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Series Solutions of Delay Integral Equations via a Modified Approach of Homotopy Analysis Method

Shaheed N. Huseen, Ali S. Tayih
Mathematics Department, Faculty of Computer Science and Mathematics, University of Thi-Qar, Thi-Qar, Iraq

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Abstract
In this paper, the series solutions of a non-linear delay integral equations are considered by a modified approach of homotopy analysis method (MAHAM). We split the function $f(x)$ into infinite sums. The outcomes of the illustrated examples are included to confirm the accuracy and efficiency of the MAHAM. The exact solution can be obtained using special values of the convergence parameter.

Keywords: Delay integral equations; Homotopy analysis method; Series solutions.

الحلول المتسلسلة للمعادلات التكاملية التباطؤية باستخدام أسلوب معدل لطريقة التحليل الهوموتوبي

شهيد ناصر حسين ، علي سمير تايه
قسم الرياضيات، كلية علوم الحاسوب والرياضيات، جامعة ذي قار، ذي قار، العراق

الخلاصة
في هذا البحث، قمنا بدراسة الحلول المتسلسلة للمعادلات التكاملية التباطؤية الخطية وغير الخطية باستخدام أسلوب معدل لطريقة التحليل الهوموتوبي. عملنا على تجزئة الدالة $f(x)$ الى مجموع غير منتهي. النتائج التي حصلنا عليها من خلال الأمثلة التوضيحية اثبتت مدى كفاءة ودقة الطريقة المستخدمة. وقد حصلنا على الحلول المضبوطة باستخدام قيم خاصة لمعلمة التقارب.

1. Introduction

Delay integral equations (DIEs) are used extensively in the applied and mathematical sciences for modeling various phenomenon; for instance, biomathematics, biological models and medical science [1-4], dynamical systems, physics and physical models [5–7], population growth, and infectious diseases [8,9]. Delay integral equations are solved by different methods, such as block pulse functions [10, 11], homotopy perturbation method (HPM) [12], and variation iteration method (VIM) [13]. The homotopy analysis method (HAM), proposed by Liao [14], is a powerful technique to solve non-linear problems. In recent years, this method has been effectively applied to numerous problems in science and engineering [15-18]. All of these successful applications verified the validity, effectiveness, and flexibility of the HAM. Recently, some modifications of HAM were published to facilitate and accurate the calculations, accelerate the rapid convergence of the series solutions, and reduce the size of work [19-28]. It is the aim of this paper to suggest a new powerful modification of the HAM,

*Email: shn_n2002@yahoo.com

namely a modified approach for homotopy analysis method (MAHAM) to solve the non-linear delay integral equations of the form

$$y(x) = f(x) + \lambda \int_{x_0}^{\theta(x)} K(x,s)F\big(y(\sigma(s))\big)ds, x \in [x_0, X], \quad (1)$$

where $f(x), K(x,s)$, and $\theta(x)$ are given, λ is constant, $\sigma(s)$ represents a general delay function, and $y(x)$ is unknown. If $\theta(x) = b$, where b is fixed, then equation (1) is called delay Fredholm integral equation (DFIE); otherwise, it is called delay Volterra integral equation (DVIE). The MAHAM demonstrates an accurate solution if compared with other numerical methods. The obtained results suggest that this newly improved technique introduces a powerful improvement for solving non-linear delay integral equations. The outline of the paper is structured as follows: The essential idea of the method is presented in Section 2. The applications of the method and numerical examples are detailed in Section 3. Lastly, Section 4 gives conclusions.

2. The Fundamental of MAHAM

Consider the non-linear delay integral equation (1).

In the MAHAM, we split $f(x)$ into an infinite sum, as follows

$f(x) = \sum_{i=0}^{\infty} f_i(x)$

Let us construct the so-called zeroth-order deformation equation:

$$(1-q)\left[\emptyset(x,q) - \sum_{i=0}^{\infty} f_i(x)\, q^i\right] = qh\left[\emptyset(x,q) - \sum_{i=0}^{\infty} f_i(x)\, q^i - \lambda \int_{x_0}^{\theta(x)} K(x,s)F\big(\emptyset(\sigma(s),q)\big)ds\right] \quad (2)$$

where $q \in [0,1], h \neq 0$. When $q = 0$ and $q = 1$, equation (2) becomes

$\emptyset(x;0) = f_0(x)$ and $\emptyset(x;1) = y(x)$,

respectively. Thus, as q increase from 0 to 1, the solution $\emptyset(x;q)$ varies continuously from the initial guess $y_0(x) = f_0(x)$ to the solution$y(x)$.

By expanding $\emptyset(x;q)$ in Taylor series with respect to q, one has

$$\emptyset(x;q) = f_0(x) + \sum_{m=1}^{\infty} y_m(x) q^m, \quad (3)$$

where $y_m(x) = \frac{1}{m!}\frac{\partial^m}{\partial q^m}\emptyset(x;0)$ is the mth-order homotopy-derivative of $\emptyset$.

Assume that the series (3) converges at $q = 1$, then

$$y(x) = \emptyset(x;1) = f_0(x) + \sum_{m=1}^{\infty} y_m(x) \quad (4)$$

is the homotopy-series solution.

We define the vector so that $y_r(x) = \{y_0(x), y_1(x), y_2(x), \dots, y_r(x)\}$.

By differentiating equation (2) m times with respect to q, then setting $q = 0$ and dividing them by $m!$, we obtain the so-called m^{th} order deformation equation

$$y_m(x) = f_m + x_m(y_{m-1}(x) - f_{m-1}) + h(y_{m-1}(x) - f_{m-1}) - h\lambda \int_{x_0}^{\theta(x)} K(x,s)F\big(y_{m-1}(\sigma(s))\big)\, ds \quad (5)$$

where

$$x_m = \begin{cases} 0 & m \le 1 \\ 1 & otherwise \end{cases} \quad (6)$$

3. Applications

Example 1. Consider the linear delay Volterra integral equation [29]

$$y(x) = e^x - xe^{x-1} + xe^{-1} + \int_0^x xy(s-1)ds \quad (7)$$

with the exact solution $u(x) = e^x$ over the interval [0,1].

For MAHAM solution, we split $f(x) = e^x - xe^{x-1} + xe^{-1}$ as $f(x) = \sum_{i=0}^{\infty} f_i(x)$ with $f_0(x) = e^x$,$f_1(x) = -xe^{x-1} + xe^{-1}$, $f_i(x) = 0$, $i \ge 2$.

Using initial approximation $y_0(x) = f_0(x) = e^x$,

we construct the zero order deformation equation

$$(1-q)\left[\emptyset(x,q)-\sum_{i=0}^{\infty} f_i(x)\, q^i\right] = qh\left[\emptyset(x,q)-\sum_{i=0}^{\infty} f_i(x)\, q^i - \int_0^x x\emptyset(s-1,q)ds\right] \quad (8)$$

The m^{th}order deformation equation is

$$y_m(x) = f_m + x_m(y_{m-1}(x) - f_{m-1}) + h(y_{m-1}(x) - f_{m-1}) - h\int_0^x x\, y_{m-1}(s-1)ds$$

where x_m is define by (6).

Consequently, we obtain the following components of the solution

$$y_0 = e^x$$

$$y_1 = \frac{x}{e} - e^{-1+x}x - \frac{(-1+e^x)hx}{e}$$

$$y_2 = -\frac{(-1+e^x)h(1+h)x}{e} - \frac{h(1+h)x(-2+\frac{1}{2}(-2+x)(-2e^x+ex))}{e^2}$$

⋮

Then, the mth order series approximation solution of MAHAM can be written as

$$Y_m(x,h) = \sum_{i=0}^{m} y_i\,(x,h) \quad (9)$$

Equation (9) is a series solution to Eq. (7) in terms of the convergence control parameter h. To determine the valid region of h, the h-curves are drawn in Figure 1 by the 3rd order MAHAM at various values of x.

We note that when $h = -1$ then $Y_m(x) = e^x$ for $m \geq 1$, that is, we obtain the exact solution at all order approximations of MAHAM. Hence, the results obtained by MAHAM are more accurate than those of Block2 and Block3 [29]. As shown in Figure 2, the 3rd order approximate solution series obtained by the MAHAM at $h = -1.001$ is in good agreement with the exact solution.

It should be noted that at $h = -1.001$, the absolute error is highly decreased when modifying the solution by taking more terms into consideration. Figures 3 and 4 illustrate this fact.

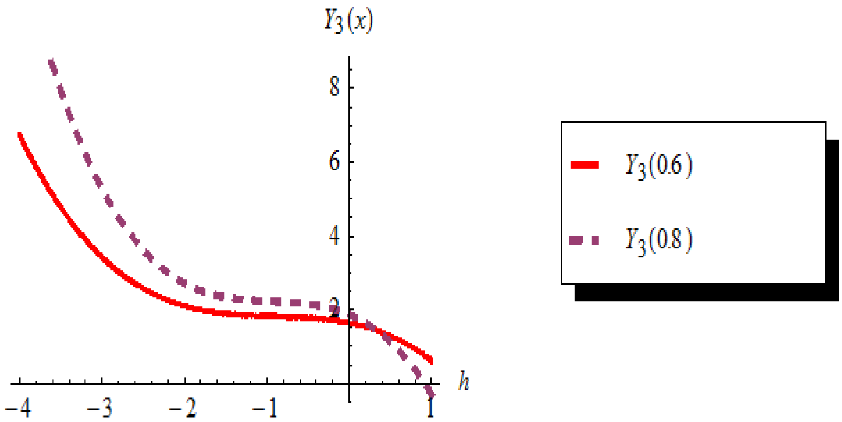

Figure 1- h-curve of Y_3 solution at different values of x for Example 1.

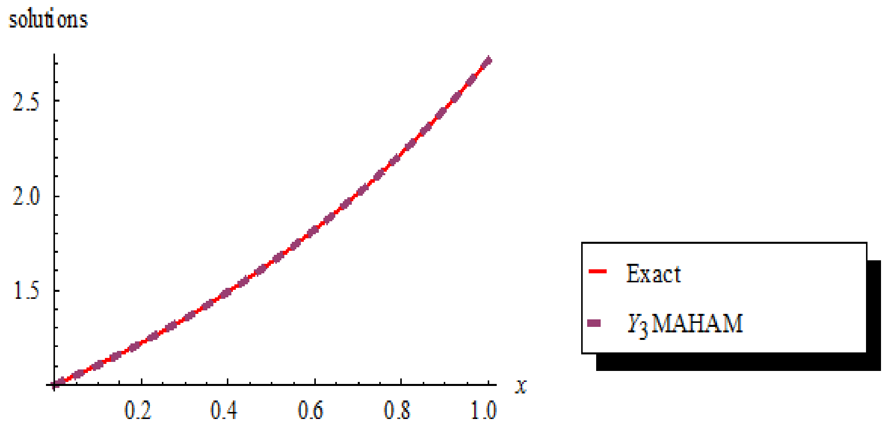

Figure 2-Comparison of Y_3 solution $(h = -1.001)$ and the exact solution for Example 1.

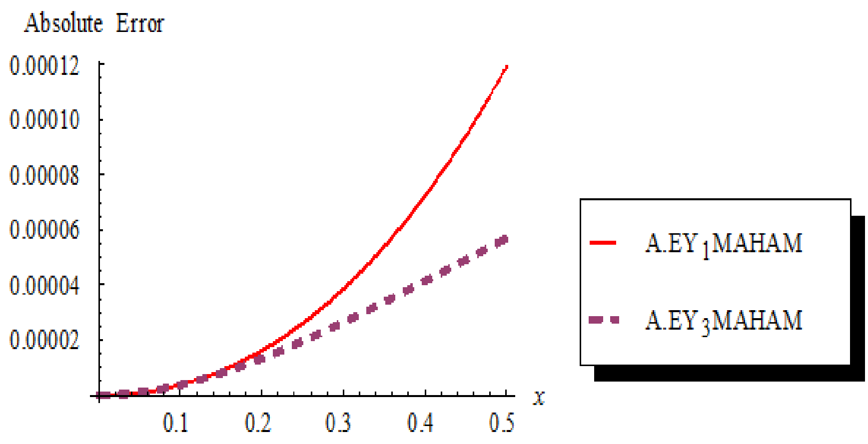

Figure 3-The absolute errors of 1st and 3^{rd} order $(h = -1.001)$ approximations of MAHAM for Example 1 at $0 \leq x \leq 0.5$.

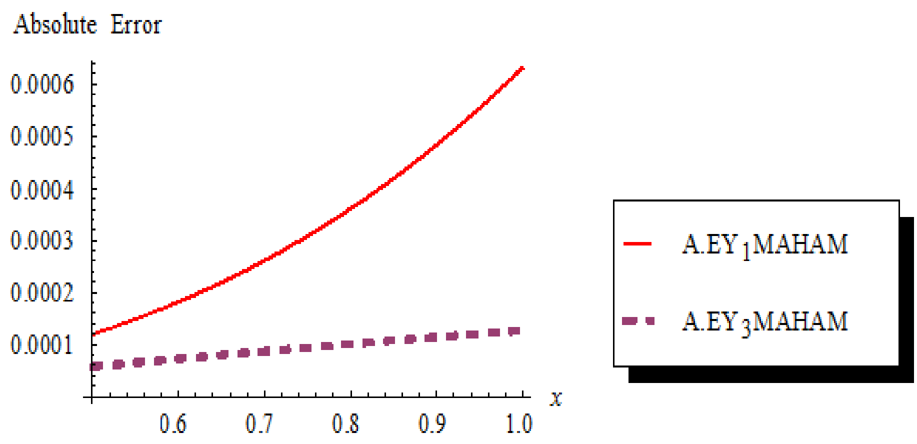

Figure 4-The absolute errors of 1st and 3^{rd} order $(h = -1.001)$ approximations of MAHAM for Example 1
at $0.5 \leq x \leq 1$.

Example 2 Consider the linear delay Fredholm integral equation [12]

$$y(x) = x^2 + \frac{5}{3}x + \int_{-1}^{1} x\, s\, y(s-1)ds \tag{10}$$

with the exact solution $u(x) = x^2 + x$ over the interval [0,1].

For MAHAM solution, we split $f(x) = x^2 + \frac{5}{3}x$ as $f(x) = \sum_{i=0}^{\infty} f_i(x)$
with $f_0(x) = x^2 + \frac{5}{3}x,\ \ f_i(x) = 0\,, i \geq 1$.
Using the initial approximation $y_0(x) = f_0(x) = x^2 + \frac{5}{3}x$,
we construct the zero order deformation equation

$$(1-q)\left[\emptyset(x,q) - \sum_{i=0}^{\infty} f_i(x)\, q^i\right] = qh\left[\emptyset(x,q) - \sum_{i=0}^{\infty} f_i(x)\, q^i - \int_{-1}^{1} x\, s\emptyset(s-1,q)ds\right] \quad (11)$$

The m^{th}order deformation equation is

$$y_m(x) = f_m + x_m(y_{m-1}(x) - f_{m-1}) + h(y_{m-1}(x) - f_{m-1}) - h\int_{-1}^{1} x\, s\, y_{m-1}(s-1)ds$$

where x_m is defined by (6).
Consequently, we obtain the following components of the solution

$$y_0 = x^2 + \frac{5}{3}x$$

$$y_1 = \frac{2hx}{9}$$

$$y_2 = -\frac{4h^2x}{27} + \frac{2}{9}h(1+h)x$$

$\vdots$

Then, the mth order series approximation solution of MAHAM can be written as

$$Y_m(x,h) = \sum_{i=0}^{m} y_i\,(x,h) \quad (12)$$

Equation (12) is a series solution to the Eq. (10) in terms of the convergence control parameter h. To determine the valid region of h, the h-curves are drawn in Figure 5 by the 3rd order MAHAM at various values of x.
We note that when $h = -3$ then $Y_m(x) = x^2 + x$ for $m \geq 1$, that is, we obtain the exact solution at all orders approximations of MAHAM. Hence, the results obtained by MAHAM are more accurate than that of the homotopy perturbation method [12].
It should be noted that at $h = -3.1$, the absolute error is highly decreased when modifying the solution by taking more terms into consideration. Table 1 illustrates this fact.

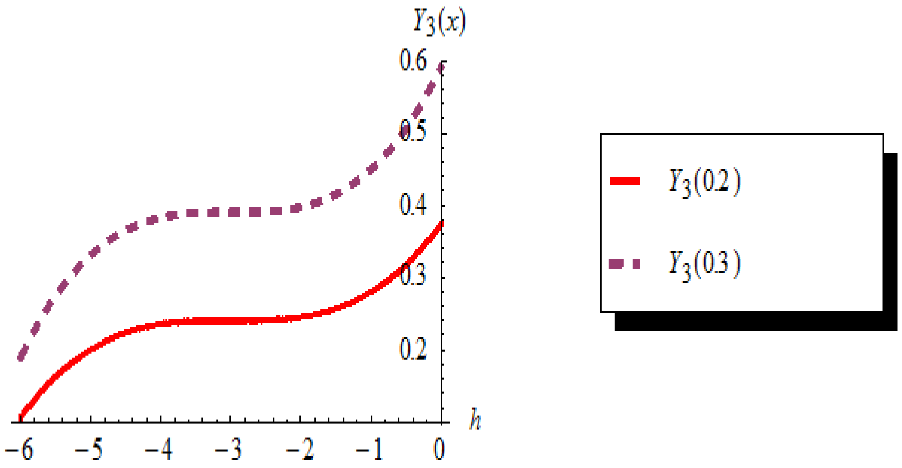

Figure 5- h-curve of Y_3 solution at different values of x for Example 2.

Table 1- The absolute errors of Y_1, Y_3, Y_5 and Y_7 of MAHAM at $h = -3.1$ for Example 2

x	Y_1 MAHAM	Y_3 MAHAM	Y_5 MAHAM	Y_7 MAHAM
0.0	0.0000000000000	0.0000000000000	0.0000000000000	0.0000000000000
0.1	2.222222×10^{-3}	2.469135×10^{-6}	2.743484×10^{-9}	3.047520×10^{-12}
0.2	4.444444×10^{-3}	4.938271×10^{-6}	5.486968×10^{-9}	6.095041×10^{-12}
0.3	6.666666×10^{-3}	7.407407×10^{-6}	8.230452×10^{-9}	9.144740×10^{-12}
0.4	8.888888×10^{-3}	9.876543×10^{-6}	1.097393×10^{-8}	1.219002×10^{-11}
0.5	1.111111×10^{-2}	1.234567×10^{-5}	1.371742×10^{-8}	1.524291×10^{-11}
0.6	1.333333×10^{-2}	1.481481×10^{-5}	1.646090×10^{-8}	1.828948×10^{-11}
0.7	1.555555×10^{-2}	1.728395×10^{-5}	1.920438×10^{-8}	2.133626×10^{-11}
0.8	1.777777×10^{-2}	1.975308×10^{-5}	2.194787×10^{-8}	2.438027×10^{-11}
0.9	2.000000×10^{-2}	2.222222×10^{-5}	2.469135×10^{-8}	2.743272×10^{-11}
1.0	2.222222×10^{-2}	2.469135×10^{-5}	2.743484×10^{-8}	3.048583×10^{-11}

Example 3. Consider the nonlinear delay Volterra integral equation [13]

$$y(x) = e^x + \frac{2}{3}\left(1 - e^{\frac{3}{2}x}\right) + \int_0^x y(s)y\left(\frac{s}{2}\right)ds \tag{13}$$

with the exact solution $u(x) = e^x$ over the interval [0,1].

For MAHAM solution, we split $f(x) = e^x + \frac{2}{3}\left(1 - e^{\frac{3}{2}x}\right)$ as $f(x) = \sum_{i=0}^{\infty} f_i(x)$ with

$f_0(x) = e^x$,$f_1(x) = \frac{2}{3}\left(1 - e^{\frac{3}{2}x}\right)$, $f_i(x) = 0\,, i \geq 2$

Using the initial approximation $y_0(x) = f_0(x) = e^x$,

we construct the zero order deformation equation

$$(1-q)\left[\emptyset(x,q) - \sum_{i=0}^{\infty} f_i(x)\, q^i\right] = qh\left[\emptyset(x,q) - \sum_{i=0}^{\infty} f_i(x)\, q^i - \int_0^x \emptyset(s,q)\emptyset(\tfrac{s}{2},q)ds\right] \tag{14}$$

The m^{th}order deformation equation is

$$y_m(x) = f_m + x_m(y_{m-1}(x) - f_{m-1}) + h(y_{m-1}(x) - f_{m-1}) - h\int_0^x y_{m-1}(s)y_{m-1}\left(\frac{s}{2}\right)ds$$

where x_m is defined by (6).

Consequently, we obtain the following components of the solution

$y_0 = e^x$

$y_1 = \dfrac{2}{3}(1 - e^{3x/2}) - \dfrac{2}{3}(-1 + e^{3x/2})h$

$y_2 = -\dfrac{2}{3}(-1 + e^{3x/2})h(1+h) - \dfrac{4}{81}h(1+h)^2(14 - 12e^{3x/4} - 6e^{3x/2} + 4e^{9x/4} + 9x)$

$\vdots$

Then, the mth order series approximation solution of MAHAM can be written as

$$Y_m(x,h) = \sum_{i=0}^{m} y_i\,(x,h) \tag{15}$$

Equation (15) is a series solution to Eq. (13) in terms of the convergence control parameter h. To determine the valid region of h, the h-curves are drawn in Figure 6 by the 3rd order MAHAM at various values of x.

We note that when $h = -1$ then $Y_m(x) = e^x$ for $m \geq 1$, that is, we obtain the exact solution at all orders approximations of MAHAM. Hence, the results obtained by MAHAM are more accurate than that of the variation iteration method [13].

It should be noted that at $h = -1.001$, the absolute error is highly decreased when modifying the solution by taking more terms into consideration. Table 2 illustrates this fact.

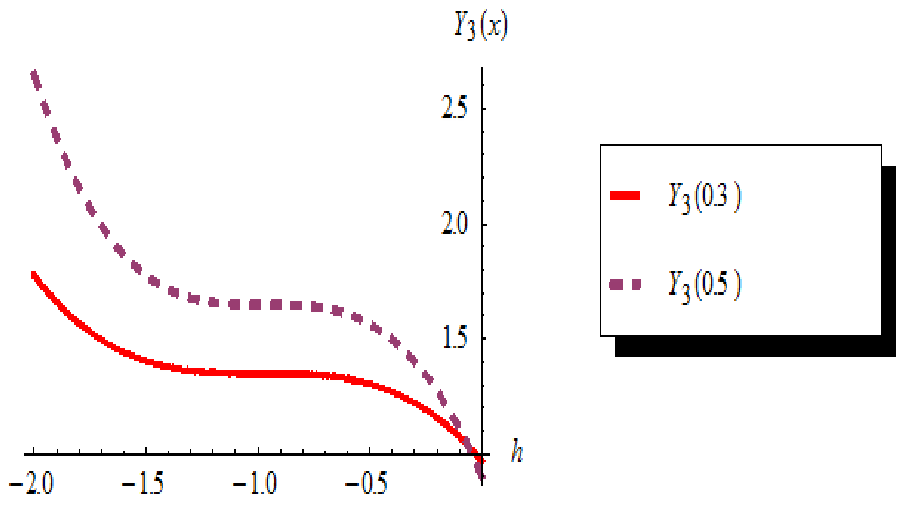

Figure 6-h-curve of Y_3 solution at different values of x for Example 3.

Table 2- The absolute errors of Y_1, Y_2 and Y_3 of MAHAM at $h = -1.001$ for Example 3

x	Y_1 MAHAM	Y_2 MAHAM	Y_3 MAHAM
0.0	0.00000000000000000	0.00000000000000000	0.00000000000000000
0.1	$1.0788949515 \times 10^{-4}$	$1.0770780245 \times 10^{-7}$	$4.7145620740 \times 10^{-10}$
0.2	$2.3323920505 \times 10^{-4}$	$2.3165317708 \times 10^{-7}$	$3.4068747822 \times 10^{-9}$
0.3	$3.7887479032 \times 10^{-4}$	$3.7302271205 \times 10^{-7}$	$1.2088819945 \times 10^{-8}$
0.4	$5.4807920026 \times 10^{-4}$	$5.3288407197 \times 10^{-7}$	$3.0953220431 \times 10^{-8}$
0.5	$7.4466667774 \times 10^{-4}$	$7.1209239171 \times 10^{-7}$	$6.5924737002 \times 10^{-8}$
0.6	$9.7306874077 \times 10^{-4}$	$9.1116308320 \times 10^{-7}$	$1.2484326572 \times 10^{-7}$
0.7	$1.2384340787 \times 10^{-3}$	$1.1301009856 \times 10^{-6}$	$2.1800558513 \times 10^{-7}$
0.8	$1.5467446151 \times 10^{-3}$	$1.3681730011 \times 10^{-6}$	$3.5885234472 \times 10^{-7}$
0.9	$1.9049503537 \times 10^{-3}$	$1.6236078161 \times 10^{-6}$	$5.6483841692 \times 10^{-7}$
1.0	$2.3211260468 \times 10^{-3}$	$1.8932018228 \times 10^{-6}$	$8.5853441555 \times 10^{-7}$

Example 4 :Consider the linear delay Volterra integral equation [29]

$$\boldsymbol{y}(x) = sin(x) + x^2 cos(x-1) - x^2 cos(-1) + \int_0^x x^2\, y(s-1)ds \qquad (16)$$

with the exact solution $u(x) = sin(x)$ over the interval [0,1].
For MAHAM solution, we split $f(x) = sin(x) + x^2 cos(x-1) - x^2 cos(-1)$
as $f(x) = \sum_{i=0}^{\infty} f_i(x)$ with $f_0(x) = sin(x)$ and $f_1(x) = x^2 cos(x-1) - x^2 cos(-1)$,
$f_i(x) = 0\,, i \geq 2$.
Using the initial approximation $y_0(x) = f_0(x) = sin(x)$,
we construct the zero order deformation equation

$$(1-q)\left[\emptyset(x,q) - \sum_{i=0}^{\infty} f_i(x)\, q^i\right] = qh\left[\emptyset(x,q) - \sum_{i=0}^{\infty} f_i(x)\, q^i - \int_0^x x^2 \emptyset(s-1,q)ds\right] \qquad (17)$$

The m^{th}order deformation equation is

$$y_m(x) = f_m + x_m(y_{m-1}(x) - f_{m-1}) + h(y_{m-1}(x) - f_{m-1}) - h\int_0^x x^2\, y_{m-1}(s-1)ds$$

where x_m is defined by (6).
Consequently, we obtain the following components of the solution

$y_0 = \mathrm{Sin}[x]$

$y_1 = -x^2\mathrm{Cos}[1] - hx^2(\mathrm{Cos}[1] - \mathrm{Cos}[1-x]) + x^2\mathrm{Cos}[1-x]$

$$y_2 = -h(1+h)x^2(\text{Cos}[1] - \text{Cos}[1-x]) - \frac{1}{3}h(1+h)x^2(-\text{Cos}[1] - (-1+x)^3\text{Cos}[1] + 6\text{Cos}[2] + 6(-1+x)\text{Cos}[2-x] - 3\text{Sin}[2] + (3-3(-2+x)x)\text{Sin}[2-x])$$

⋮

Then, the mth order series approximation solution of MAHAM can be written as

$$Y_m(x,h) = \sum_{i=0}^{m} y_i\,(x,h) \tag{18}$$

Equation (18) is a series solution to Eq. (16) in terms of the convergence control parameter h. To determine the valid region of h, the h-curves are drawn in Figure 7 by the 3rd order MAHAM at various values of x.

We note that when $h=-1$ then $Y_m(x) = sin(x)$ for $m \geq 1$, that is, we obtain the exact solution at all order approximations of MAHAM. Hence, the results obtained by MAHAM are more accurate than those of Block2 and Block3 [29]. As shown in Figure 8, the 3rd order approximate solution series obtained by the MAHAM at $h=-1.001$ is in good agreement with the exact solution.

It should be noted that at $h=-1.001$, the absolute error is highly decreased when modifying the solution by taking more terms into consideration. Figures 9 and 10 illustrate this fact.

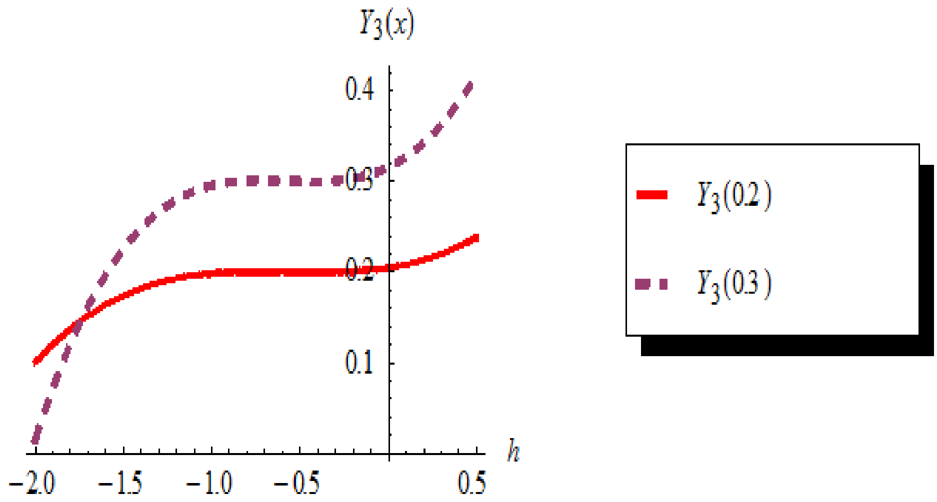

Figure 7- h-curve of Y_3 solution at different values of x for Example 4.

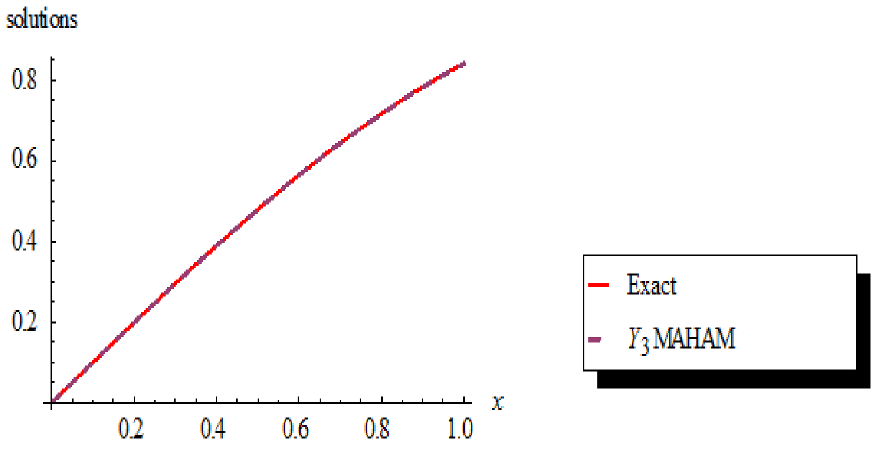

Figure 8-Comparison of Y_3 solution ($h=-1.001$)and the exact solution for Example 4.

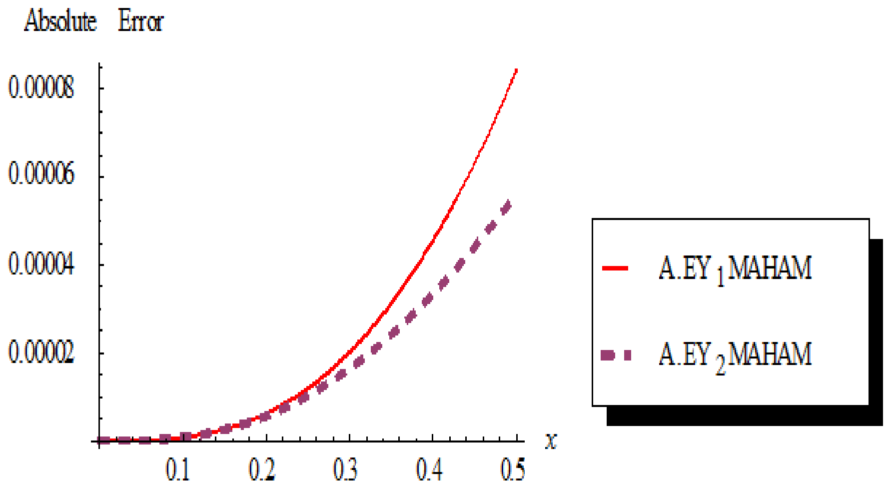

Figure 9-The absolute errors of 1st and 2^{nd} order $(h = -1.001)$ approximations of MAHAM for Example 4 at $0 \le x \le 0.5$.

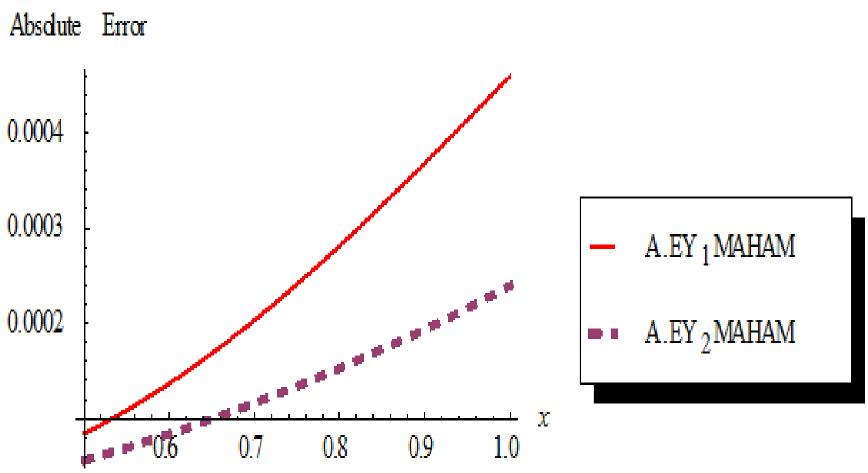

Figure 10-The absolute errors of 1st and 2^{nd} order $(h = -1.001)$ approximations of MAHAM for Example 4 at $0.5 \le x \le 1$.

Example 5: Consider the nonlinear delay Volterra integral equation

$$y(x) = f(x) + \int_0^{0.9x} (s + x)[y(s)]^3 ds \ , \tag{19}$$

where $f(x)$ is chosen so that its exact solution is $y(x) = -x + x^2$ [30].

Now

$f(x) =$
$-x + x^2 + 0.282123x^5 - 0.6200145000000001x^6 + 0.47070488571428576x^7 - 0.12213652982142859x^8$

For HAM solution:

We construct the zero order deformation equation

$$(1-q)[\emptyset(x,q) - y_0(x)] = qh\left[\emptyset(x,q) - f(x) - \int_0^x (s + x)\, \emptyset^3(s,q) ds\right] \tag{20}$$

The m^{th} order deformation equation is

$y_m(x) =$
$x_m y_{m-1}(x) +$
$h\left[y_{m-1}(x) - (1 - x_m)f(x) - \int_0^x (s\ +\ x)\ \left(\sum_{n=0}^{m-1} y_{(m-1)-n}\,(s) \sum_{k=0}^{n} y_{n-k}\,(s) y_k(s)\ \right)ds\right]$
where x_m is defined by (6).
Using the initial approximation $y_0(x) = f(x)$,
we obtain the following components of the solution
$y_0 = -x + x^2 + 0.282123x^5 - 0.6200145000000001x^6 + 0.47070488571428576x^7 - 0.12213652982142859x^8$.
$y_1 = -h(0. - 0.282123x^5 + 0.6200145x^6 - 0.4707048857142857x^7 + 0.12213652982142859x^8 + \cdots$
$y_2 = -h(1 + h)(0. - 0.282123x^5 + 0.6200145x^6 - 0.4707048857142857x^7 + 0.12213652982142859x^8 + \cdots$
⋮
Then, the m^{th} order series approximation solution expression by HAM can be written in the form

$$Y_m(x, h) = \sum_{i=0}^{m} y_i\,(x, h) \tag{21}$$

Equation (21) is a family of approximation solutions to equation (19) in terms of the convergence parameter h. To find the valid region of h, the h-curves given by the 3rd order HAM at different values of x are drawn in Figure 11.

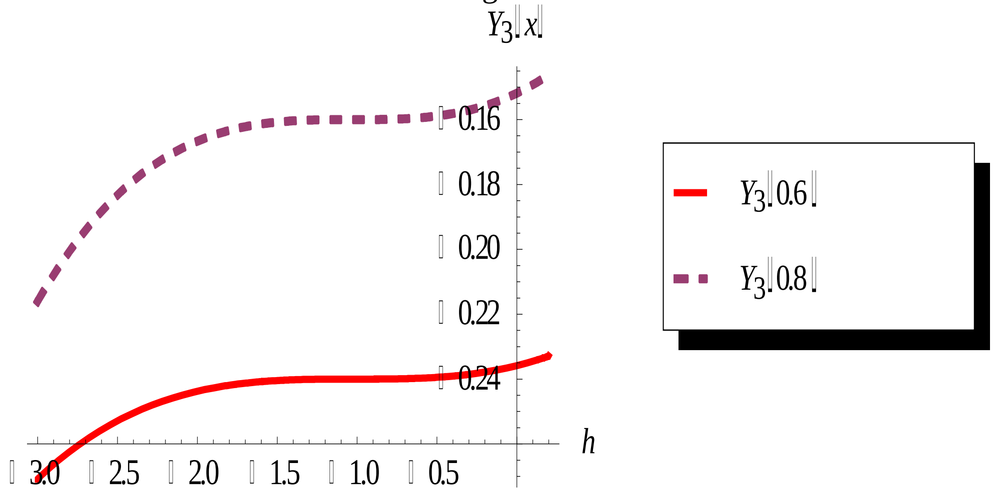

Figure 11-h-curve of Y_3 solution at different values of x for Example 5.

For MAHAM solution:
Wesplit
$f(x) =$
$-x + x^2 + 0.282123x^5 - 0.6200145000000001x^6 + 0.47070488571428576x^7 - 0.12213652982142859x^8$ as $f(x) = \sum_{i=0}^{\infty} f_i(x)$ with
$f_0(x) = -x + x^2$
,$f_1(x) = 0.282123x^5 - 0.6200145000000001x^6 + 0.47070488571428576x^7 - 0.12213652982142859x^8$, $f_i(x) = 0\,, i \geq 2$
Using the initial approximation $y_0(x) = f_0(x) = -x + x^2$,
we construct the zero order deformation equation

$$(1 - q)\left[\emptyset(x, q) - \sum_{i=0}^{\infty} f_i(x)\, q^i\right] = qh\left[\emptyset(x, q) - \sum_{i=0}^{\infty} f_i(x)\, q^i - \int_0^x (s\ +\ x)\ \emptyset^3(s, q)ds\right]$$
(22)

The m^{th}order deformation equation is
$y_m(x) = f_m + x_m(y_{m-1}(x) - f_{m-1}) + h(y_{m-1}(x) - f_{m-1}) -$

$$h\int_0^x (s+x)\left(\sum_{n=0}^{m-1} y_{(m-1)-n}(s)\sum_{k=0}^{n} y_{n-k}(s)y_k(s)\right)ds$$

where x_m is defined by (6).

Consequently, we obtain the following components of the solution

$y_0 = -x + x^2$

$$y_1 = -h(0.08197517296361026x^9 + 0.08197517296361026hx^9 - 0.27681342514807544x^{10} - 0.27681342514807544hx^{10} + \cdots$$

$$y_2 = -h(0.08197517296361027hx^9 + 0.08197517296361027h^2x^9 - 0.27681342514807544hx^{10} - 0.27681342514807544h^2x^{10} + \cdots$$

$\vdots$

Then, the m^{th} order series approximation solution expression by MAHAM can be written in the form

$$Y_m(x,h) = \sum_{i=0}^{m} y_i(x,h) \qquad (23)$$

Equation (23) is a family of approximation solutions to equation (19) in terms of the convergence parameter h. To find the valid region of h, the h-curves given by the 3rd order MAHAM at different values of x are drawn in Figure 12.

It should be noted that the results obtained by MAHAM are more accurate than that of the HAM. Table 3 illustrates this fact.

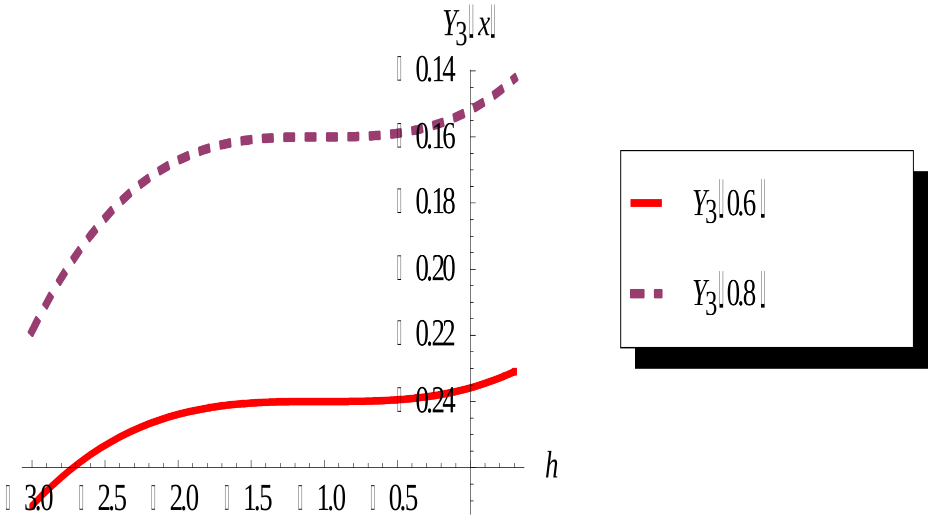

Figure 12- h-curve of Y_3 solution at different values of x for Example 5.

Table 3-The absolute errors of Y_3 of HAM and MAHAM at $h = -1$.

x	A.E Y_3 HAM	A.E Y_3 MAHAM
0.1	0.0000000000000	0.0000000000000
0.2	1.915134×10^{-15}	0.0000000000000
0.3	9.878209×10^{-13}	0.0000000000000
0.4	6.524361×10^{-11}	0.000000000000
0.5	1.3607733×10^{-9}	0.0000000000000
0.6	1.3303303×10^{-8}	0.0000000000000
0.7	$7.48816076 \times 10^{-8}$	0.0000000000000
0.8	2.7294426×10^{-7}	2.775557×10^{-17}
0.9	6.9125831×10^{-7}	2.775557×10^{-17}
1.0	1.2700852×10^{-6}	$6.9388939 \times 10^{-17}$

4. Conclusions

In this article, a modified approach derived from Liao's homotopy analysis technique (MAHAM) was introduced to solve non-linear delay integral equations. The MAHAM gives approximate and exact solutions in a few iterations. Finally, we can say that the MAHAM is a powerful and efficient method for linear and nonlinear delay integral equations.

References

[1] C. Iancu, "A numerical method for approximating the solution of an integral equation from biomathematics", *Studia Univ. Babes-Bolyai, Mathematica*, vol. 43, pp. 37–45, 1988.

[2] R. Precup, "Monotone technique to the initial values problem for a delay integral equation from biomathematics", *Studia Univ. Babes-Bolyai Math.*, vol. 40, pp. 63–73, 1995.

[3] Q. Hu, "Multilevel correction for discrete collocation solutions of Volterra integral equations with delay arguments", *Appl. Numer. Math.*, vol. 31, pp. 159–171, 1999.

[4] C. T. H. Baker and M. S. Derakhshan, "Convergence and stability of quadrature methods applied to Volterra equations with delay", *IMA J. Numer. Anal.*, vol. 13, pp. 67–91, 1993.

[5] M. H. Alsnar, "The numerical stability of multistep methods for Volterra Integral equations with many delay", *Internat. J. Comput. Math.*, vol. 81, pp. 1257–1263, 2004.

[6] H. Brunner, "Collocation and continuous implicit Runge-Kutta methods for a class of delay Volterra integral equations", *J. Comput. Appl. Math.*, vol. 53, pp. 61–72, 1994.

[7] B. Cahlon and D. Schimidt, "Stability criteria for certain delay integral equations of Volterra type", *J. Comput. Appl. Math.*, vol. 84, pp.161–188, 1997.

[8] Cañada and A. Zertiti, Method of upper and lower solutions for nonlinear delay integral equations modelling epidemics and population growth, M3AS, *Math. Models and Methods in Applied Sciences*, vol. 4, pp. 107–120, 1994.

[9] L. R.Williams and R.W. Leggett, "Nonzero solutions of nonlinear integral equations modelling infectious disease", *SIAM J. Math. Anal.*, vol. 13, pp. 112–121, 1982.

[10] H. R. Marzban and M. Razzaghi, "Optimal control of linear delay systems via hybrid of block-pulse and Legendre polynomials", *Journal of the Franklin Institute,* vol. 341, pp. 279-293, 2004.

[11] E. Messina, E. Russo and A. "Vecchio, A convolution test equation for double delay integral equations", *Journal of Computational and Applied Mathematics*, vol. 228, pp. 589-599, 2009.

[12] Salih S.H., Hassan I. H. and Al Heety F.A., "Numerical solution of linear delay Fredholm integral equations by using homotopy perturbation method", *Eng.&Tech. Journal*, vol.30, no.4, 2012.

[13] M. Avaji,J. S. Hafshejani, S. S. Dehcheshme and D. F. Ghahfarokh,"Solution of delay Volterra integral equations using the variational iteration method." *Journal of Applied Sciences(Faisalabad)* vol. 12.2, pp. 196-200, 2012.

[14] S. J. Liao, Proposed homotopy analysis techniques for the solution of nonlinear problems, Ph.D. dissertation, *Shanghai Jiao Tong University, Shanghai*,1992.

[15] S. J. Liao, "An approximate solution technique which does not depend upon smallparameters: a special example", *Int. J. Non-linear Mech.*, vol. 30, pp.371-380, 1995.

[16] S. J. Liao, "An approximate solution technique which does not depend upon small parameters (Part 2): an application in fluid mechanics", *Int. J. Non-linear Mech.*, vol. 32, pp. 815- 822, 1997.

[17] S. J. Liao, "An explicit, totally analytic approximation of Blasius viscous flow problem", *Int. J.Non-Linear Mech.*, vol. 34, pp. 759-778, 1999.

[18] S. J. Liao, "A uniformly valid analytic solution of 2D viscous flow past a semi-infinite flat plate". *J. Fluid Mech.*, vol. 385, pp. 101-128, 1999.
[19] Z. Ayati, J. Biazar, B. Gharedaghi,"The application of modified homotopy analysis method for solving linear and non-linear inhomogeneous Klein-Gordon equations", *Acta Universitatis Apulensis*, no. 39, pp. 31-40, 2014.
[20] A.S. Bataineh, "Avoid the uncontrollability problems of the nonzero endpoint conditions via homotopy analysis method", *Appl. Comput. Math.*, vol.13, no.1, pp. 1-13, 2014.
[21] M. A. El-Tawil and S. N. Huseen, "The q-Homotopy Analysis Method (q-HAM)", *International Journal of Applied mathematics and mechanics*, vol. 8, no. 15, pp. 51-75, 2012.
[22] H.N. Hassan and El-Tawil M.A., "A new technique of using homotopy analysis method for solving high-order nonlinear differential equations", *Mathematical Method in the Applied Sciences* , vol.34, no.6, pp.728-742, 2011.
[23] S. N. Huseen and S. R. Grace, "Approximate Solutions of Nonlinear Partial Differential Equations by Modified q-Homotopy Analysis Method (mq-HAM)", *Hindawi Publishing Corporation, Journal of Applied Mathematics*, 2013, Article ID 569674, 9 pages http:// dx.doi.org/10.1155/ 2013/ 569674.
[24] L. Akinyemi, "q-Homotopy analysis method for solving the seventh-order time-fractional Lax's Korteweg-de Vries and Sawada-Kotera equations". *Comp. Appl. Math.*, vol. 38, no. 4, pp. 1-22, 2019.
[25] M. Senol , O. Iyiola , H. Kasmaei, L. Akinyemi, "Efficient analytical techniques for solving time-fractional nonlinear coupled Jaulent-Miodek system with energy-dependent Schrodinger potential". *Adv Differ Equ.* vol. 2019, pp. 1-21, 2019.
[26] S. N. Huseen, "On Analytical Solution of Time-Fractional Type Model of the Fisher'sEquation", *Iraqi Journal of Science*, vol. 61, no. 6, pp. 1419-1425, 2020.
[27] L. Akinyemi , O. Iyiola, U. Akpan, "Iterative methods for solving fourth- and sixth-order time-fractional Cahn-Hillard equation". *Math Meth Appl Sci.* pp. 1-25, 2020, https : // doi. org/ 10 .1002 / mma.6173.
[28] L. Akinyemi, S.N. Huseen, "A powerful approach to study the new modified coupled Korteweg-de Vries system", *Mathematics and Computers in Simulation*, vol. 177, pp. 556–567, 2020.
[29] M. Mustafa and T. A. Latiff Ibrahem, "Numerical solution of Volterra integral equations with delay using Block methods", *Al-Fatih Journal*, vol. 36, pp. 89-96, 2008.
[30] M. Zarebnia and L. Shiri, "Convergence of Numerical Method For the Solution of Nonlinear Delay Volterra Integral Equations", *Int. J. Industrial Mathematics*, vol. 9, no. 2, 2017.