B}) = 0 \ \ \forall k \neq t.$

Theorem 4.2
Let $\mathbb{G}$ be the generalized matrix algebra that satisfies the conditions:
1. $\forall \ M\epsilon\mathcal{M}, NMN = 0\forall \ N\epsilon N \Longrightarrow M = 0.$
2. $\forall \ N\epsilon\mathcal{N}, NMN = 0 \ \forall M\epsilon\mathcal{M} \Longrightarrow N = 0.$
3. For $A\epsilon\mathcal{A}, AM = 0 \ \forall M\epsilon\mathcal{M} \Longrightarrow A = 0.$
4. $\forall \ B\epsilon\mathcal{B}, MBN = 0 \ \forall \ M\epsilon\mathcal{M}$ and $N\epsilon\mathcal{N} \Longrightarrow B = 0.$

Then, any bijective higher Jordan (Triple)product homomorphism $\emptyset$ from $\mathbb{G}$ onto ring $\mathcal{R}'$ is additive.

Corollary 4.3: Let $\mathcal{A}$ be an algebra (unital, not necessarily prime) over a commutative ring $\mathcal{R}$. Then, any bijective Jordan higher (Triple) product homomorphisms from $M_n(\mathcal{A})$, $n \geq 2$ onto arbitrary ring $\mathcal{R}'$ is additive .

We shall introduce the following Lemmas to prove Theorem 4.2. From now on, $\emptyset$ satisfies the hypothesis of Theorem 4.2.

Lemma 4.4: $\emptyset_n(0) = 0$.

Proof: Since $\emptyset_i$ is surjective, then $\exists \ A\epsilon G, \emptyset_i(A) = 0$.

$$\emptyset_n(0) = \emptyset_n(0A0) = \sum_{i=1}^{n} \emptyset_i(0)\emptyset_i(A)\emptyset_i(0)$$

Lemma 4.5
$\emptyset_n(A_{11} + A_{12} + A_{21} + A_{22}) = \emptyset_n(A_{11}) + \emptyset_n(A_{12}) + \emptyset_n(A_{21}) + \emptyset_n(A_{22})$,
for every $A_{ij}\epsilon\mathbb{G}_{ij}$.

Proof: Let $s\epsilon\mathbb{G}$ be an element such that
$\emptyset_n(s) = \emptyset_n(A_{11}) + \emptyset_n(A_{12}) + \emptyset_n(A_{21}) + \emptyset_n(A_{22})$. Then for every $X_{ij}, 1 \leq i, j \leq 2$

$$\begin{aligned}\emptyset_n(X_{ij}sX_{ij}) &= \textstyle\sum_{i=1}^{n} \emptyset_i(X_{ij})\emptyset_i(s)\emptyset_i(X_{ij}) \\ &= \textstyle\sum_{i=1}^{n} \sum_{lt} \emptyset_i(X_{ij})\emptyset_i(A_{lt})\emptyset_i(X_{ij}) \\ &= \textstyle\sum_{l,t} \sum_{i=1}^{n} \emptyset_i(X_{ij})\emptyset_i(A_{lt})\emptyset_i(X_{ij}) \\ &= \textstyle\sum_{l,t} \emptyset_n(X_{ij}A_{lt}X_{ij})\end{aligned}$$

$\emptyset_n(X_{ij}sX_{ij}) = \emptyset_n(X_{ij}A_{ji}X_{ij})$

Since $\emptyset_i$ is one-to-one,
$X_{ij}sX_{ij} = X_{ij}A_{ji}X_{ij}$, that is $X_{ij}(s - A_{ij})X_{ij} = 0$,
Since $\mathcal{A}$ and $\mathcal{B}$ are unital algebra, we have $s_{11} = A_{11}$ and $s_{22} = A_{22.}$
From $X_{12}(s - A_{12})X_{12} = 0$, we have $s_{12} = A_{12}$, by condition (2).
Similarly, we have $s_{21} = A_{21}$.
Lemma 4.6: $\emptyset_n$ is additive on $\mathbb{G}_{12}$.
Proof: Let $A_{11} = I_A$. The identity element of A and $C_{22} = I_B$ is that of B, then we have
$\emptyset_n(A_{11}) + \emptyset_n(A_{12} + B_{12}) = \emptyset_n(A_{11} + A_{12} + B_{12}C_{22})$, and by lemma 4.5
$= \emptyset_n((A_{11} + C_{22} + A_{12})(A_{11} + B_{12})(A_{11} + C_{22} + A_{12}))$
$= \emptyset_n((A_{11} + C_{22} + A_{12})(A_{11})(A_{11} + C_{22} + A_{12}))$
$+\emptyset_n((A_{11} + C_{22} + A_{12})(B_{12})(A_{11} + C_{22} + A_{12}))$
$= \emptyset_n(A_{11} + A_{12}) + \emptyset_n(A_{11}B_{12}C_{22})$
$= \emptyset_n(A_{11}) + \emptyset_n(A_{12}) + \emptyset_n(A_{11}B_{12})$
$= \emptyset_n(A_{11}) + \emptyset_n(A_{12}) + \emptyset_n(B_{12})$
From which it follows that $\emptyset_n((A_{12} + B_{12}) = \emptyset_n(A_{12}) + \emptyset_n(B_{12})$.
Lemma 4.7: $\emptyset_n$ is additive on $\mathbb{G}_{21}$.
Proof: Suppose that $s \in \mathbb{G}$,$\emptyset_n(s) = \emptyset_n(A_{21}) + \emptyset_n(B_{21})$. For every X_{ij}, we have
$\emptyset_n(X_{ij}sX_{ij}) = \sum_{i=1}^{n} \emptyset_i(X_{ij})(\emptyset_i(A_{21}) + \emptyset_i(B_{21})\emptyset_i(X_{ij})$
$= \sum_{i=1}^{n} \emptyset_i(X_{ij})\emptyset_i(A_{21})\emptyset_i(X_{ij}) + \sum_{i=1}^{n} \emptyset_i (X_{ij})\emptyset_i(B_{21})\emptyset_i(X_{ij})$
$= \emptyset_n(X_{ij}\, A_{21}\, X_{ij}) + \emptyset_n(X_{ij}B_{21}X_{ij})$. Then by lemma 4.6 , we have
$\emptyset_n(X_{12}\, S\, X_{12}) = \emptyset_n(X_{12}A_{21}X_{12}) + \emptyset_n(X_{12}B_{21}X_{12})$
$= \emptyset_n(X_{12}A_{21}X_{12} + X_{12}B_{21}X_{12})$
Hence, we have $X_{12}\, s\, X_{12} = X_{12}A_{21}X_{12} + X_{12}B_{21}X_{12}$.
Then by condition (4), we get
$S_{12} = A_{12} + B_{21}$ .Also we have
$\emptyset_n(X_{11}S\, X_{11}) = \emptyset_n(X_{22}\, S\, X_{22}) = B_{21}(X_{21}\, SX_{21})$ =0
Then, it follows that $S_{11} = S_{22} = 0$ and $S_{12} = 0$, by condition (1), and hence $\emptyset_n(A_{21} + B_{21}) = \emptyset_n(S_{21}) = \emptyset_n(A_{21}) + \emptyset_n(B_{21})$.
Lemma 4.8
$\emptyset_n(A_{11} + B_{11}) = \emptyset_n(A_{11}) + \emptyset_n(B_{11})$ for every $A_{11},\ B_{11}\epsilon\mathbb{G}_{11}$.
Proof: We first claim that for every $C_{11}\epsilon\mathbb{G}_{11}$ and $D_{12}\epsilon\mathbb{G}_{12}$, we have
$$\emptyset_n(C_{11}D_{12}) = \emptyset_n(P_{11})\emptyset_n(C_{11})\emptyset_n(D_{12}) + \emptyset_n(D_{12})\emptyset_n(C_{11})\emptyset_n(P_{11}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.1)$$
where P_{11} is the identity element in $\mathcal{A}$. One get by lemma 4.5,
$\emptyset_n(C_{11}) + \emptyset_n(C_{11}D_{12}) = \emptyset_n(C_{11} + C_{11}D_{12})$
$= \emptyset_n((P_{11} + D_{12})C_{11}(P_{11} + D_{12}))$
$= \sum_{i=1}^{n} \emptyset_i (P_{11} + D_{12})\emptyset_i(C_{11})\emptyset_i(P_{11} + D_{12})$
$= \sum_{i=1}^{n}(\emptyset_i (P_{11}) + \emptyset_i(D_{12}))\emptyset_i(C_{11})(\emptyset_i(P_{11}) + \emptyset_i(D_{12}))$
$= \sum \emptyset_i(P_{11})\emptyset_i(C_{11})\emptyset_i(P_{11}) + \emptyset_i(D_{12})\emptyset_i(C_{11})\emptyset_i(D_{12})$
$+\emptyset_i(P_{11})\emptyset_i(C_{11})\emptyset_i(D_{12}) + \emptyset_i(D_{12})\emptyset_i(C_{11})\emptyset_i(P_{11})$
$= \emptyset_n(C_{11}) + \emptyset_n(P_{11})\emptyset_n(C_{11})\emptyset_n(D_{12}) + \emptyset_n(D_{12})\emptyset_n(C_{11})\emptyset_n(P_{11})$
Then, (4.1) holds
Now, let $S\epsilon\mathbb{G}$, $\emptyset_n(S) = \emptyset_n(A_n) + \emptyset_n(B_{11})$
Since for every $X_{ij}\epsilon\mathbb{G}_{ij}$,
$\emptyset_n(X_{ij}S\, X_{ij}) = \sum_{k=1}^{n} \emptyset_k(X_{ij})\emptyset_k(S)\emptyset_k(X_{ij}) = \sum_{k=1}^{n} \emptyset_k(X_{ij}A_{11}X_{ij}) + \emptyset_k(X_{ij}B_{11}X_{ij})$.
We have $\emptyset_n(X_{12}S\, X_{12}) = \emptyset_n(X_{21}S\, X_{21}) = \emptyset_n(X_{22}\, S\, X_{22})$
So $S_{12} = S_{21} = S_{22} = 0$
By conditions (1) and (2) we also have
$\emptyset_n(S_{11}X_{12}) = \sum_{k=1}^{n} \emptyset_k(P_{11})\emptyset_k(S_{11})\emptyset_k(X_{12})$
$+\emptyset_k(X_{12})\emptyset_k(S_{11})\emptyset_k(P_{11})$
$= \sum_{k=1}^{n} \emptyset_k(P_{11})(\emptyset_k(A_{11}) + \emptyset_k(B_{11}))\emptyset_k(X_{12})$
$+\emptyset_k(X_{12})(\emptyset_k(A_{11}) + \emptyset_k(B_{11}))\emptyset_k(P_{11})$
$= \emptyset_n(A_{11}X_{12}) + \emptyset_n(B_{11}X_{12}) = \emptyset_n(A_{11}X_{12} + B_{11}X_{12})$

Then, we get $S_{11}X_{12} = (A_{11} + B_{11})X_{12}$. Hence by condition (3),
$\emptyset_n(A_{11} + B_{11}) = \emptyset_n(A_{11}) + \emptyset_n(B_{11})$.
Lemma 4.9: $\emptyset_n$ is additive on $\mathbb{G}_{22}$.
Proof: Let A_{22} and B_{22} be elements of $\mathbb{G}_{22}$ and $s\epsilon\mathbb{G}$ be an element such that
$\emptyset_n(s) = \emptyset_n(A_{22}) + \emptyset_n(B_{22})$.
Since for every $X_{ij}\epsilon\mathbb{G}_{ij}$,
$\emptyset_n\big(X_{ij}\, S\, X_{ij}\big) = \sum_{k=1}^{n} \emptyset_k(X_{ij})\emptyset_k(S)\emptyset_k(X_{ij})$
$=\sum_{k=1}^{n} \emptyset_k\big(X_{ij}A_{22}X_{ij}\big) + \emptyset_k(X_{ij}B_{22}X_{ij})$ … …………………………(4.2),
then we have
$\emptyset_n(X_{11}\, S\, X_{11}) = \sum \emptyset_k(X_{11})\,\emptyset_k(A_{22})\emptyset_k(X_{11})$
$+\emptyset_k(X_{11})\emptyset_k(B_{22})\emptyset_k(X_{11}) = 0$
Hence, $X_{11}SX_{11} = 0$ and $X_{11}S_{11}X_{11} = 0$. Then, it follows that $S_{11} = 0$. Since $\emptyset_n(X_{12}S\, X_{12}) = 0$ by (4.2), we have $X_{12}S\, X_{12} = 0$ and hence $S_{21} = 0$. Hence, by the same way, $S_{12} = 0$.
then $\emptyset_n(S_{22}) = \emptyset_n(A_{22}) + \emptyset_n(B_{22})$
Considering, for every $X_{12}, Y_{12}\epsilon G$
$\emptyset_n(X_{12}\, S_{22}Y_{21}) = \emptyset_n((X_{12} + Y_{21})S_{22}(X_{12} + Y_{21}))$
$= \sum_{k=1}^{n} \emptyset_k(X_{12} + Y_{21})\emptyset_k(S_{22})\emptyset_k(X_{12} + Y_{21})$
$= \sum_{k=1}^{n} \emptyset_k(X_{12} + Y_{21})(\emptyset_k(A_{22}) + \emptyset_k(B_{22}))\emptyset_k(X_{12} + Y_{21})$
$= \emptyset_n\big((X_{12} + Y_{21})A_{22}(X_{12} + Y_{21})\big) + \emptyset_n((X_{12} + Y_{21})B_{22}(X_{12} + Y_{21})$
$= \emptyset_n(X_{12}A_{22}Y_{21}) + \emptyset_n(X_{12}B_{22}Y_{21})$
$= \emptyset(X_{12}A_{22}Y_{21} + X_{12}B_{22}Y_{21})$
We have $X_{12}S_{22}Y_{21} = X_{12}A_{22}Y_{21} + X_{12}B_{22}Y_{21}$. Therefore, by condition (4), it follows that $S_{22} = A_{22} + B_{22}$,
$\emptyset_n(A_{22} + B_{22}) = \emptyset_n(A_{22}) + \emptyset_n(B_{22})$. We now prove our main result.
Proof of Theorem 4.2: Let $A = A_{11} + A_{12} + A_{21} + A_{22}$ and $B = B_{11} + B_{12} + B_{21} + B_{22}$ by the element of G. Then by lemmas 4.5, 4.6, 4.7, 4.8, and 4.9, we have
$\emptyset_n(A + B) = \emptyset_n(A_{11} + B_{11} + A_{12} + B_{12} + A_{21} + B_{21} + A_{22} + B_{21}$
$= \emptyset_n(A_{11} + B_{11}) + \emptyset_n(A_{12} + B_{12}) + \emptyset_n(A_{21} + B_{21}) + \emptyset_n(A_{22} + B_{22})$.
$= \emptyset_n(A_{11}) + \emptyset_n(B_{11}) + \emptyset_n(A_{11}) + \emptyset_n(B_{12}) + \emptyset_n(A_{21}) + \emptyset_n(B_{21}) + \emptyset_n(A_{22}) + \emptyset_n(B_{22})$
$= \emptyset_n(A) + \emptyset_n(B)$.

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