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The Dual Notion of St-Polyform Modules

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Abstract

In the year 2018, the concept of St-Polyform modules was introduced and studied by Ahmed, where a module M is called St-Polyform, if for every submodule N of M and for any homomorphism f:N⟶M, ker$f$ is St-closed submodule in N. The novelty of this paper is that it dualizes this class of modules to a form that we denote as CSt-Polyform modules. Accordingly, some results that appeared in the original paper are dualized. For example, we prove that in the class of hollow modules, every CSt-Polyform module is Coquasi-Dedekind. In addition, several important properties of CSt-Polyform module are established, while further characterization of CSt-Polyform is provided. Moreover, many relationships of CSt-Polyform modules with other related concepts are considered, such as the copolyform, epiform, CSt-semisimple, κ-nonsingular modules, while some others will be introduced, such as the non-CSt-singular and G. Coquasi-Dedekind modules.

Keywords: St-Polyform modules, CSt-Polyform modules, P-small submodules, St-closed submodules, CSt-closed submodules.

المقاسات المتعددة الصيغ المضادة من النمط–CSt

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الخلاصة

في عام 2018 ، أعطيت المقاسات المتعددة الصيغ من النمط–St من قبل منى عباس أحمد. ان أصالة هذا البحث تكمُن في تقديم رديفاً لهذا الصنف من المقاسات، أُطلق عليه أسم المقاسات المتعددة الصيغ المضادة من النمط CSt

تم استخدام هذا الرديف في الحصول على نتائج رديفة لما موجود في المصدر المشار اليه أعلاه، حيث تم البرهنة على أنه في صنف المقاسات المجوفة؛ كل مقاس متعدد الصيغ مضاد يكون مقاسا" شبه ديدكاند مضاد. كما أُعطيت العديد من الخصائص المهمة وتشخيص أخر لهذا النوع من المقاسات. كذلك درسنا علاقة المقاسات المتعددة الصيغ المضادة بمقاسات اخرى مثل: مقاسات الصيغة الشاملة، مقاسات شبه بسيطة من النمط–CSt، مقاسات غير شاذة من النمط–κ، ومقاسات اخرى تم تقديمها في هذا البحث مثل مقاسات غير شاذة من النمط–CSt ومقاسات شبه ديديكاند معممة.

1. Introduction

Throughout this paper, all rings are commutative with non-zero identity elements and all modules are unitary left R-modules. The aim of this paper is to dualize the concept of St-

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Polyform modules which was first studied by Ahmed [1]. For the sake of completeness, we begin with some definitions and notations that will be followed in this paper. A non-zero submodule N of M is called essential (semi-essential) if $N \cap P \neq (0)$ for each non-zero submodule (prime submodule) P of M [2, 3]. A submodule P of M is called prime, if whenever $rm \in P$ for $r \in R$ and $m \in M$, then either $m \in P$ or $r \in (P:M)$. A submodule N of M is called closed, if N has no proper essential extensions inside M [2, P.18]. The concept of St-closed submodules is stronger than that of closed submodules, where a submodule N of M is said to be St-closed (simply $N \leq_{stc} M$), if N has no proper semi-essential extensions inside M [4]. A submodule N of M is called small in M (denoted by $L \ll M$), if for every proper submodule K of M, $N+K \neq M$ [2, P.20]. A submodule W is called coessential of N in M (denoted by $W \leq_{ce} N$ in M), if whenever $\frac{N}{W} \ll \frac{M}{W}$ then N=W [5, P.20]. Hadi and Ibrahiem introduced P-small submodules as an extension to the concept of small submodules, where a proper submodule N of an R-module M is called P-small (simply $N \ll_P M$), if $N+P \neq M$ for every prime submodule P of M [6]. A generalization of coessential submodules appeared in another study [7], where a submodule L is called cosemi-essential of N in M, if $\frac{N}{L} \ll_P \frac{M}{L}$. A submodule N is called coclosed in M (simply $N \leq_{cc} M$), if N has no proper coessential submodule in M [8]. Ahmed introduced the concept of CSt-closed submodule which is stronger than coclosed submodules, where a submodule N is called CSt-closed (simply $N \leq_{CSt} M$), if N has no proper cosemi-essential extensions inside M, that is; if $\frac{N}{A} \ll_P \frac{M}{A}$, then N=A for all submodules A of M contained in N [7]. An R-module M is called St-Polyform , if for every submodule N of M and for any homomorphism $f: N \longrightarrow M$, $\ker f$ is St-closed submodule in N. Equivalently, M is an St-Polyform if for every non-zero submodule N of M and for each non-zero homomorphism $f: N \longrightarrow M$, $\ker f$ is not semi-essential submodule of N [1].

In this paper, the authors introduce and study the duality of St-Polyform modules, named here as CSt-Polyform modules. In Section 2, some remarks and supporting examples are given, which reflect the main properties of CSt-Polyform modules. Other characteristics of CSt-Polyform modules are established see Theorem (2.10). The conditions under which CSt-Polyform and copolyform can be equivalent are studied; see the results (2.5) and (2.7). Several results about St-Polyform modules have corresponding duals for CSt-Polyform modules; see Propositions (3.4), (3.10) and (3.17). In addition, we determine a commutative ring having a faithful CSt-Polyform module, see Proposition (2.8). Moreover, the relationships of CSt-Polyform module with other related concepts are considered; see the results (3.2), (3.4), (3.8), (3.10), (3.15), (3.16), (3.21) and (3.23).

2. CSt-Polyform Modules

In this section, we dualize the class of St-Polyform and call it CSt-Polyform module.

Definition (2.1): An R-module M is called CSt-Polyform, if for each proper submodule N of M and for all homomorphism $f: M \longrightarrow M/N$, f(M) is CSt-closed submodule in M/N. A ring R is said to be CSt-Polyform if R is CSt-Polyform R-module.

In the following, we give some examples and remarks. Before that, a submodule N of an R-module M is called corational, if $\mathrm{Hom}_R(M,N/K)=0$ for all submodule K of N, and an R-module M is called copolyform, if every small submodule of M is corational [9].

Examples and Remarks (2.2)

1. Every CSt-Polyform module is copolyform, since every CSt-closed submodule is coclosed [7]; hence, the result follows directly from the definition of CSt-Polyform module.

2. The converse of (1) is not true in general; for example, the Z-module Z is copolyform. In fact, the only small submodule of Z is (0), which is corational in Z. On the other hand, Z is not CSt-Polyform. To show that, consider the submodule (4) of Z. Let f: $Z \rightarrow Z/(4)$ be a

homomorphism. Note that Z/(4) ≅ Z_4 and $\mathrm{Hom}_R(Z,Z_4)=0$. On the other hand, (0) $\not\leq_{CSt}$ Z_4 [7], thus Z is not CSt-Polyform.

3. Every simple module is CSt-Polyform module. In fact, the only proper submodule of any module M is (0), so for all non-zero homomorphism f:M⟶ M/(0), f(M) is either zero, which is a contradiction, or M. Since M is not P-small submodule of itself, therefore M is CSt-Polyform.

4. For each prime number P, Z_{P^2} is not CSt-Polyform Z-module, since it is not copolyform, such as Z_4, Z_9, Z_{25}, Z_{49}. In fact, pZ_{P^2} is a small submodule of Z_{P^2} but not corational in Z_{P^2}, since $\mathrm{Hom}_Z(Z_{P^2}, pZ_{P^2}/(0))\neq 0$.

5. Z_6 is a CSt-Polyform Z-module; see Example (3.5).

6. Z_4 is not CSt-Polyform Z-module, since Z_4 is not copolyform, so by (2.2)(1), Z_4 is not CSt-Polyform.

Remark (2.3): If a submodule N of M is CSt-Polyform module, and N is essential submodule of M, then M is not necessarily CSt-Polyform; for example, suppose that M=Z_{P^2} and N= pZ_{P^2}. Note that pZ_{P^2} is CSt-Polyform Z-module, because pZ_{P^2} is simple for each prime number p, see Remark (2.2)(3). On the other hand, N is essential in Z_{P^2}, but Z_{P^2} is not CSt-Polyform Z-module.

Now, we provide conditions under which the converse of Remark (2.2)(1), will be satisfied. Before that, a module M is called almost finitely generated, if M is not finitely generated and every proper submodule of M is finitely generated [6].

Lemma (2.4): Let M be an almost finitely generated module and N ≤ M, then:

1. N is a P-small submodule of M if and only if N is small.

2. N$\leq_{CSt}$ M if and only if N$\leq_{cc}$ M.

Proof:

1. See [6].

2. The result follows directly by (1).

Proposition (2.5): Let M be an almost finitely generated R-module. M is a CSt-Polyform module if and only if M is copolyform.

Proof: The necessity is fulfilled by just Remark (2.2)(1). For sufficiency, suppose that M is an almost finitely generated module. Let N≤M and f: M→M/N be a homomorphism. Since M is copolyform, then f(M) $\leq_{cc}$ M/N. By using Lemma (2.4)(2), f(M) $\leq_{CSt}$ M/N. Hence, M is CSt-Polyform module.

Example (2.6): Z_{P^∞} is copolyform module [10]. Also, it is almost finitely generated. Hence, by Proposition (2.5), Z_{P^∞} is a CSt-Polyform module.

Following [11], an R-module M is called multiplication, if for each submodule N of M, there exists an ideal I if R such that N=IM.

Proposition (2.7): In the class of multiplication (or finitely generated or almost finitely generated modules), CSt-Polyform coincides with the class of copolyform modules.

Proof: The difference between CSt-Polyform and copolyform concepts are depend on the difference between CSt-closed and coclosed submodules. Beside that the last two classes are coincide under multiplication, finitely generated, and almost finitely generated conditions as we can see in [7] and Lemma (2.4)(2). For that reason CSt-Polyform and copolyform modules are coincide under the same conditions.

Recall that a ring R is called semiprime, if for each element r∈R, whenever r^2=0, then r=0 [2, P.2]. The CSt-Polyform R-module can be used as a useful condition in the following proposition.

Proposition (2.8): If a commutative ring R has a faithful CSt-Polyform R-module, then R is semiprime ring.

Proof: Suppose that R is a commutative ring that has a faithful CSt-Polyform module, say M. For each non-zero element x∈R, define f_x: M→M by f(m)=xm ∀m∈M. We can easily show

that $f_x^2(M) \subseteq f_x(M)$. We claim that $f_x(M)/f_x^2(M) \ll_p M/f_x^2(M)$. In fact, assume that $f_x(M)/f_x^2(M)+N/f_x^2(M)=M/f_x^2(M)$, where N is a submodule of M containing $f_x^2(M)$. That is, $(xM/x^2M)+(N/x^2M)=(M/x^2M)$, which implies that xM+N=M. We should prove that $xM \subseteq N$; let $xt \in xM$ and $t \in M$. Since xM+N=M, then t=xy+n, where $y \in M$ and $n \in N$. By multiplying the two sides by x, we get $xt=x^2y+xn$. But $x^2M \subseteq N$, therefore $x^2y \in N$, also $xn \in N$, thus $xt \in N$, that is $xM \subseteq N$, hence $f_x(M)/f_x^2(M) \ll_p M/f_x^2(M)$. Since M is CSt-Polyform, then $f_x(M)= f_x^2(M)$, that is $xM=x^2M$ for all non-zero $x \in R$. To prove that R is semiprime, let $r \in R$ with $r^2=0$. Note that $r^2M=rM=0$. This implies that $r \in ann_R M$, but M is faithful, thus r=0. This completes the proof.

The following theorem gives another characterization of CSt-Polyform module. Before that, we need to give the following lemma.

Lemma (2.9): If a submodule N of an R-module M is P-small and CSt-closed, then N=(0).

Proof: Since $N \ll_P M$, then $N/(0) \ll_P M/(0)$. But $N \ll_{CSt} M$, hence N=(0).

Theorem (2.10): An R-module M is CSt-Polyform if and only if, for each proper submodule N of M and for all non-zero homomorphism f: M⟶ M/N, f(M) is not P-small submodule of M/N.

Proof: Let M be a CSt-Polyform module, and assume that there exists a proper submodule N of M and a non-zero homomorphism f: N⟶M/N with f(M) is P-small submodule of M. By assumption $f(M) \ll_{CSt} M/N$, hence f(M)=(0), by Lemma (2.9), i.e. f=0. But this is a contradiction, thus f(M) is not P-small submodule of M. Conversely, suppose that there exists a submodule K of M and a non-zero homomorphism f:M⟶M/N such that f(M) is not CSt-closed in M/K. Put f(M) ≡ K/N, where K is a submodule of M, such that $N \leq K \leq M$. Since f(M) is not CSt-closed in M/K, so there exists a proper submodule L/K of N/K such that $\frac{N/K}{L/K} \ll_P \frac{M/K}{L/K}$. By the 3rd isomorphism theorem, $N/L \ll_P M/L$. Define a homomorphism g: M/K⟶M/L by g(m+K)=m+L $\forall m \in M$. Clearly, g is an epimorphism. Now, $(gof)(M) = g(f(M)) = g(N/K) = N/L$. But $N/L \ll_P M/L$, so we get a contradiction with our assumption, thus $f(M) \ll_{CSt} M/K$, hence the result follows.

By using Theorem (2.10), we can prove the following.

Proposition (2.11): If an R-module M is CSt-Polyform module, then M/N is CSt-Polyform module for every proper submodule N of M.

Proof: Let N and L be submodules of M such that $N \subsetneqq L \subsetneqq M$. Assume that f: $M/N \to \frac{M/N}{L/N}$ is a non-zero homomorphism with $L/N \subsetneqq M/N$. We have to show that f(M/N) is not P-small submodule of $\frac{M/N}{L/N}$. Consider the following sequence of functions:

$$M \xrightarrow{\pi} M/N \xrightarrow{f} \frac{M/N}{L/N} \xrightarrow{g} M/L$$

where π is the natural epimorphism and g is the usual isomorphism. Since M is CSt-Polyform, so by Theorem (2.10), $(gof o\pi)(M)$ is not P-small submodule of M/L. This implies that $(gof)(M/N)$ is not P-small submodule of $\frac{M/N}{L/N}$, hence f(M) is not P-small submodule of $\frac{M/N}{L/N}$. By Theorem (2.10), M/N is CSt-Polyform module.

The following result can be concluding from Proposition (2.11). Also, it can be proved as follows, before that we need to give the following lemma.

Lemma (2.12): Let L and N be submodules of an R-module M such that $L \subsetneqq N \subsetneqq M$. If $L \ll_P N$, then $L \ll_P M$.

Proof: Suppose that there exists a prime submodule P of M with L+P=M. Note that by assumption, P is a prime submodule of N. According to [12, Prop.(1.7), P.11], P∩N is prime in N. Now, N=M∩N=(L+P)∩N=L+(P∩N). But this is a contradiction since $L \ll_P N$, therefore $L \ll_P M$.

Corollary (2.13): Every direct summand of CSt-Polyform module is CSt-Polyform.

Proof: Let M be a CSt-Polyform module and N be a direct summand of M. Let f: N→N/K be a homomorphism, K≨N. Now, consider the following:

$M \xrightarrow{\rho} N \xrightarrow{f} N/K \xrightarrow{i} M/K$

where ρ : $M \longrightarrow N$ is a projection homomorphism and i: $N/K \longrightarrow M/K$ is the inclusion homomorphism. Since M is CSt-Polyform , then by Theorem (2.10), $(iof o\rho)$(M) is not P-small submodule of M/K. That is, $(iof o\rho)$(M) = (iof)(N) = f(N) is not P-small submodule of N/K. By Lemma (2.12), f(N) is not P-small submodule of M/K. Thus, N is CSt-Polyform module.

3. CSt-Polyform modules and other related concepts

This section deals with the relationships of CSt-Polyform modules with other related concepts, such as epiform, CSt-semisimple, non-CSt-singular, κ -non-CSt-singular, and Coquasi-Dedekind modules.

Following [10], a non-zero module M is called epiform, if each non-zero homomorphism f: M→M/N with N is a proper submodule of M, which is an epimorphism. For example, the Z-module Z_{P^∞} is epiform [10].

Remark (3.1): It is clear that every epiform module is CSt-Polyform. In fact, f(M)= M/N in the definition of epiform, which is CSt-closed in itself [7], so it is a CSt-Polyform module. The converse is not true in general; for example, Z_6 is CSt-Polyform Z-module, as we showed in Example (2.2)(5), but not epiform [10].

Under certain conditions, CSt-Polyform module can be epiform; before that, an R-module M is called prime hollow (simply P_r -hollow) if each proper prime submodule of M is small [13].

Theorem (3.2): Let M be a P_r -hollow module. M is a CSt-Polyform module if and only if M is epiform.

Proof: Assume that M is CSt-Polyform and let f: M→M/N be a non-zero homomorphism with a proper submodule N of M. Assume that f(M)≠M/N. Since M is a CSt-Polyform, then f(M) is not P-small submodule of M/N. On the other hand, M is P_r –hollow, implies M/N is a P_r–hollow module [13]. This implies that f(M) is P-small submodule of M/N. But this is a contradiction, thus f(M)=M/N and, consequently, M is an epiform module. The converse is clear.

Note that Theorem (3.2) represents an analogue of that appeared in [10] for copolyform modules.

Recall that a module M is called CSt-semisimple, if every submodule of M is CSt-closed [7]. Before giving the next result, we need the following.

Lemma (3.3): Any factor of a CSt-semisimple module is CSt-semisimple.

Proof: Let M be an R-module and K ≤ L ≤ M with L/K ≤ M/K. Assume that $\frac{L/K}{N/K}$ is P-small submodule of $\frac{M/K}{N/K}$. By the 3rd isomorphism theorem, L/N $\ll_P$ M/N. Since M is CSt-semisimple, then N $\ll_{CSt}$ L in M. This implies that L=N, hence L/K = N/K, and the proof is complete.

The following result is a dual of that for copolyform modules which appeared in [1, Rem (33)].

Proposition (3.4): Every CSt-semisimple is CSt-Polyform module.

Proof: The result follows by the definition of CSt-semisimple and Lemma (3.3).

Example (3.5): It is clear that Z_6 is CSt-semisimple. Since every submodule of Z_6 is St-colsed in Z_6 , thus Z_6 is CSt-Polyform .

Note: We conclude the following implications:

CSt-semisimple module ⟹ CSt-Polyform module ⟹ Copolyform module

Following [14], a module M is called noncosingular, if for any non-zero module N and for every non-zero homomorphism f: M→N, Im (f) is not small submodule of N.

As a stronger of a noncosingular concept, we introduce the following.

Definition (3.6): An R-module M is called non-CSt-singular, if for any non-zero module N and for every non-zero homomorphism f: M→N, Im (f) is not P-small submodule of N.

Compare the following proposition with [10, Prop.(2.5)].

Proposition (3.7): If M is a P_r-hollow and non-CSt-singular module, then M is epiform.

Proof: Let f: M→M/N be a non-zero homomorphism with a proper submodule N of M. Since M is a non-CSt-singular, then f(M) is not P-small submodule of M/N. Also, by [6,Rem.(3.2)(6)], M/N is a P_r –hollow module. Thus, f(M) = M/N, that is M is an epiform module.

Proposition (3.8): Every non-CSt-singular module is CSt-Polyform module.

Proof: Let M be a non-CSt-singular module and f:M→M/N be a non-zero homomorphism, where N is a proper submodule of M. If M=(0), then there is nothing to prove. Otherwise, clearly, M/N is non-zero module. Since M is non-CSt-singular, then f(M) is not P-small submodule of M/N. By Theorem (2.10), the result follows.

In [1], the author introduced the concept of κ-non St-singular module, where M is called κ-non St-singular, if for any non-zero homomorphism f∈$\mathrm{End_R}$(M), ker f $\leq_{sem}$ M.

Dually, we have the following.

Definition (3.9): An R-module M is called κ-non-CSt-singular, if for any non-zero endomorphism f of M, Im f is not P-small submodule of M.

It is clear that every non-CSt-singular module is κ-non-CSt-singular.

The following proposition represents a dual of that which is appeared in [1, Prop. (40)].

Proposition (3.10): Every CSt-Polyform module is κ-non-CSt-singular.

Proof: Let M be a CSt-Polyform and f: M→M/N with proper submodule N. By assumption, f(M) is not P-small submodule of M. Put N=(0), then we obtain that f: M→M and f(M) not P-small submodule of M. That is M is κ-non-CSt-singular.

Note (3.11): We can summarize the relations mentioned in the previous results and argument by the following implications; before that, a module M is called τ-noncosingular, if for every non-zero module N and every non-zero homomorphism f: M→N, Im f is not small submodule in N [15]. It is clear that every noncosingular module is τ-noncosingular.

non-CSt-singular ⟹ κ-non-CSt-singular ⟹ noncosingular ⟹ τ-noncosingular

non-CSt-singular ⟹ CSt-Polyform ⟹ non-CSt-singular ⟹ Copolyform

CSt-Polyform ⟹ κ-non-CSt-singular ⟹ noncosingular ⟹ τ-noncosingular

Following [16]; An R-module M is called fully prime module, if every proper submodule of M is prime. The following theorem gives some relations of CSt-Polyform with other modules under the class of fully prime modules. Before that, we need the following lemma.

Lemma (3.12): If M is fully prime module, and N≤M, then:

1. N≪M if and only if N$\ll_p$M.

2. N$\leq_{cc}$ M if and only if N$\leq_{CSt}$M.

Proof

1. The necessity follows by [7]. For the converse, let N be a submodule of M such that N+L=M, where L≤ M. If L is a proper submodule of M, then by assumption, L is prime. This implies that N is not P-small submodule, but this is a contradiction, thus N=L, hence N$\leq_{cc}$ M.

2. The necessity is clear. Conversely, suppose that N $\leq_{CSt}$ M and let L≤N with N/L≪ M/L. Since M is fully prime, then by (1), N/L$\ll_p$ M/L. By assumption, N=L, that is N$\leq_{cc}$ M.

Theorem (3.13): Let M be a fully prime R-module, then M is a κ-non-CSt-singular module if and only if M is a τ-noncosingular module.

Proof: Assume that M is a κ-non-CSt-singular module and let f be a non-zero endomorphism of M, then f(M) is not P-small submodule of M. Since M is fully prime, so by Proposition (3.12)(1), f(M) is not small submodule of M, thus M is τ-noncosingular module. The proof of the sufficiency follows by the direct implication between small and P-small.

Remark (3.14): Note that Theorem (3.13) is also satisfied when the class of "fully prime module" is replaced by finitely generated (or almost finitely generated or multiplication) module. In fact, the proof has a similar argument both by using [6, Prop.(1.4)] and Lemma (2.4).

Theorem (3.15): Let M be a finitely generated (or multiplication of almost finitely generated) module. Consider the following statements:

1. M is a copolyform module.
2. M is a CSt-Polyform module.
3. M is a κ-non-CSt-singular module.
4. M is a τ-noncosingular module.

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$(3) $\Leftrightarrow$ (4).

Proof:

(1) $\Leftrightarrow$ (2): It is as the proof of Proposition (2.5).
(2) $\Rightarrow$ (3): It is as the proof of Proposition (3.10).
(3) $\Leftrightarrow$ (4): It is as the proof of Remark (3.14).

An R-module M is called Noetherian, if every submodule of M is finitely generated [2].

Corollary (3.16): Let M be a Noetherian module. Consider the following statements:

1. M is a copolyform module.
2. M is a CSt-Polyform module.
3. M is a κ-non-CSt-singular module.
4. M is a τ-noncosingular module.

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$(3) $\Leftrightarrow$ (4).

Proof: In the Noetherian module, every submodule is finitely generated; in particular, M is finitely generated of itself, so as a similar proof of Theorem (3.15), the result follows.

Recall that an R-module M is called Coquasi-Dedekind, if for every proper submodule N of M, $\mathrm{Hom}_R(M,N)=0$. Equivalently, M is Coquasi-Dedekind if every non-zero endomorphism of M is epimorphism [17]. We think that there is no direct implication between CSt-Polyform and Coquasi-Dedekind modules; in fact, we cannot prove that. However, under certain conditions, we could do that as the following proposition shows.

Proposition (3.17): In the class of hollow modules, every CSt-Polyform module is Coquasi-Dedekind.

Proof: Suppose that M is a hollow CSt-Polyform module and $f\in \mathrm{End}_R(M)$, $f\neq 0$. Let N be a proper submodule of M. Consider the following:

$$M \xrightarrow{f} M \xrightarrow{\pi} M/N$$

where π is the natural epimorphism. Note that $\pi of\neq 0$. If it is not epimorphism, it follows that, since M is CSt-Polyform, then $(\pi \circ f)(M)$ is not P-small submodule of M/N. This implies that $(\pi \circ f)(M)$ is not P-small submodule of M [6, Prop.(1.3)]. Hence, $(\pi of)(M)$ is not small submodule of M. But this is a contradiction, since M is hollow, therefore f is epimorphism. That is, M is Coquasi-Dedekind.

Example (3.18): Z_6 is CSt-Polyform Z-module, see Example (2.2)(5). However, it is not Coquasi-Dedekind [16, Cor. (2.3.6)]. It is natural to realize that, since Z_6 is not hollow.

In the following, we introduce a generalization of the Coquasi-Dedekind module.

Definition (3.19): A non-zero module M is called generalized Coquasi-Dedekind (simply G. Coquasi-Dedekind), if every non-zero endomorphism of M is not P-small submodule of M. i.e. $\forall$ $f\in End_R(M)$, $f\neq 0$, f(M) is not P-small submodule of M.

Remark (3.20): It is clear that every Coquasi-Dedekind module is G. Coquasi-Dedekind. Since if M is Coquasi-Dedekind, then every non-zero endomorphism is epimorphism, this means that f(M) is not P-small submodule of M, thus M is G. Coquasi-Dedekind. The converse is not true in general; for example: Z_6 is G. Coquasi-Dedekind, since every endomorphism of Z_6 is not P-small submodule of Z_6. On the other hand, Z_6 is not Coquasi-Dedekind [16, Cor. (2.3.6)].

Proposition (3.21): Every CSt-Polyform module is G. Coquasi-Dedekind.

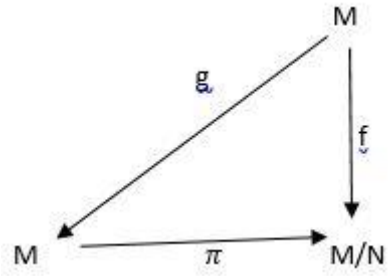
Proof: Let M be a CSt-module and f: M $\rightarrow$ M is a non-zero endomorphism. Since M is CSt-Polyform and M $\cong$ M/(0), then f(M) is not P-small submodule of M/(0), hence M is G. Coquasi-Dedekind.

Remark (3.22): The converse of Proposition (3.21) is not true in general; for example, the Z-module Z is G. Coquasi-Dedekind. In fact, every non-zero endomorphism of Z is not P-small submodule of M, while Z is not CSt-Polyform, see Example (2.2)(2).

In the following theorem, we use a condition under which the converse of Proposition (3.21) is true. Before that, an R-module M is called quasi-projective, if for every submodule N of M and any homomorphism f:M$\rightarrow$M/N, it can be lifted to a homomorphism g:M$\rightarrow$M [5, P.29].

Theorem (3.23): Let M be a quasi-projective module, then M is CSt-Polyform if and only if M is a G. Coquasi-Dedekind module.

Proof: The proof of the necessity of this theorem is provided by Proposition (3.21). For the sufficiency, Let N be a proper submodule of M and f: M $\rightarrow$ M/N, $f\neq 0$. Consider the following diagram:



where π is the natural epimorphism. Since M is quasi projective, then there exists $g\in End(M)$ such that $\pi o g=f$. But M is G. Coquasi-Dedekind, then g(M) is not P-small submodule of M. This implies that $(\pi o g)(M)$ is not P-small submodule of M/N [6, Prop.(1.3)]. But $\pi o g=f$, therefore f(M) is not P-small submodule of M/N, thus M is CSt-Polyform.

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