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Qualitative Analysis of some Types of Neutral Delay Differential Equations

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Abstract

In this paper, we conduct some qualitative analysis that involves the global asymptotic stability (GAS) of the Neutral Differential Equation (NDE) with variable delay, by using Banach contraction mapping theorem, to give some necessary conditions to achieve the GAS of the zero solution.

Keywords: Nonlinear Neutral Differential Equation, Banach Fixed Point Theorem, Variable Delays, Global Asymptotically Stable

التحليل النوعي لبعض انواع المعادلات التباطؤيه المحايده

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الخلاصه

في هذا البحث، قمنا بدراسه بعض الصفات التحليل النوعي مثل الاستقراريه والاستقراريه بصوره محاذيه للمعادلات التفاضليه التباطؤيه المحايده مع تباطؤ متغير وباستخدام نظريه بناخ الانكماشيه لتحقيق الاستقراريه للحل الصفري.

1-Introduction

Several techniques are used to research the zero solution stability of NDE with delay, such as the fixed points theory, Lyapunov functions, and characteristic equations. Each technique has its cons and pros. The fixed point theorem has been used in order to achieve the zero solution is GAS by providing certain conditions. In 2004, Raffoul [1] studied the travail solution stability of

$\dot{y}(t) = -a(t)y(t) + c(t)\dot{y}(t - g(t)) + q(y(t), y(t - g(t)))$.

Furthermore, in 2013, Liu and Yan [2] gave some conditions for achieving GAS of the $\dot{y}(t) = -a(t)y(t) + c(t)y(t - \sigma_1(t)) + q(t, y(t), y(t - \sigma_2(t)))$.

Also, in 2017,Tunc [3] studied the GAS of $\dot{y}(t) = -a(t)y(t) + b(t)g(y(t)) + c(t)f\left(\dot{y}(t - \sigma_1(t))\right) + q(t, y(t), y(t - \sigma_2(t)))$.

Finally, in 2018, Ardjoun and Djoudi [4] studied the GAS of nonlinear neutral dynamic equations with variable delay.

The goal of this study is to use the fixed-point theorem to analyze the GAS of the following NDE

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$$\dot{z}(t) = -h(t)z(t) + \sum_{j=1}^{L} d_j(t)\, \acute{z}\left(t - \sigma_j(t)\right) + g\left(t, z\left(t - \sigma(t)\right)\right) \ldots \ldots (1-1),$$

which satisfies these conditions:

A_1- $h, d_j, g \in C([0,\infty),\mathbb{R})$, such that $\sigma(t), \sigma_j(t) \in F$ with $t - \sigma(t) \to \infty$ and $t - \sigma_j(t) \to \infty$ as $t \to \infty, j = 1, 2, \ldots L$.

A_2- $g(t,0) = 0$, and there exists functions $T_1 \in C\left([0,\infty),(0,\infty)\right)$ such that

$$|g(t,z_1) - g(t,z_2)| \le T_1(t)|z_1 - z_2|$$

for all $z_i \in \mathbb{R}, I = 1, 2$, and $t \in [0,\infty)$, that is the function g satisfied Lipschiz condition.

A_3- On $[0,\infty)$, the function h is bounded and

$$\liminf_{t\to\infty} \int_0^t h(e)de > -\infty.$$

A_4- There exists $\mu \in (0,1)$ satisfies the following:

$$\int_0^t e^{-\int_w^t h(e)\,de} \left[\left|\sum_{j=1}^{L} d_j(w)\right| + T_1(w)\right] dw \le \mu$$

and

$$|h(t)| \int_0^t e^{-\int_w^t h(e)de} \left[\left|\sum_{j=1}^{L} d_j(w)\right| + T_1(w)\right] dw + \left|\sum_{j=1}^{L} d_j(t)\right| + T_1(t) \le \mu \ , t \ge 0.$$

2- Preliminary

In this section, we give the Banach fixed point theorem which is the most important theorem that we use to obtain the main result of this study.

Theorem 2.1 (Banach fixed point theorem) [5, 6]

If $S: Y \to Y$ is a contraction and *Y* is a Banach space, then there is a unique point $y^* \in Y$ which is fixed byS.

3- Main Result

In this section, we give the theorem that verifies the GAS of equation (1-1).

Theorem 3-1:

Assume that the conditions $(A_1) - (A_4)$ hold, then the equation of zero solution $(1-1)$ is GAS in C^1 if and only if

$$\int_0^\infty h(e)de = \infty \ldots \ldots (3-1)$$

Proof:

We begin of If side :

Assuming that the equation of zero solution (1-1) is GAS in C^1, then to prove that (3-1) holds, suppose that it does not hold, then there exists

$$K_\circ = \liminf_{t\to\infty} \int_o^t h(e)de$$

$$E_\circ = \sup_{t\in[0,\infty)} \left\{e^{-\int_0^t h(e)de}\right\}$$

and

$$Q_\circ = \sup_{t\in[0,\infty)}\{h(t)\}$$

It follows from (A_3) that $K_\circ \in (-\infty,\infty)$, $E_\circ \in (0,\infty)$ and $Q_\circ \in [0,\infty)$. So, there exists $\{b_n\} \subset [0,\infty)$ which is an increasing sequence such that

$$\lim_{n\to\infty} b_n = \infty$$

and

$$K_\circ = \lim_{n\to\infty} \int_0^{b_n} h(e)de \dots\dots (3-2)$$

which denotes

$$I_n = \int_0^{b_n} e^{\int_0^w h(e)de} \left[\left| \sum_{j=1}^{L} d_j(w) \right| + T_1(w) \right] dw, \qquad n = 1, 2, \dots$$

By (A_4), it follows that

$$I_n = e^{\int_0^{b_n} h(e)de} \int_0^{b_n} e^{-\int_w^{b_n} h(e)de} \left[\left| \sum_{j=1}^{L} d_j(w) \right| + T_1(w) \right] dw < \mu e^{\int_0^{b_n} h(e)de}$$

The above equation with (3.2) shows that $\{I_n\}$ is bounded, and a convergent subsequence exists, assuming that $\{I_n\}$ is convergent.

So, there is a positive integer u, such that for each $n \in N, n > u$,

$$\int_{b_u}^{b_n} e^{\int_0^w h(e)de} \left[\left| \sum_{j=1}^{L} d_j(w) \right| + T_1(w) \right] dw < \frac{1-\mu}{8P(e^{-d}+1)} \dots (3-3)$$

and

$$e^{\int_{b_u}^{b_n} h(e)de} > \frac{1}{2},\ e^{-\int_0^{b_n} h(e)de} < e^{-d} + 1,\ e^{\int_0^{b_u} h(e)de} < e^{d} + 1 \dots (3-4)$$

where

$$P = \max\{E_\circ(e^d + 1), Q_\circ E_\circ(e^d + 1), 1\}.$$

For any $\beta > 0$, consider that the solution of equation (1-1) is $z(t) = z(t, b_u, \Omega)$ with $\|\Omega\|_{b_u} < \beta$ and $|\Omega(b_u)| > \frac{\beta}{2}$. It follows from (3-1), (3-2), (3-3), (A_3) and (A_4) for $t \in [b_u, \infty)$ that

$$\begin{aligned}
|z(t)| &\le |\Omega(b_u)| e^{-\int_{b_u}^{t} h(e)de} \\
&\quad + \int_{b_u}^{t} e^{-\int_w^t h(e)de} \left[\left\| \sum_{j=1}^{L} d_i(w)\, \acute{z}\left(w - \sigma_j(w)\right) + g(w, z(w - \sigma(w))) \right\| \right] dw \\
&\le |\Omega(b_u)| e^{-\int_0^t h(e)de} e^{\int_0^{b_u} h(e)de} + \|z\|_{b_u} \int_{b_u}^{t} e^{-\int_w^t h(e)de} \left[\left| \sum_{j=1}^{L} d_j(w) \right| + T_1(w) \right] dw \\
&\le \beta E_\circ(e^d + 1) + \mu \|z\|_{b_u} \\
&\le P\beta + \mu \|z\|_{b_u}
\end{aligned}$$

and

$$\begin{aligned}
|\acute{z}(t)| &\le |\Omega(b_u)||h(t)| e^{-\int_{b_u}^{t} h(e)de} + \left| \sum_{j=1}^{L} d_i(t)\, \acute{z}\left(t - \sigma_j(t)\right) \right| + |g(t, z(t - \sigma(t)))| \\
&\quad + |h(t)| \int_{b_u}^{t} e^{-\int_w^t h(e)de} \left[\left\| \sum_{j=1}^{L} d_i(w)\, \acute{z}\left(w - \sigma_j(w)\right) + g(w, z(w - \sigma(w))) \right\| \right] dw \\
&\le Q_\circ \beta E_\circ(e^d + 1) + \|z\|_{b_u} |h(t)| \int_{b_u}^{t} e^{-\int_w^t h(e)de} \left[\left| \sum_{j=1}^{L} d_j(w) \right| + T_1(w) \right] dw + \left| \sum_{j=1}^{L} d_j(t) \right| + T_1(t)
\end{aligned}$$

$$\leq P\beta + \mu\|z\|_{b_u}$$

Hence

$$\|z\|_{b_u} \leq P\beta + \mu\|z\|_{b_u}$$

So that

$$\|z\|_{b_u} \leq \frac{P}{1-\mu}\beta \ldots\ldots (3-5)$$

From (3-2), (3-3), (3-5) and (A_2) we get for $n > u$,

$$|z(b_n)| \geq |\Omega(b_u)|e^{-\int_{b_u}^{b_n} h(e)de}$$

$$- e^{-\int_0^{b_n} h(e)de} \int_{b_u}^{b_n} e^{-\int_0^{w} h(e)de} \left[\left|\sum_{j=1}^{L} d_i(w)\, \acute{z}\left(w - \sigma_j(w)\right) + g(w, z(w - \sigma(w))\right|\right] dw$$

$$\geq |\Omega(b_u)|e^{-\int_{b_u}^{b_n} h(e)de} + \|z\|_{b_u} e^{-\int_0^{b_n} h(e)de} \int_{b_u}^{b_n} e^{-\int_0^{t} h(e)de} \left[\left|\sum_{j=1}^{L} d_j(w)\right| + T_1(w)\right] dw$$

$$> \frac{1}{4}\beta - \frac{P}{1-\mu}\beta(e^{-d}+1)\frac{1-\mu}{8P(e^{-d}+1)}$$

$$= \frac{1}{8}\beta$$

This contradicts to the fact that

$$\lim_{t\to\infty} b_n = \infty$$

Thus, the zero solution of (1-1) is GAS in C^1.

Second, we prove the side ($\Leftarrow$). Assume that $\int_0^{\infty} h(e)de = \infty$ holds, for any $b \in [0, \infty)$.

Let

$$Y = \left\{z \in C^1([m_b, \infty)) : \lim_{t\to\infty} z(t) = \lim_{t\to\infty} \acute{z}(t) = 0\right\}$$

where $m_b = \inf_{t\in[b,\infty)}\{t - \tau_j(t), t - \tau(t)\}, j = 1, 2, \ldots, L$, for each $b \in [0, \infty)$ and with the norm $\|z\|_b = \sup_{t\in[m_b,\infty)}\{|z(t)|, |\acute{z}(t)|\}$ for $z \in Y$, Y is a Banach space. Let $M = \{z \in Y, z(t) = \Omega(t) \text{ for } t \in [m_b, b]\}$, where $\Omega \in \Phi_b = \left\{\Omega \in \mathbb{F}_b, \acute{\Omega}(b) = h(b)\Omega(b) + \sum_{i=1}^{r} d_i(b)\, \acute{\Omega}(b - \sigma_i(b)) + g(b, \Omega(b - \sigma(b)))\right\}$.

It is clear that M is a non-empty and closed subset of Y.

Now we define the mapping $\mathbb{V}: M \to C([m_b, \infty))$ by $\mathbb{V}_z(t) = \Omega(t)$ for $t \in [m_b, b]$ and

$$\mathbb{V}_z(t) = \Omega(b)e^{-\int_b^t h(e)de} + \int_b^t e^{-\int_w^t h(e)de} \left[\sum_{j=1}^{L} d_j(w)\, \acute{z}\left(w - \sigma_j(w)\right) + g(w, z(w - \sigma(w)))\right] dw \ldots (3-6)$$

for $t \in [b, \infty)$.

First, we show that $\mathbb{V}: M \to M$ is a self-mapping.

We can get by using (3-6),

$$\acute{\mathbb{V}}_z(t) = -\Omega(b)h(b)e^{-\int_b^t h(e)de} + \sum_{j=1}^{L} d_i(t)\, \acute{z}\left(t-\sigma_j(t)\right) + g(t, z(t-\sigma(t))$$
$$- h(t)\int_b^t e^{-\int_w^t h(e)de}\left[\sum_{j=1}^{L} d_i(w)\, \acute{z}\left(w-\sigma_j(w)\right) + g(w, z(w-\sigma(w))\right] dw \ldots (3-7)$$
$$= h(t)\mathbb{V}_z(t) + \sum_{j=1}^{L} d_j(t)\, \acute{z}\left(t-\sigma_j(t)\right) + g(t, z(t-\sigma(t))$$

By (3-6), and using the definition of Φ_b, we get

$$\acute{\mathbb{V}}_z(b) = -\Omega(b)h(b) + \sum_{j=1}^{L} d_j(b)\, \acute{\Omega}\left(b-\sigma_j(b)\right) + g(b, \Omega(b-\sigma(b)) = \acute{\Omega}(b)$$

Then, $\mathbb{V}_z \in C^1([m_b, \infty))$ for $z \in M$ and $\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \acute{z}(t) = 0$.

By assumption (A_1) and the definition of limit, the following is implied

$$\lim_{t\to\infty} t - \sigma_j(t) = \infty, j = 1 \ldots L$$

For any $\varepsilon > 0$, there exists $q > 0$, such that $\sup\left\{\left|z\left(t-\sigma_j(t)\right)\right|, \left|z(t-\sigma(t))\right|\right\} < \varepsilon$, for $t \geq q \ldots (3-8)$

From (3-6), (3-8) and assumptions (A_1) and (A_4), it follows for $t \geq q$ and $z \in M$ that

$$|\mathbb{V}_z(t)| = \left|\Omega(b)e^{-\int_b^t h(e)de} + \int_b^t e^{-\int_w^t h(e)de}\left[\sum_{j=1}^{L} d_j(w)\, \acute{z}\left(w-\sigma_j(w)\right) + g(w, z(w-\sigma(w))\right] dw\right|$$
$$\leq |\Omega(b)|e^{-\int_b^t h(e)de} + \int_b^u e^{-\int_w^t h(e)de}\left[\sum_{j=1}^{L} d_j(w)\, \acute{z}\left(w-\sigma_j(w)\right) + g(w, z(w-\sigma(w))\right] dw$$
$$+ \int_u^t e^{-\int_w^t h(e)de}\left[\left|\sum_{j=1}^{L} d_j(w)\, \acute{z}\left(w-\sigma_j(w)\right)\right| + \left|g(w, z(w-\sigma(w)) - g(w,0)\right|\right] dw$$
$$\leq e^{-\int_b^t h(e)de}\left[|\Omega(b)| + \int_b^u e^{-\int_b^t h(e)de}\left[\sum_{j=1}^{L} d_j(w)\, \acute{z}\left(w-\sigma_j(w)\right) + g(w, z(w-\sigma(w))\right] dw\right]$$
$$+ \int_u^t e^{-\int_w^t h(e)de}\left[\left|\sum_{j=1}^{L} d_j(w)\, \acute{z}\left(w-\sigma_j(w)\right)\right| + T_1(w)\left|z(w-\sigma(w))\right|\right] dw$$

$$\leq e^{-\int_b^t h(e)de}\left[|\Omega(b)| + \int_b^u e^{-\int_b^e h(e)de}\left(\left|\sum_{j=1}^{L} d_j(w)\,\acute{z}\left(w-\sigma_j(w)\right) + g(w, z(w-\sigma(w))\right|\right)dw\right] + \varepsilon\int_u^t e^{-\int_b^e h(e)de}\left[\left|\sum_{j=1}^{L} d_j(w)\right| + T_1(w)\right]dw$$

$$\leq e^{-\int_b^t h(e)de}\left[|\Omega(b)| + \int_b^u e^{-\int_b^e h(e)de}\left(\left|\sum_{j=1}^{L} d_j(w)\,\acute{z}\left(w-\sigma_j(w)\right) + g(w, z(w-\sigma(w))\right|\right)dw\right] + \varepsilon\mu$$

Otherwise, from (3-8), it follows that there exists $u_1 > u$ such that for $t > u_1$

$$e^{-\int_b^t h(e)de}\left[|\Omega(b)| + \int_b^u e^{-\int_b^e h(e)de}\left(\left|\sum_{j=1}^{L} d_j(w)\,\acute{z}\left(w-\sigma_j(w)\right) + g(w, z(w-\sigma(w))\right|\right)dw\right] < \varepsilon$$

hence, $\lim_{t\to\infty} \mathbb{V}_z(t) = 0$, for $z \in M$.

In addition, it follows from (3-7) and (A_2) that

$$\left|\acute{\mathbb{V}}_z(t)\right| \leq |h(t)\mathbb{V}_z(t)| + \left|\sum_{j=1}^{L} d_j(t)\,\acute{z}\left(t-\sigma_j(t)\right)\right| + \left|g(t, z(t-\sigma(t)) - g(t,0)\right|$$
$$\leq |h(t)\mathbb{V}_z(t)| + \left|\sum_{j=1}^{L} d_j(t)\,\acute{z}\left(t-\sigma_j(t)\right)\right| + T_1(t)\left|z(t-\sigma(t))\right|$$

Now, (A_3) and (A_4) lead to $\lim_{t\to\infty} \acute{\mathbb{V}}_z(t) = 0$ for $z \in M$. Therefore, $\mathbb{V}_z \in M$.

In the second step, we show that $\mathbb{V}: M \to M$ is a mapping contraction.

For $x, z \in M$, by following (3-6), (A_2), and (A_4) we show that

$$|\mathbb{V}_x(t) - \mathbb{V}_z(t)| \leq \int_b^t e^{-\int_w^t h(e)de}\left[\left|\sum_{j=1}^{L} d_j(w)\right|\left|\acute{x}\left(w-\sigma_j(w)\right) - \acute{z}\left(w-\sigma_j(w)\right)\right| + \left|g(w, x(w-\sigma(w)) - g(w, z(w-\sigma(w))\right|\right]dw$$

$$\leq \|x-z\|_b \int_b^r e^{-\int_w^t h(e)de}\left[\left|\sum_{j=1}^{L} d_j(w)\right| + T_1(w)\right]dw \;\ldots\ldots (3-9)$$

$$\leq \mu\|x-z\|_b, \qquad t \in [b,\infty)$$

Now from (3-7), (3-9), (A_2), and (A_4) we get that, for $t \in [b,\infty)$,

$$\left|\acute{\mathbb{V}}_x(t)-\acute{\mathbb{V}}_z(t)\right| \le |h(t)||\mathbb{V}_x(t)-\mathbb{V}_z(t)| + \left|\sum_{j=1}^{L} d_j(t)\right| \left|\acute{x}\left(t-\sigma_j(t)\right)-\acute{z}\left(t-\sigma_j(t)\right)\right| + \left|f(t,x(t-\sigma(t))-f(t,z(t-\sigma(t))\right| \ldots\ldots (3-10)$$

$$\le \|x-z\|_b \left[\left||h(t)|\right| \int_b^t e^{-\int_w^t h(e)de} \left[\left|\sum_{j=1}^{L} d_j(w)\right| + T_1(w)\right]\right] dw + \left|\sum_{j=1}^{L} d_j(t)\right| + L_1(t)$$

$$\le \mu\|x-z\|_b$$

When we complete the proof, we can obtain that $\mathbb{V}: M \to M$ is a mapping contraction, then $\mathbb{V}$ has z wich is a unique fixed point in M by the theorem of Banach fixed point, that is the equation (1-1) has a unique solution through (b,Ω) and satisfies $\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \acute{z}(t) = 0 \ldots\ldots (3-11)$.

The last proof of the theorem is to prove that the zero solution of equation (1-1) is stable in C^1.

First, suppose that the following conditions are true

$E = \sup_{t\in[b,\infty)}\left\{e^{-\int_b^t h(e)de}\right\}$, $Q = \sup_{t\in[b,\infty)}\{|h(t)|\}$.

From (3-1) and (A_3), we get $E, Q \in (0,\infty)$. For any $\varepsilon > 0$, let $\eta > 0$ such that

$$\eta < \varepsilon \min\left\{1, \frac{1-\mu}{E}, \frac{1-\mu}{EQ}\right\}$$

If $z(t) = z(t,b,\Omega)$ is a solution of equation (1-1) with $\|\Omega\|_b < \eta$, then $z(t) = \mathbb{V}_z(t)$ on $[b,\infty)$.

We claim that $\|z\|_b < \varepsilon$. Otherwise, there exists $b_1 > b$ such that $\sup\{|z(b_1)|, |\acute{z}(b_1)|\} = \varepsilon$ and $\sup\{|z(t)|, |\acute{z}(t)|\} < \varepsilon$ for $t \in [m_b, b_1)$. If $|z(b_1)| = \varepsilon$, then it follows from (3-6), (A_2) and (A_4) that

$$|z(b_1)| \le |\Omega(b)| e^{-\int_b^{b_1} h(e)de} + \int_b^{b_1} e^{-\int_w^{b_1} h(e)de} \left[\left|\sum_{j=1}^{L} d_j(w)\, \acute{z}\left(w-\sigma_j(w)\right) + g(w, z(w-\sigma(w)))\right|\right] dw$$

$$\le E\eta + \varepsilon \int_b^{b_1} e^{-\int_w^{b_1} h(e)de} \left[\left|\sum_{j=1}^{L} d_j(w)\right| + T_1(w)\right] dw$$

$$\le E\eta + \varepsilon\mu$$

$$< \varepsilon$$

This is a discrepancy.

Now, if $|\acute{z}(b_1)| = \varepsilon$, then from (3-7), (A_2), and (A_4), we obtain

$$|\acute{z}(b_1)| = |\Omega(b)|h(b_1)e^{-\int_b^{b_1} h(e)de} + \left|\sum_{j=1}^{L} d_i(b_1)\,\acute{z}\left(b_1 - \sigma_j(b_1)\right)\right| + \left|g(b_1, z\left(b_1 - \sigma(b_1)\right)\right|$$

$$+ |h(b_1)| \int_b^{b_1} e^{-\int_w^{b_1} h(e)de} \left[\sum_{j=1}^{L} d_i(w)\,\acute{z}\left(w - \sigma_j(w)\right)\right.$$

$$\left. + g(w, z\left(w - \sigma(w)\right)\right] dw$$

$$\leq EQ\eta + \varepsilon|h(b_1)| \int_b^{b_1} e^{-\int_w^{b_1} h(e)de} \left[\sum_{j=1}^{L} d_j(w) + T_1(w)\right] dw + \sum_{j=1}^{L} d_j(b_1) + T_1(b_1)$$

$$\leq E\eta Q + \varepsilon\mu$$

$$< \varepsilon$$

It is also a discrepancy. Hence, the zero solution of equation (1-1) is stable in C^1. Besides, equation (3-11) implies that the trivial solution of equation (1-1) is GAS in C^1. The proof is complete.

Now, we give an application.

Example 3.2

In equation (1-1), let $h(t) = \frac{1}{1+t}$, $d_j(t) = \frac{1}{10\,L(1+t)}$, $j = 1, 2, \ldots L$, $\sigma_j(t) = 3 + 2\sin t$, $\sigma(t) = 3 + 2\cos t$, and $g\left(t, z(t)\right) = \ln\left(1 + \frac{|z|}{10(1+t)}\right)$, where $h, d_j, g \in C\,([0,\infty), \mathbb{R})$. A direct calculation shows that $|h(t)| \leq 1$, $\int_0^\infty h(u)du = \infty$ when $t \in [0,\infty)$, and A_2 holds when $T_1(t) = \frac{1}{10\,(1+t)}$.

Now, let $\mu = \frac{1}{5}$, then for $t \in [0,\infty)$,

$\int_0^t e^{-\int_w^t h(e)\,de}\left[\left|\sum_{j=1}^{L} d_j(w)\right| + T_1(w)\right]dw = \frac{2t}{10(1+t)} \leq \mu$ and

$|h(t)| \int_0^t e^{-\int_w^t h(e)de}\left[\left|\sum_{j=1}^{L} d_j(w)\right| + T_1(w)\right]dw + \left|\sum_{j=1}^{L} d_j(t)\right| + T_1(t) \leq \frac{2t+2}{10(1+t)} \leq \mu$,

Hence (A_1)- (A_4) hold, then by theorem (3-1), the zero solution is GAS.

In conclusion, we used Banach contraction mapping theorem to give some necessary conditions to achieve the global asymptotic stability of NDE.

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