DOI: 10.24996/ijs.2022.63.4.22

ISSN: 0067-2904

Convergence Theorems of Three-Step Iteration Algorithm in CAT (0) Spaces

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Received: 8/10/2020 Accepted: 16/5/2021 Published: 30/4/2022

Abstract
In this paper, a modified three-step iteration algorithm for approximating a joint fixed point of non-expansive and contraction mapping is studied. Under appropriate conditions, several strong convergence theorems and Δ-convergence theorems are established in a complete CAT (0) space. a numerical example is introduced to show that this modified iteration algorithm is faster than other iteration algorithms. Finally, we prove that the modified iteration algorithm is stable. Therefore these results are extended and improved to a novel results that are stated by other researchers. Our results are also complement to many well-known theorems in the literature. This type of research can be played a vital role in computer programming, game theory and computational analysis.

Keywords: CAT (0) space, Joint fixed points, Non-expansive mapping, Contraction, Strong convergence, Δ-convergence, Stable.

نظريات التقارب لخوارزمية التكرار من ثلاث خطوات في فضاء CAT(0)

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الخلاصة:
في هذا البحث, تم دراسة خوارزمية تكرار معدلة مكونة من ثلاث خطوات لتقريب نقاط صامدة مشتركة لتطبيق اللمتمددة (nonexpansive mapping) وتطبيق الانكماش (contraction) . تحت شروط مناسبة, تم انشاء بعض نظريات التقارب و Δ− تقارب في فضاء CAT(0) وكذلك تقديم مثال رقمي يظهربأن خوارزمية التكرار المعدلة هي اسرع من خوارزميات التكرار الاخرى. أخيرآ, تم اثبات بأن خوارزمية التكرار المعدلة مستقرة وذات نتائج افضل من نتائج الخوارزميات الاخرى , كما أنها تكمل العديد من النظريات المعروفة في الادبيات. هذا النوع من النتائج تلعب يلعب هذا النوع دورآ مهما و حيوياً في برمجة الكمبيوتر ونظرية الالعاب وكذلك التحليل الحسابي

1. Introduction and Preliminaries

The metric space E is called a CAT (0) space, if it is geodesically connected and at least each geodesic triangle in E is as thin as its comparison triangle in the Euclidean plane. Abass, M., et al. in [1] indicated that an R-trees, Pre-Hilbert space, Euclidean buildings are examples of CAT (0) space.

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Fixed point theory in CAT (0) spaces was first studied through Kirk [1]. He proved that each non-expansive (single valued) mapping that is defined on bounded closed convex subset of a complete CAT (0) spaces permanently has a fixed point. Therefore, the fixed point theory for single valued as well as multi-valued mappings in CAT (0) spaces has intensively been evolved by numerous authors [2]. The convergence for non-expansive mappings in CAT (0) spaces was studied by authors in [3]. Thereafter, Khan and Abbas [4] studied the strong and Δ-convergence in CAT (0) space for an iteration process that is independent of the Ishikawa iteration process as well as several results are obtained for two non-expansive mappings. It is important to note that fixed point theorems in CAT (0) space can be stratified to graph theory, computer science and biology [1].

Let (G, d) be a metric space and $u, v \in G$ with $d(u, v) = x$. A geodesic path from u to v means that an isometry $c: [0, x] \to c([0,1]) ⊂ G$ such that $c(0) = u$ and $c(x) = v$. The image of every geodesic path between u and v is called geodesic segment. Each point y in the segment is appeared by$\omega u \oplus (1-\omega)v$, where $\omega \in [0, 1]$ that is $[u, v] = \{\omega u \oplus (1-\omega)v: \omega \in [0, 1]\}$. The space (G, d) is called a geodesic if each two points of G are joined through a geodesic segment, and G is uniquely geodesic if there exists properly one geodesic joining u and v for every$u, v \in G$. A subset H of G is called convex if H has each geodesic segment that joins any two points in H [5].

A geodesic triangle $\Delta(u_1, u_2, u_3)$ is a geodesic metric space (G, d) that consists of three points u_1, u_2, u_3 in G (the vertices Δ) and a geodesic segment between every pair of vertices (the edges of Δ). A comparison triangle $\bar{\Delta}(\overline{u_1}, \overline{u_2}, \overline{u_3})$ in W^2 for $\Delta(u_1, u_2, u_3)$ *is* a triangle in 2-dimensional Euclidean plane W^2 with $\overline{u_1}, \overline{u_2}, \overline{u_3} \in W^2$ such that $d(u_1, u_2) = |\overline{u_1} - \overline{u_2}|_{W^2}$, $d(u_1, u_3) = |\overline{u_1} - \overline{u_3}|_{W^2}$, $d(u_2, u_3) = |\overline{u_2} - \overline{u_3}|_{W^2}$, where $|.|_{W^2}$ is the Eulidean norm on W^2 [6].

Definition [5]: A geodesic space is called CAT (0) space if the whole geodesic triangles achieve the following comparison axiom.

Let Δ be a geodesic triangle in G and $\bar{\Delta} ⊂ W^2$ be a comparison triangle for Δ. Therefore, Δ is called to achieve the CAT (0) inequality if for all $u, v \in \Delta$ and for all $\bar{u}, \bar{v} \in \bar{\Delta}$, $d(u, v) \le d_{W^2}(\bar{u}, \bar{v})$.

If u, v_1, v_2 are points in CAT (0) and $v_0 = \frac{1}{2}(v_1 \oplus v_2)$, then the CAT (0) inequality leads to

$$d(u, v_0)^2 \le \frac{1}{2} d(u, v_1)^2 + \frac{1}{2} d(u, v_2)^2 - \frac{1}{4} d(v_1, v_2)^2$$

Which is the CN (Courbure Negative) inequality of Bruhat and Tits. In general, a geodesic space is a CAT (0) space if and only if it accomplishes (CN) [4].

Lemma (1)[7]: Let (G, d) be a CAT (0) space. Then,

$$d\big((1-k)a \oplus k\mathrm{b}, \mathrm{c}\big)^2 \le (1-k)d(a, c)^2 + kd(b, c)^2 - k(1-k)d(a, b)^2$$

for all $k \in [0,1]$ and $a, b, c \in G$.

Let $\{u_n\}$ be a bounded sequence in a CAT (0) space$G$. We set

$r(u, \{u_n\}) = \lim_{n\to\infty} \sup d(u, u_n)$, for $u \in G$,

The asymptotic radius $r(\{u_n\})$ of $\{u_n\}$ is given through

$r(\{u_n\}) = \inf\{r(u, \{u_n\}): u \in G\}$,

and the asymptotic center $A(\{u_n\})$ of $\{u_n\}$ is defined as

$$A(\{u_n\}) = \{u \in G: r(u, \{u_n\}) = r(\{u_n\})\}.$$

It is well known that $A(\{u_n\})$ has punctually one point in CAT (0) space.

A new iteration algorithm is recently introduced by Vatan Karakaya et al. in [8], which is given as follows:

$$v_1 = v \in H$$
$$v_{n+1} = (1 - \alpha_n - \gamma_n)u_n + \alpha_n T u_n + \gamma_n T w_n$$
$$u_n = (1 - \rho_n - \mu_n)w_n + \rho_n T w_n + \mu_n T v_n$$

$$w_n = (1-\sigma_n)v_n + \sigma_n T v_n,\ \forall n \in N. \quad (1)$$

Where H is a nonempty convex subset of a normed space. $\{\alpha_n\}, \{\gamma_n\}, \{\rho_n\}, \{\mu_n\}, and\ \{\sigma_n\}$ are real sequences in $[0,1]$ that satisfy

$$(\alpha_n + \gamma_n)_{n=0}^{\infty}, (\rho_n + \mu_n)_{n=0}^{\infty} \in [0,1], \forall n \in N, \sum_{n=0}^{\infty} (\alpha_n + \gamma_n) = \infty.$$

In this work, we further modify the iteration(1) that is done by the work of Vatan Karakaya et al. in [8], for joint fixed points of two mapping non-expansive and contraction mapping in a complete CAT (0) space as follows:

Let (E,d) be a metric space and H be a nonempty closed convex subset of E. Consider $T: H \to H$ be a mapping and $u \in H$ be a fixed point of T that means$Tu = u$. The set of joint fixed points of T and S will be denoted by F which is$F = \{ u \in H : Tu = Su = u\}$. If there is a continuous mapping $P: E \to H$ such as $Pu = u, \forall u \in H$ then H is called retract of E . A mapping $P: E \to E$ is called a retraction if $P^2 = P$. If P is a retraction, then $Pv = v$, $\forall\ v$ in the range of P.

$$\begin{aligned} v_1 &= v \in H \\ v_{n+1} &= P\big((1-\alpha_n-\gamma_n)u_n \oplus \alpha_n S u_n \oplus \gamma_n S w_n\big) \\ u_n &= P\big((1-\rho_n-\mu_n)w_n \oplus \rho_n T w_n \oplus \mu_n T v_n\big) \\ w_n &= P\big((1-\sigma_n)v_n \oplus \sigma_n T v_n\big),\ n \in N. \end{aligned} \quad (2)$$

Where $\{\alpha_n\}, \{\gamma_n\}, \{\rho_n\}, \{\mu_n\}, and\ \{\sigma_n\}$ are real sequences in $[0,1]$ that satisfy

$$(\alpha_n + \gamma_n)_{n=0}^{\infty}, (\rho_n + \mu_n)_{n=0}^{\infty} \in [0,1], \forall n \in N, \sum_{n=0}^{\infty} (\alpha_n + \gamma_n) = \infty.$$

The purpose of this paper is to study the modified three-step iteration algorithm (2) for approximating a joint fixed point and to discuss the existence and convergence theorems for the above iteration algorithm of CAT (0) spaces. We also compare the rate of convergence between the modified iteration algorithm (2) and the iteration algorithm (3) as well as by utilizing a numerical example we show that our iteration algorithm is faster than other iteration algorithms. The notion of stability of fixed point iteration procedures will be discussed in this work. The theme of stability, as an application of theory of fixed point, have been intensively investigated by numerous authors [9, 10]. As well as it is useful in diverse domains of mathematics such as difference equations, differential equations, numerical analysis, integral equations, and game theory . For more details see [11].

Now we give the following definitions and lemmas that are needed to prove our convergence results.

Definition (2)[7]: Let H be a nonempty subset of a metric space (E,d). A mapping $T : H \to H$ is called a contraction if there exists $\vartheta \in (0,1)$ such that $d(Ta,Tb) \le \vartheta d(a,b)$, for all $a, b \in H$ and is called nonexpansive mapping if $d(Ta,Tb) \le d(a,b)$, for all $a, b \in H$. Every contraction mapping is non-expansive mapping.

Definition (3)[12]: A sequence $\{u_n\}$ in a CAT(0) space E is Δ-convergence to $u \in E$ if u is the unique asymptotic center of $\{v_n\}$ $\forall$ subsequence $\{v_n\}$ of $\{u_n\}$. Here, we set $\Delta - \lim_{n\to\infty} u_n = u$ and u is the Δ-limit of $\{u_n\}$.

Definition (4)[12]: Two mappings T, S is called to accomplish the condition (N) if there is a nondecreasing function $\xi: [0,\infty) \to [0,\infty)$ with $\xi(0) = 0$, and $\xi(Þ) > 0, \forall\ Þ \in (0,\infty)$ such that

$d(a,Ta) \ge \xi\big(d(a,F)\big)$or $d(a,Sa) \le \xi\big(d(a,F)\big)$ $\forall\ a \in E$, where $d(a,F) = \inf\{\|a - h^*\|, h^* \in F = F(T) \cap F(S) \neq \emptyset\}$.

Lemma (5)[7]: Let E be a complete CAT (0) space and $a \in E$. Presume $\{s_n\}$ is a sequence in $[z,c]$ for $z, c \in (0,1)$ and $\{a_n\}, \{b_n\}$ are sequences in E such as $\lim_{n\to\infty} sup\ d(a_n, h^*) \le$

t, $\lim_{n\to\infty} sup\ d(b_n, h^*) \le t$ and $\lim_{n\to\infty} d((1-s_n)\, a_n \oplus s_n b_n) = t$ for some $t \ge 0$. Thus $\lim_{n\to\infty} d(\, a_n, b_n) = 0$.

Lemma (6)[13]: Let H be a closed convex subset of a complete CAT (0) space E and $T: H \to H$ be a nonexpansive mapping. Therefore $\Delta - \lim_{n\to\infty} v_n = v$ and $\lim_{n\to\infty} d(v_n, Tv_n) = 0$, lead to $v_n \in H$ and $Tv = v$.

Lemma (7)[12]: If $\{v_n\}$ is a bounded sequence in a complete CAT (0) space with $A(\{v_n\}) = \{v\}$, there is a subsequence $\{u_n\}$ of $\{v_n\}$ with $A(\{u_n\}) = \{u\}$ and $d(v_n, u)$ converges, therefore $v = u$.

Lemma (8)[8]: If τ is a real number that satisfies $0 \le \tau < 1$ and $(\Omega_n)_{n\in N}$ is a sequence of positive number such as $\lim_{n\to\infty} \Omega_n = 0$, then for any $(\Omega_n)_{n\in N}$ accomplishing $u_{n+1} \le \tau u_n + \Omega_n$,
$n = 1,2,\ldots$, one has $\lim_{n\to\infty} u_n = 0$.

2. Convergence Results

In this section, several results for the iteration algorithm (2) to converge to a joint fixed point of non-expansive mapping and contraction in a complete CAT (0) spaces are proved.

We assume that H is a nonempty closed convex subset of a complete CAT (0) space E,and let $F := F(T) \cap F(S)$ be the set of joint fixed point of two mappings T and S. Now, we start the following:

Lemma (9): Let $T: H \to H$ be a mapping which accomplishes contraction and $S: H \to H$ be a mapping which accomplishes non-expansive. If $\{v_n\}$ is a sequence defined by (2), then
$1 - \lim_{n\to\infty} d(v_n, k^*)$ exists for all $k^* \in F$.
$2 - \lim_{n\to\infty} d(v_n, Tv_n) = \lim_{n\to\infty} d(v_n, Sv_n) = 0$

Proof:

We start to prove that $\lim_{n\to\infty} d(v_n, k^*)$ exists, we have

$$\begin{aligned} d(w_n, k^*) &= d(P((1-\sigma_n)v_n \oplus \sigma_n Tv_n, Pk^*)) \\ &\le (1-\sigma_n)d(v_n, k^*) \oplus \sigma_n d(PTv_n, Pk^*) \\ &\le (1-\sigma_n)d(v_n, k^*) + \sigma_n d(Tv_n, k^*) \\ &\le (1-\sigma_n)d(v_n, k^*) + \vartheta\sigma_n d(v_n, k^*) \\ &\le (1-\sigma_n + \vartheta\sigma_n)d(v_n, k^*) \\ &\le d(v_n, k^*) \end{aligned}$$

$$\begin{aligned} d(u_n, k^*) &= d\left(P\left((1-\rho_n-\mu_n)w_n \oplus \rho_n Tw_n \oplus \mu_n Tv_n, Pk^*\right)\right) \\ &\le (1-\rho_n-\mu_n)d(w_n, k^*) + \rho_n d(Tw_n, k^*) + \mu_n d(Tv_n, k^*) \\ &\le (1-\rho_n-\mu_n)d(w_n, k^*) + \vartheta\rho_n d(w_n, k^*) + \vartheta\mu_n d(v_n, k^*) \\ &\le d(v_n, k^*). \end{aligned}$$

Therefore,

$$\begin{aligned} d(v_{n+1}, k^*) &= d\left(P\left((1-\alpha_n-\gamma_n)u_n \oplus \alpha_n Su_n \oplus \gamma_n Sw_n, Pk^*\right)\right) \\ &\le (1-\alpha_n-\gamma_n)d(u_n, k^*) + \alpha_n d(Su_n, k^*) + \gamma_n d(Sw_n, k^*) \\ &\le (1-\alpha_n-\gamma_n)d(u_n, k^*) + \alpha_n d(u_n, k^*) + \gamma_n d(w_n, k^*) \\ &= d(v_n, k^*). \end{aligned}$$

Which shows that $d(v_n, k^*)$ is decreasing and bounded below, thererfore $\lim_{n\to\infty} d(v_n, k^*)$ exists.

To prove (2), we have $\lim_{n\to\infty} d(u_n, k^*)$ exists for each $k^* \in F$. Assume that $d(v_n, k^*) = h, \forall h \ge 0$.

If $h = 0$, the proof is straight forward.

Now, assume that $h > 0$. Since $d(w_n, k^*) \le d(v_n, k^*)$ and $d(u_n, k^*) \le d(v_n, k^*)$. Therefore, we have

$\lim_{n\to\infty} sup\, d(w_n, k^*) \le h$ and $\lim_{n\to\infty} sup\, d(u_n, k^*) \le h$.

As well as we have

$d(Sw_n, k^*) \leq d(w_n, k^*)$ *and* $d(Su_n, k^*) \leq d(u_n, k^*)$.
Thus, $\lim_{n\to\infty} sup\, d(Sw_n, k^*) \leq h$ *and* $\lim_{n\to\infty} sup\, d(Su_n, k^*) \leq h$.
Moreover, $\lim_{n\to\infty} d(v_{n+1}, k^*) = h$, and

$$
\begin{aligned}
h &= \lim_{n\to\infty} d(v_{n+1}, k^*) \\
&\leq \lim_{n\to\infty} d\left(P\big((1-\alpha_n-\gamma_n)u_n \oplus \alpha_n Su_n \oplus \gamma_n Sw_n, k^*\big)\right) \\
&\leq \lim_{n\to\infty} d\big((1-\gamma_n)u_n + \gamma_n Sw_n, k^*\big).
\end{aligned}
$$

By Lemma (5), we get
$\lim_{n\to\infty} d(u_n, Sw_n) = 0$.
Now,

$$
\begin{aligned}
d(v_{n+1}, k^*) &= d\left(P\big((1-\alpha_n-\gamma_n)u_n \oplus \alpha_n Su_n \oplus \gamma_n Sw_n, k^*\big)\right) \\
&\leq d\big((1-\gamma_n)u_n + \gamma_n Sw_n, k^*\big) \\
&\leq d(u_n, k^*) + \gamma_n d(u_n, Sw_n).
\end{aligned}
$$

So, we deduce$\lim_{n\to\infty} d(u_n, k^*) = h$, and

$$
\begin{aligned}
d(u_n, k^*) &\leq d(u_n, Sw_n) + d(Sw_n, k^*) \\
&\leq d(u_n, Sw_n) + d(w_n, k^*).
\end{aligned}
$$

Then, we deduce $\lim_{n\to\infty} d(w_n, k^*) = h$.
We also have

$$
\begin{aligned}
h &= \lim_{n\to\infty} d(w_n, k^*) \\
&= \lim_{n\to\infty} d(P((1-\sigma_n)v_n \oplus \sigma_n Tv_n, k^*)) \\
&= \lim_{n\to\infty} d((1-\sigma_n)v_n + \sigma_n Tv_n, k^*).
\end{aligned}
$$

From Lemma (5), we get

$$\lim_{n\to\infty} d(v_n, Tv_n) = 0.$$

Next, $d(w_n, v_n) \leq (1-\sigma_n)d(v_n, v_n) + \sigma_n\, d(Tv_n, v_n)$.
This gives,

$$\lim_{n\to\infty} d(w_n, v_n) = 0.$$

Moreover, from
$d(u_n, v_n) \leq (1-\rho_n-\mu_n)d(w_n, v_n) + \rho_n d(Tw_n, v_n) + \mu_n d(Tv_n, v_n)$
We get,

$$\lim_{n\to\infty} d(u_n, v_n) = 0,$$

and
$d(v_n, Sv_n) \leq d(v_n, u_n) + d(u_n, Sw_n) + d(Sw_n, Sv_n)$,
This leads to,

$$\lim_{n\to\infty} d(v_n, Sv_n) = 0$$

Theorem (10): Let $T, S: H \to H$ with joint fixed point $k^* \in F$ be as in Lemma (9). Presume that $\{v_n\}$ is a sequence defined by (2), then $\{v_n\}$ converges strongly to the joint fixed point.
Proof: To prove that $\{v_n\}$ converges to the joint fixed point $Tk^* = Sk^* = k^*$. We have

$$
\begin{aligned}
d(v_{n+1}, k^*) &= d\left(P\big((1-\alpha_n-\gamma_n)u_n \oplus \alpha_n Su_n \oplus \gamma_n Sw_n, Pk^*\big)\right) \\
&\leq (1-\alpha_n-\gamma_n)d(u_n, k^*) + \alpha_n d(Su_n, k^*) + \gamma_n d(Sw_n, k^*) \\
&\leq (1-\alpha_n-\gamma_n)(1-\rho_n-\mu_n+\vartheta\rho_n+\vartheta\mu_n)d(v_n, k^*) + \alpha_n(1-\rho_n-\mu_n+\vartheta\rho_n+\vartheta\mu_n)d(v_n, k^*) + \gamma_n(1-\sigma_n+\vartheta\sigma_n)d(v_n, k^*) \\
&\leq (1-(\alpha_n+\gamma_n)(1-\vartheta))\, d(v_n, k^*).
\end{aligned}
$$

By induction

$$d(v_{n+1}, k^*) \le \prod_{i=0}^{n} [(1-(\alpha_i+\gamma_i)(1-\vartheta))]\, d(v_0, k^*)$$
$$\le d(v_0, k^*) e^{-(1-\vartheta)\sum_{i=0}^{n}(\alpha_i+\gamma_i)}$$

$\forall n \in N$, since $0 < \vartheta < 1, \alpha_n, \gamma_n \in [0,1]$ & $\sum_{n=0}^{\infty}(\alpha_n + \gamma_n) = \infty$, we have $\lim_{n\to\infty} sup\, d(v_{n+1}, k^*) \le \lim_{n\to\infty} sup\, d(v_0, k^*) e^{-(1-\vartheta)\sum_{i=0}^{n}(\alpha_i+\gamma_i)} = 0$. Hence, $\lim_{n\to\infty} d(v_{n+1}, k^*) = 0, for\ all\ \ n \in N$.

Theorem (11): Let $T, S: H \to H$ with joint fixed point $k^* \in F$ be as in Lemma (9) and T, S accomplish condition (N). Presume that $\{v_n\}$ is a sequence defined by (2), then $\{v_n\}$ converges strongly to the joint fixed point.

Proof: From Lemma (9), we have $\lim_{n\to\infty} d(v_n, k^*)$ exists and $\lim_{n\to\infty} d(v_n, Tv_n) = \lim_{n\to\infty} d(v_n, Sv_n) = 0$. Then from condition (N), we obtain

$$\lim_{n\to\infty} f(d(v_n, F)) \le \lim_{n\to\infty} d(v_n, Tv_n) = 0$$

or

$$\lim_{n\to\infty} f(d(v_n, F)) \le \lim_{n\to\infty} d(v_n, Sv_n) = 0$$

Thus, we have $\lim_{n\to\infty} f\big(d(v_n, F)\big) = 0$, since ξ is a nondecreasing function accomplishing with $\xi(0) = 0$ and $\xi(Þ) > 0$, $\forall\, Þ \in (0, \infty)$, we have $\lim_{n\to\infty} d(v_n, F) = 0$.

Now, to show that $\{v_n\}$ is a cauchy sequence in H. Let $\epsilon > 0$, since $\lim_{n\to\infty} d(v_n, F)$, then there exsits a positive integer n_0, such that

$$d(v_n, F) < \frac{\epsilon}{4}, \qquad for\ all\ n \ge n_0$$

In particular, inf $\{d(v_{n_0}, k^*); k^{**} \in F\} < \frac{\epsilon}{4}$. Therefore there exists $k^{**} \in F$, such that $d(v_{n_0}, k^{**}) < \frac{\epsilon}{2}$.

Now, for all $n, m \ge n_0$, we get

$$d(v_{n+m}, v_n) \le d(v_{n+m}, k^{**}) + d(k^{**}, v_n)$$
$$\le 2d(v_{n_0}, k^{**})$$
$$< 2(\tfrac{\epsilon}{2}) = \epsilon.$$

Therefore $\{v_n\}$ is a cauchy sequence in H of CAT (0) space E. Thus, the completeness of E implies that $\{v_n\}$ must be converge to q in H. Next, to show that $q \in F, \lim_{n\to\infty} d(v_n, F) = 0$ gives that $d(q, F) = 0$ and the closedness of F forces $q \in F$.

Theorem (12): Let $T, S: H \to H$ with joint fixed point $k^* \in F$ be as in Lemma (9). Assume that $\{v_n\}$ is a sequence defined by (2), then $\{v_n\}$ converges strongly to the joint fixed point iff $\lim_{n\to\infty} d(v_n, F) = 0$, where $d(v_n, F) = \inf\{d(v, k^*):\ k^* \in F\}$.

Proof: Necessity is clear.

Conversely, presume that $\lim_{n\to\infty} d(v_n, F) = 0$, we have $d(v_{n+1}, k^*) \le d(v_n, k^*), \forall k^* \in F$. This leads to

$$d(v_{n+1}, F) \le d(v_n, F)$$

So that, $d(v_n, F)$ exists. Thus, the result is followed from Theorem (11).

Theorem (13): Let $T, S: H \to H$ with joint fixed point $k^* \in F$ be as in Lemma (9). Let $\{v_n\}$ be the sequence defined by (2) with the real sequences $\{\alpha_n\}, \{\gamma_n\}, \{\rho_n\}, \{\mu_n\}, and\ \{\sigma_n\} \in [0,1]$ for all $n \in N$. Therefore, the sequence $\{v_n\}$ Δ-converges to the joint fixed point.

Proof: We have seen from Lemma (9) that $\lim_{n\to\infty} d(v_n, k^*)$ exists for all $k^* \in F$ and

$$\lim_{n\to\infty} d(v_n, Tv_n) = \lim_{n\to\infty} d(v_n, Sv_n) = 0.$$

Now, we show that $\varphi_\Delta(v_n) := \cup_{\{z_n\} ⊂ \{u_n\}} A([z_n]) ⊂ F(T,S)$.

Let $\varphi_\Delta(v_n) \coloneqq \bigcup_{\{z_n\} \subseteq \{v_n\}} A([z_n]) \subseteq F$. To show that $\varphi_\Delta(v_n) \subseteq F$, assume that $z \in \varphi_\Delta(v_n)$. Therefore, there is a subsequence $\{z_n\}$ of$\{v_n\}$. Such that $A([z_n]) = \{z\}$. By Lemma (6) for all subsequence$\{y_n\}$ of $\{z_n\}$ such as $\Delta - \lim_{n\to\infty} y_n = y$ and $y \in H$. So that from Lemma (7) we get$z = y$. This shows that $\varphi_\Delta(v_n) \subseteq F$.
Finally, we have to show that $\{v_n\}$ Δ-convergence to the joint fixed point of F. It is enough to show that $\varphi_\Delta(v_n)$ is a singleton, let $\{z_n\}$ be a subsequence of$\{v_n\}$. By Lemma (6), there exists a subsequence $\{y_n\}$ of $\{z_n\}$ such as $\Delta - \lim_{n\to\infty} y_n = y$ and $y \in H$. Let $A([z_n]) = \{z\}$ and$A([v_n]) = \{v\}$. Since $z \in \varphi_\Delta(v_n)$ and$d(v_n, z)$ are convergent, then from Lemma (7), we get $v = z$. So , $\varphi_\Delta(v_n)$ has punctually one point. Hence, $\{v_n\}$ is Δ-convergence to the joint fixed point of F.

3. Applications

Theorem (14): Let $T, S: H \to H$ with joint fixed point $k^* \in F$ be as in Lemma (9). Let $\{v_n\}$ be the sequence defined by (2) with the real sequences $\{\alpha_n\}, \{\gamma_n\}, \{\rho_n\}, \{\mu_n\},\ and\ \{\sigma_n\} \in [0,1]$ such that $0 < B < \alpha_n + \gamma_n, for\ all\ n \in N$. Then, $\{v_n\}$ is stable.
Proof: Let $\{z_n\}$ be an arbitrary sequence in CAT (0) space E. Define

$$\varepsilon_n = d(z_{n+1}, P((1-\alpha_n-\gamma_n)a_n \oplus \alpha_n S a_n \oplus \gamma_n S b_n))$$

Where $a_n = (1-\rho_n-\mu_n)b_n \oplus \rho_n T b_n \oplus \mu_n T z_n)$ & $b_n = (1-\sigma_n)z_n \oplus \sigma_n T z_n, \forall n$.
Presume that $v_n \to k^*$as $n \to \infty$ and $\lim_{n\to\infty} \varepsilon_n = 0$. Therefore, we belay that $\lim_{n\to\infty} z_n = k^*$

$$\begin{aligned}
d(z_{n+1}, k^*) = {} & d\left(z_{n+1}, P\big((1-\alpha_n-\gamma_n)a_n \oplus \alpha_n S a_n \oplus \gamma_n S b_n\big)\right) \\
& + d\left(P\big((1-\alpha_n-\gamma_n)a_n \oplus \alpha_n S a_n \oplus \gamma_n S b_n\big), Pk^*\big)\right) \\
& \le \varepsilon_n + d\left(P\big((1-\alpha_n-\gamma_n)a_n \oplus \alpha_n S a_n \oplus \gamma_n S b_n\big), Pk^*\big)\right) \\
& \le (1-\alpha_n-\gamma_n)d(a_n, k^*) + \alpha_n d(Sa_n, k^*) + \gamma_n d(Sb_n, k^*) \\
& \le (1-\alpha_n-\gamma_n)(1-\rho_n-\mu_n+\vartheta\rho_n+\vartheta\mu_n)d(z_n, k^*) + \alpha_n(1-\rho_n-\mu_n+\vartheta\rho_n+ \\
& \qquad \vartheta\mu_n)d(z_n, k^*) + \gamma_n(1-\sigma_n+\vartheta\sigma_n)d(z_n, k^*) \\
& \le \varepsilon_n + (1-(\alpha_n+\gamma_n)(1-\vartheta))\, d(z_n, k^*) \\
& \le \varepsilon_n + (1-B(1-\vartheta))\, d(z_n, k^*)
\end{aligned}$$

Therefore, from Lemma (8), we get $\lim_{n\to\infty} z_n = k^*$.
Conversely, assume that $\lim\limits_{n\to\infty} z_n = k^*$, we have to show that $\lim_{n\to\infty} \varepsilon_n = 0$.

$$\begin{aligned}
\varepsilon_n &= d\left(z_{n+1}, P\big((1-\alpha_n-\gamma_n)a_n \oplus \alpha_n S a_n \oplus \gamma_n S b_n\big)\right) \\
&\le d(z_{n+1}, k^*) + d\left(P\big((1-\alpha_n-\gamma_n)a_n \oplus \alpha_n S a_n \oplus \gamma_n S b_n\big), Pk^*\big)\right) \\
&\le d(z_{n+1}, k^*) + (1-(\alpha_n+\gamma_n)(1-\vartheta))\, d(z_n, k^*)
\end{aligned}$$

Since, $\vartheta \in (0,1), \alpha_n + \gamma_n \in [0,1],\ \forall n,$

$$0 < 1 - (\alpha_n+\gamma_n)(1-\vartheta) < 1$$

By taking the limit of both sides and utilizing the hypothesis $\lim_{n\to\infty} z_n = k^*$, we get$\lim_{n\to\infty} \varepsilon_n = 0$.

Theorem (15): Let $T: H \to H$ be a mapping which accomplishes contraction and $S: H \to H$ be a mapping which accomplishes non-expansive. Let $\{\alpha_n\}, \{\gamma_n\}, \{\rho_n\}, \{\mu_n\}, and\ \{\sigma_n\}$ be real sequences in $[0,1]$. Then $\{y_n\}$ defined by:

$$\begin{aligned}
y_1 &= z \in H \\
y_{n+1} &= P\big((1-\alpha_n-\gamma_n)y_n \oplus \alpha_n S a_n \oplus \gamma_n S b_n\big) \\
a_n &= P\big((1-\rho_n-\mu_n)y_n \oplus \rho_n T b_n \oplus \mu_n T y_n\big) \\
b_n &= P\big((1-\sigma_n)y_n \oplus \sigma_n T y_n\big),\ n \in N \qquad (3)
\end{aligned}$$

is Δ-convergent to the joint fixed point.

4. Numerical Example

In this section, we give a theorem and numerical example to support our results .We start by the following definition.

Definition (15) [14]: Let $(б_n)$ and $(Л_n)$ be two sequences of real numbers that converge to $б$ and $Л$ if

$$\lim_{n\to\infty} \frac{\|б_n - б\|}{\|Л_n - Л\|} = 0$$

Then we said that the convergent of $(б_n)$ is faster than the convergent of$(Л_n)$.

Theorem (16): Let $T: H \to H$ be a contraction mapping and $S: H \to H$ be a non-expansive mapping. Assume that the iteration algorithm (2) and the iteration algorithm (3) converge to the same joint fixed point k^*. Then the convergent of the iteration algorithm (2) is faster than the convergent of the iteration algorithm (3).

Proof: Let $k^* \in F$. Therefore, for iteration algorithm (2)

$$\begin{aligned} d(w_n, k^*) &= d(P((1-\sigma_n)v_n \oplus \sigma_n T v_n, Pk^*)) \\ &\le (1-\sigma_n)d(v_n, k^*) \oplus \sigma_n d(PTv_n, Pk^*) \\ &\le (1-\sigma_n)d(v_n, k^*) + \sigma_n d(Tv_n, k^*) \\ &\le (1-\sigma_n)d(v_n, k^*) + \vartheta\sigma_n d(v_n, k^*) \\ &\le (1-\sigma_n + \vartheta\sigma_n)d(v_n, k^*) \end{aligned}$$

Presume that

$ღ = (1-\sigma_n + \vartheta\sigma_n)$

so, $d(w_n, k^*) \le ღ d(v_n, k^*)$.

$$\begin{aligned} d(u_n, k^*) &= d\Big(P\big((1-\rho_n-\mu_n)w_n \oplus \rho_n T w_n \oplus \mu_n T v_n, Pk^*\big)\Big) \\ &\le (1-\rho_n-\mu_n)d(w_n, k^*) + \rho_n d(Tw_n, k^*) + \mu_n d(Tv_n, k^*) \\ &\le (1-\rho_n-\mu_n)ღ d(v_n, k^*) + \vartheta\rho_n d(w_n, k^*) + \vartheta\mu_n d(v_n, k^*) \\ &\le (1-\rho_n-\mu_n)ღ d(v_n, k^*) + \vartheta\rho_n ღ d(v_n, k^*) + \vartheta\mu_n d(v_n, k^*) \\ &\le [(1-\rho_n-\mu_n)ღ + \vartheta\rho_n ღ + \vartheta\mu_n] d(v_n, k^*) \end{aligned}$$

Now by assuming that

$\lambda = [(1-\rho_n-\mu_n)ღ + \vartheta\rho_n ღ + \vartheta\mu_n]$

We have , $d(u_n, k^*) \le \lambda d(v_n, k^*)$.

Therefore,

$$\begin{aligned} d(v_{n+1}, k^*) &= d\Big(P\big((1-\alpha_n-\gamma_n)u_n \oplus \alpha_n S u_n \oplus \gamma_n S w_n, Pk^*\big)\Big) \\ &\le (1-\alpha_n-\gamma_n)d(u_n, k^*) + \alpha_n d(Su_n, k^*) + \gamma_n d(Sw_n, k^*) \\ &\le (1-\alpha_n-\gamma_n)d(u_n, k^*) + \alpha_n d(u_n, k^*) + \gamma_n d(w_n, k^*) \\ &\le (1-\alpha_n-\gamma_n)\lambda d(v_n, k^*) + \alpha_n \lambda d(v_n, k^*) + \gamma_n ღ d(v_n, k^*) \\ &\le [(1-\alpha_n-\gamma_n)\lambda + \alpha_n\lambda + \gamma_n ღ] d(v_n, k^*) \\ &= (1-(\alpha_n+\gamma_n))\lambda)\, d(v_n, k^*) \\ &\le (1-(\alpha\_n+\gamma\_n)\lambda)^n\, d(v_n, k^*). \end{aligned}$$

Let $б_n = (1-(\alpha\_n+\gamma\_n)\lambda)^n\, d(v_n, k^*)$.

Now, by iteration algorithm (3)

$$\begin{aligned} d(b_n, k^*) &= d(P((1-\sigma_n)y_n \oplus \sigma_n T y_n, Pk^*)) \\ &\le ღ d(y_n, k^*) \end{aligned}$$

$$\begin{aligned} d(a_n, k^*) &= d\Big(P\big((1-\rho_n-\mu_n)y_n \oplus \rho_n T b_n \oplus \mu_n T y_n, Pk^*\big)\Big) \\ &\le (1-\rho_n-\mu_n)d(y_n, k^*) + \rho_n d(Tb_n, k^*) + \mu_n d(Ty_n, k^*) \\ &\le \lambda d(y_n, k^*) \end{aligned}$$

Therefore,

$$d(y_{n+1}, k^*) = d\Big(P\big((1-\alpha_n-\gamma_n)y_n \oplus \alpha_n S a_n \oplus \gamma_n S b_n, Pk^*\big)\Big)$$

$$\leq (1-\alpha_n-\gamma_n)d(y_n,k^*)+\alpha_n d(Su_n,k^*)+\gamma_n d(Sb_n,k^*)$$
$$\leq (1-(\alpha_n+\gamma_n)+\alpha_n\lambda+\gamma_n\,\omega)\,d(y_n,k^*)$$
$$\leq (1-(\alpha_n+\gamma_n))d(y_n,k^*)$$
$$\leq (1-(\alpha_n+\gamma_n))^n d(y_n,k^*)$$

Let $Л_n=(1-(\alpha_n+\gamma_n))^n d(y_n,k^*)$, then we have

Now,

$$\frac{\beta_n}{Л_n}=\frac{(1-(\alpha_n+\gamma_n))\lambda)^n d(v_n,k^*)}{(1-(\alpha_n+\gamma_n))^n d(y_n,k^*)}=\lambda^n\frac{d(v_n,k^*)}{d(y_n,k^*)}\to 0\ as\ n\to\infty.$$

Therefore, the convergent of $\{v_n\}$ is faster than the convergent of $\{y_n\}$ to k^*.

Example (17): Consider the two mappings $T,S:R\to R$ by $T(u)=\frac{1+u}{2}$ and $S(u)=\frac{1+4u}{5},\forall u\in R$. So that, one can easily see that T is contraction and S is nonexpansive nonself mapping. It is clear that $F(T)=F(S)=\{1\}$ of the mappings T and S. Set $\alpha_n=0.01,\gamma_n=0.10$, $\rho_n=0.03,\mu_n=0.05$ and $\sigma_n=0.3$. By using MATLAB program, we have seen that the iteration algorithm which is defined by (2) is faster than the iteration algorithm that defined by (3) for initial points $u_1=12$. Finally, the convergence demeanors of the iteration algorithm (2) and (3) are appeared in Figure 1.

Table 1-Numerical results correspondent to $u_1=12$ for 107 steps

n	Iteration (2)	Iteration (3)	n	Iteration (2)	Iteration (3)	n	Iteration (2)	Iteration (3)
0	12	12	36	-	1.1657	72	-	1.0025
1	7.3048	10.7900	37	-	1.1475	73	-	1.0022
2	4.6136	9.7131	38	-	1.1313	74	-	1.0020
3	3.0712	8.7547	39	-	1.1168	75	-	1.0018
4	2.1817	7.9016	40	-	1.1040	76	-	1.0016
5	1.6804	7.1425	41	-	1.0926	77	-	1.0014
6	1.3900	6.4668	42	-	1.0824	78	-	1.0012
7	1.2235	5.8654	43	-	1.0733	79	-	1.0011
8	1.1281	5.3302	44	-	1.0652	80	-	1.0010
9	1.0734	4.8539	45	-	1.0581	81	-	1.0009
10	1.0421	4.4300	46	-	1.0517	82	-	1.0008
11	1.0421	4.0527	47	-	1.0460	83	-	1.0007
12	1.0138	3.7169	48	-	1.0409	84	-	1.0006
13	1.0079	3.4180	49	-	1.0364	85	-	1.0005
14	1.0045	3.1521	50	-	1.0324	86	-	1.0005
15	1.0026	2.9153	51	-	1.0289	87	-	1.0004
16	1.0015	2.7046	52	-	1.0257	88	-	1.0004
17	1.0009	2.5171	53	-	1.0229	89	-	1.0003
18	1.0005	2.3502	54	-	1.0203	90	-	1.0003
19	1.0003	2.2017	55	-	1.0181	91	-	1.0003
20	1.0002	2.0695	56	-	1.0161	92	-	1.0002
21	1.0001	1.9519	57	-	1.0143	93	-	1.0002
22	1.0001	1.8472	58	-	1.0128	94	-	1.0002
23	1.0000	1.7540	59	-	1.0114	95	-	1.0002
24	-	1.6710	60	-	1.0101	96	-	1.0002
25	-	1.5972	61	-	1.0090	97	-	1.0001
26	-	1.5315	62	-	1.0080	98	-	1.0001
27	-	1.4731	63	-	1.0071	99	-	1.0001
28	-	1.4210	64	-	1.0063	100	-	1.0001
29	-	1.3747	65	-	1.0056	101	-	1.0001
30	-	1.3335	66	-	1.0050	102	-	1.0001
31	-	1.2968	67	-	1.0045	103	-	1.0001
32	-	1.2642	68	-	1.0040	104	-	1.0001
33	-	1.2351	69	-	1.0035	105	-	1.0001
34	-	1.2092	70	-	1.0032	106	-	1.0000
35	-	1.1862	71	-	1.0028	107	-	-

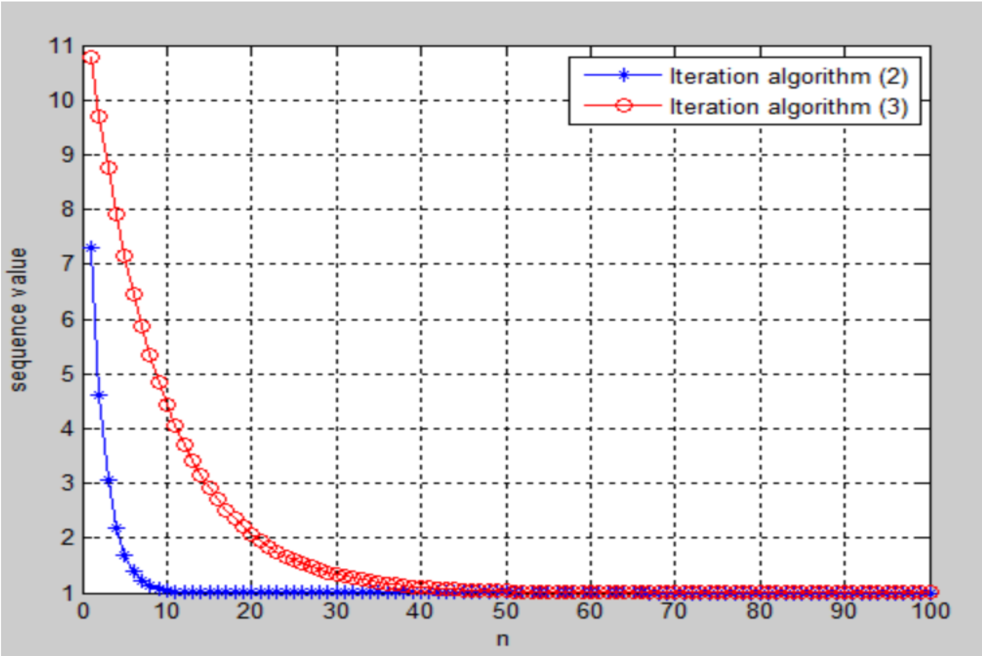

Figure 1-Convergence behaviors correspondent to $u_1 = 12$ for 100 steps.

5. CONCLUSION

In this study, we have been discussed a modified three-step iteration algorithm (2) for approximating fixed point. We have also been established several strong convergence and Δ-convergence results under appropriate conditions. In addition the stability of three-step iteration algorithm (2) for two mappings non-expansive mapping and contraction mapping is proved. The results in this paper are provided improvement and extensions of several previous works for a CAT (0) space that are given in the literature, as well as by utilizing a numerical example we also compare the rate of convergence between the modified iteration algorithm (2) and the iteration algorithm (3). The results shows that our modified iteration algorithm is faster than other iteration algorithms.

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