Al-khafaji Iraqi Journal of Science, 2021, Vol. 62, No. 6, pp: 2000-2008
DOI: 10.24996/ijs.2021.62.6.26

ISSN: 0067-2904

On a Subclass of Analytic and Univalent Functions with Positive Coefficients Defined by a Differential Operator

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Received: 30/9/2020 Accepted: 28/11/2020

Abstract

In this paper, a differential operator is used to generate a subclass of analytic and univalent functions with positive coefficients. The studied class of the functions includes:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \, , (a_n \geq 0, n \in N = \{1,2, \ldots\}),$$

which is defined in the open unit disk $U = \{z \in C: |z| < 1\}$, satisfying the following condition

$$Re\left\{\frac{z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))' + \lambda z^2(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))''}{(1-\lambda)D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z) + \lambda z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))'}\right\}$$
$$\geq \beta\left\{\frac{z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))' + \lambda z^2(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))''}{(1-\lambda)D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z) + \lambda z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))'} - 1\right\} + \alpha, \qquad (z \in U).$$

This leads to the study of properties such as coefficient bounds, Hadamard product, radius of close –to- convexity, inclusive properties, and (n, τ) –neighborhoods for functions belonging to our class.

Keywords: univalent function; coefficient bounds; inclusive properties; neighborhood; radius of close –to- convexity.
2018 Mathematics Subject Classification: 30C45, 30C50

حول فئة فرعية من الدوال التحليلية وحيدة التكافؤ ذات المعاملات الموجبة المحددة بواسطة عامل التفاضل

عقيل كتاب الخفاجي

قسم الرياضيات, كلية التربية للعلوم الصرفة, جامعة بابل

الخلاصه

في هذا البحث ، استخدم المعامل التفاضلي لتوليد فئة فرعية من الدوال التحليلية احادية التكافؤ ذات المعاملات الموجبة. اي انه دراسة فئة من الدوال

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$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \,, (a_n \geq 0, n \in N = \{1,2,\ldots\}),$$

المعرفة في قرص الوحدة المفتوح $U = \{z \in C: |z| < 1\}$. والتي تحقق الشرط التالي:

$$Re\left\{\frac{z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))' + \lambda z^2(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))''}{(1-\lambda)D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z) + \lambda z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))'}\right\} \geq \beta\left\{\frac{z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))' + \lambda z^2(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))''}{(1-\lambda)D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z) + \lambda z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))'} - 1\right\} + \alpha, \qquad (z \in U).$$

وهذا يقود إلى دراسة خصائص مثل حدود المعامل ، منتج Hadamard ، نصف قطر التحدب القريب ، الخصائص الشاملة و (n، τ) – الجوارات للدوال التي تنتمي إلى صفنا.

1. Introduction

Let $\mathbb{A}$ denotes the class of all analytic and univalent functions in the unit disk $U = \{z \in C: |z| < 1\}$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \,, (a_n \geq 0, n \in N = \{1,2,\ldots\}), \tag{1}$$

Let $g \in \mathbb{A}$ has the form

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, b_n \geq 0$$

and f of the form (1), then the convolution (or Hadamard product) $(f * g)$ of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z), \tag{2}$$

For $f \in \mathbb{A}$, Elhaddad *et al*. [1] introduced the following differential operator:

$$D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z) = z + \sum_{n=2}^{\infty}[1 + ([n]_q - 1)\lambda]^m \frac{\Gamma_q(\delta)(q^{\Upsilon};q)_{n-1}}{\Gamma_q(\sigma(n-1)+\delta)(q;q)_{n-1}} a_n z^n$$

where $0 < q < 1, n, m \in \mathbb{N}, \sigma, \delta, \gamma > 0, 0 \leq \lambda \leq 1$ and $z \in U$.

Note that:

- If $q \to 1$ and $\Upsilon = 1$, we obtain the operator defined in [2].
- If $q \to 1, \sigma = 0, \Upsilon = 0$ and $\delta = 1$, we obtain Al-Oboudi operator, see Ref. [3].
- If $q \to 1, \sigma = 0, \Upsilon = 1, \delta = 1$ and $\lambda = 1$, we obtain Sălăgean operator, see Ref. [4].
- If $q \to 1, m = 0$ and $\Upsilon = 1$, we obtain $\mathbb{E}_{\sigma,\delta}(z)$, see Ref. [5].

Using the operator $D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)$, we introduce the class of analytic functions with positive coefficients as illustrated below.

Definition 1. For ($\beta \geq 0, 0 \leq \rho < 1, 0 \leq \lambda \leq 1$), the function f given by Equation 1, is said to be in the class $\mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$ if and only if the following inequality is satisfied:

$$Re\left\{\frac{z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))' + \lambda z^2(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))''}{(1-\lambda)D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z) + \lambda z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))'}\right\} \geq \beta\left\{\frac{z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))' + \lambda z^2(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))''}{(1-\lambda)D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z) + \lambda z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))'} - 1\right\} + \rho, \qquad (z \in U).$$

Many authors have studied various classes of analytic functions with positive coefficients [4,6, 7, 8, 9, 10, 11, 12]. In this work, we introduce and study the class $\mathbb{A}(\lambda,\beta,\alpha,\gamma,q,\sigma,\delta)$ of analytic functions with positive coefficients. Also, several properties, such as coefficient bounds, Hadamard product, radius of close –to- convexity, inclusive properties, and (n, τ) –neighborhoods of functions in our class, are obtained.

2. Main results

In this section, we prove the geometric properties of functions in the class $\mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$.

Theorem 1. A function fof the form (1) belongs to the class $\mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$ if and only if

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha]\,\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)\,a_n \le 1-\rho, \tag{3}$$

where $0 \le \rho < 1, 0 \le \lambda \le 1, \beta \ge 0$ and $\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)$ is defined by

$$\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta) = [1+([n]_q-1)\lambda]^m \frac{\Gamma_q(\delta)(q^\gamma;q)_{n-1}}{\Gamma_q(\sigma(n-1)+\delta)(q;q)_{n-1}}.$$

Proof. Suppose that the inequality (3) holds true. Then we want to prove that $f \in \mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$. By Definition 1, we have

$$Re\left\{\frac{z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))' + \lambda z^2(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))''}{(1-\lambda)D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z)+\lambda z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))'}\right\}$$
$$\ge \beta\left|\frac{z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))' + \lambda z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))'}{(1-\lambda)D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z)+\lambda z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))'}-1\right|+\rho,$$

Then, using this fact, we obtain:

$$Re\left\{\frac{z(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z))' + \lambda z^2(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z))''}{(1-\lambda)D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)+\lambda z(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z))'}\left(1+\beta e^{i\emptyset}\right)-\beta e^{i\emptyset}\right\} \ge \rho,$$
$$(-\pi < \emptyset \le \pi)$$

or, equivalently

$$Re\left\{\frac{D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))' + \lambda z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))''(1+\beta e^{i\phi})}{(1-\lambda)D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z)+\lambda z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))'}\right.$$
$$\left.-\frac{\beta e^{i\phi}\left((1-\lambda)D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z)+\lambda z^2(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))'\right)}{(1-\lambda)D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z)+\lambda z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))'}\right\} \ge \rho. \tag{4}$$

Let

$$C(z) = \left[(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))' + \lambda z^2(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))''\right]\left(1+\beta e^{i\phi}\right)$$
$$-\beta e^{i\phi}\left[(1-\lambda)\left(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z)\right)+\lambda z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))'\right],$$

and

$$\mathrm{D}(z) = (1-\lambda)D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z)+\lambda z(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)f(z))'$$

using the fact that

$$|C(z)+(1-\rho)D(z)| \ge |C(z)-(1+\rho)D(z)|$$

$(0 \le \rho < 1)$

but

$|C(z)+(1-\rho)D(z)| =$

$$\left|\left[z-\sum_{n=2}^{\infty}\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)\,a_n z^n - \lambda\sum_{n=2}^{\infty}n(n-1)\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)\,a_n z^n\right]\left(1+\beta e^{i\phi}\right)\right.$$
$$-\beta e^{i\phi}\left[(1-\lambda)(z-\sum_{n=2}^{\infty}\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)\,a_n z^n+\lambda z-\lambda\sum_{n=2}^{\infty}n\,\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)\,a_n z^n\right]+(1$$
$$\left.-\rho)\left[z-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\,\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)\,a_n z^n\right]\right|$$
$$=\left|(2-\rho)z-\sum_{n=2}^{\infty}[(n+\lambda n(n-1)+(1-\rho)(1-\lambda+n\lambda)]\,\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)\,a_n z^n\right.$$
$$\left.-\beta e^{i\phi}\sum_{n=2}^{\infty}[n+\lambda n(n-1)-(1-\lambda+n\lambda)]\,\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)\,a_n z^n\right|$$

$$\geq (2-\rho)|z| - \sum_{n=2}^{\infty}\left[\left(n+\lambda n(n-1)\right)+(1-\rho)(1-\lambda+n\lambda)\right]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)\, a_n|z|^n$$
$$-\,\beta\sum_{n=2}^{\infty}[n+\lambda n(n-2)-1+\lambda]\,\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)\, a_n|z|^n.$$

Also,

$$|C(z)-(1+\rho)D(z)|$$
$$=\left|\left[z-\sum_{n=2}^{\infty}n\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)a_nz^n-\lambda\sum_{n=2}^{\infty}n(n-1)\,\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)\,a_nz^n\right]\left(1+\beta e^{i\phi}\right)\right.$$
$$-\beta e^{i\phi}\left[z-(1-\lambda)\sum_{n=2}^{\infty}\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)a_n\,z^n-\lambda\sum_{n=2}^{\infty}n\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)\,a_nz^n\right]$$
$$\left.-(1+\rho)\left[z-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\,\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)\,a_nz^n\right]\right|$$
$$=\left|-\rho z-\sum_{n=2}^{\infty}[n+\lambda n(n-1)-(1-\rho)(1-\lambda+n\lambda)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)a_n\,z^n\right.$$
$$\left.-\beta e^{i\phi}\sum_{n=2}^{\infty}[n+n\lambda(n-1)-(1-\lambda+n\lambda]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)a_nz^n\right|$$
$$\leq \rho|z|+\sum_{n=2}^{\infty}\left[\left(n+n\lambda(n-1)\right)-(1+\rho)(1-\lambda+n\lambda)\right]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)a_n|z|^n$$
$$+\beta\sum_{n=2}^{\infty}[n+n\lambda(n-1)-(1-\lambda+n\lambda)]\,\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)\,a_n|z|^n.$$

On the other hand, we have

$$|C(z)+(1-\rho)D(z)|-|C(z)-(1+\rho)D(z)|\geq$$
$$2(1-\rho)|z|-\sum_{n=2}^{\infty}\left[\left(2n+2n\lambda(n-1)\right)-2\rho(1+\lambda+n\lambda)\right.$$
$$\left.-\beta\left(2n+2n\lambda(n-1)-2(1-\lambda+n\lambda)\right)\right]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)a_n|z|^n\geq 0.$$

Or

$$\sum_{n=2}^{\infty}[n(1+\beta)+n\lambda(n-1)(1+\beta)-(1-\lambda+n\lambda)(\rho+\beta)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)a_n\leq 1-\rho$$

This is equivalent to

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\rho)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)a_n\leq 1-\rho$$

Conversely, suppose that (4) holds. Then we must show that

$$Re\left\{\frac{\left[z(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z))'+\lambda z^2(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z))''\right](1+\beta e^{i\phi})}{(1-\lambda)D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)+\lambda z(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z))'}\right.$$
$$\left.-\frac{\beta e^{i\phi}\left[(1-\lambda)D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)+\lambda z(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z))'\right]}{(1-\lambda)D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)+\lambda z(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z))'}\right\}\geq\rho.$$

By taking the values of z on the positive real axis, where $0\leq z=r<1$, we have from the above inequality that

$$Re\left\{\frac{(1-\rho)-\sum_{n=2}^{\infty}\left[n\left(1+\beta e^{i\phi}\right)(1-\lambda+n\lambda)-\left(\rho+\beta e^{i\phi}\right)(1-\lambda+n\lambda)\right]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)a_nr^{n-1}}{1-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)a_nr^{n-1}}\right\}\geq 0,$$

Since $Re\left(-e^{i\phi}\right)\geq -\left|e^{i\phi}\right|=-1$, then from the last inequality, we have:

$$Re\left\{\frac{(1-\rho)-\sum_{n=2}^{\infty}[n(1+\beta)(1-\lambda+n\lambda)-(\rho+\beta)(1-\lambda+n\lambda)]K(n,\mu,\theta)a_n r^{n-1}}{1-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)a_n r^{n-1}}\right\}\geq 0.$$

By letting $r\to 1^-$, we achieve our result.

□

Next, we get the radius of close – to – convexity for functions belonging to the class $\mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$.

Theorem 2: Let the function f, defined by (1), be in the class $\mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$, then f is close – to – convex of order $\delta(0\leq\delta<1)$ in $|z|<r(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$, where

$$r(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)=\inf_{n}\left\{\frac{(1-\delta)(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\rho)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{n(1-\rho)}\right\}^{\frac{1}{n-1}},n\geq 2. \quad (5)$$

Proof: We must that $|f'(z)-1|\leq 1-\delta$ for $|z|<r(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$, then we have

$$|f'(z)-1|\leq\sum_{n=2}^{\infty}na_n|z|^{n-1}$$

Thus, $|f'(z)-1|\leq 1-\delta$, if

$$\sum_{n=2}^{\infty}\left(\frac{n}{1-\delta}\right)a_n|z|^{n-1}\leq 1. \quad (6)$$

According to Theorem 1, we have

$$\sum_{n=2}^{\infty}\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\rho)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{(1-\rho)}a_n\leq 1, \quad (7)$$

Hence, (7) will be true if

$$\frac{n|z|^{n-1}}{1-\delta}\leq\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\rho)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{(1-\rho)}$$

Equivalently, if

$$|z|\leq\left\{\frac{(1-\delta)(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\rho)]\mathrm{K}(n,\mu,\theta)}{n(1-\rho)}\right\}^{\frac{1}{n-1}},n\geq 2, \quad (8)$$

Then the theorem is following form (8).

□

Theorem 3: Let $f\in\mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$. Then

$$D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)=\exp\left(\int_0^z\frac{\beta-\psi(t)\rho}{t(\beta-\psi(t))}dt\right),|\psi(t)|<1,z\in U$$

Proof: The case $\beta=0$ is obvious. Let $\beta\neq 0$, for $f\in\mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$, and

$$w=\frac{z(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z))'}{D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)},$$

We have $Re\,w>\beta|w-1|+\rho$,

Therefore,

$$\left|\frac{w-1}{w-\rho}\right|<\frac{1}{\beta}.$$

Or, equivalently

$$\frac{w-1}{w-\rho}=\frac{\psi(z)}{\beta},$$

where, $|\psi(z)|<1,z\in U$.

So, we have

$$\frac{(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z))'}{D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)}=\frac{\beta-\psi(z)\rho}{z(\beta-\psi(z)}$$

After integration, we get

$$\log\left(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)\right)=\int_0^z \frac{\beta-\psi(t)\rho}{t(\beta-\psi(t)}dt,$$

Thus,

$$\left(D_{\lambda,q}^{\Upsilon,m}(\sigma,\delta)f(z)\right)=exp\left[\int_0^z \frac{\beta-\psi(t)\rho}{t(\beta-\psi(t)}dt\right].$$

This completes the proof.

Theorem 4. Let $f(z)=z+\sum_{n-2}^{\infty}a_nz^n$, $g(z)=z+\sum_{n=2}^{\infty}a_nz^n$ belong to $\mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$. Then the Hadamard product of f and g, given by $(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n$, belongs to $\mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$.

Proof: Since f and $g\in\mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$, we have

$$\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\rho+\beta]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\rho}\right]a_n\leq 1,$$

and

$$\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\rho}\right]a_n\leq 1$$

And by applying the Cauchy-Schwarz inequality, we have

$$\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\rho+\beta)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\rho}\right]\sqrt{a_n}$$

$$\leq\left(\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\rho+\beta)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\rho}\right]a_n\right)^{\frac{1}{2}}$$

$$\times\left(\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\rho+\beta)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\rho}\right]a_n\right)^{\frac{1}{2}}.$$

However, we obtain

$$\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\rho+\beta)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)\sqrt{a_n}}{1-\rho}\right]\sqrt{a_n}\leq 1.$$

Now, we want to prove that

$$\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\rho+\beta)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\rho}\right]a_n\leq 1,$$

Since

$$\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\rho+\beta)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\rho}\right]a_n$$

$$=\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\rho+\beta)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)\sqrt{a_n}}{1-\rho}\right]\sqrt{a_n}.$$

hence, we get the required result.
□

Theorem 5. Let the function f, defined by (1), and g, given by $g(z)=z+\sum_{n=2}^{\infty}a_nz^n$, be in the class $\mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$. Then, the function h, defined by $h(z)=z+\sum_{n=2}^{\infty}a_n^2b_n^2z^n$, is in the class $\mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$, where $0\leq\lambda\leq 1, 0\leq\rho<1, 0\leq\Upsilon<1, \beta\geq 0, z\in U$, and

$$\gamma\leq 1-\frac{(1-\rho)^2(1+\beta)}{(1+\lambda)(2+\beta-\rho)^2(1-\mu)(1+\theta)-(1-\rho)^2}$$

Proof: We must find the largest γ, such that

$$\sum_{n=2}^{\infty}\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\gamma)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\gamma}a_n^2\leq 1. \tag{9}$$

Since f and g are in the class $\mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$, we see that

$$\sum_{n=2}^{\infty}\left\{\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\rho)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\rho}\right]a_n\right\}^2 \leq 1, \tag{10}$$

and

$$\sum_{n=2}^{\infty}\left\{\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\rho}\right]b_n\right\}^2 \leq 1, \tag{11}$$

Combining the inequalities (10) and (11) gives

$$\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\rho)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\rho}\right]a_n^2 b_n^2 \leq 1,$$

But $h \in \mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$, if and only if

$$\sum_{n=2}^{\infty}\left\{\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\gamma]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\gamma}\right\}a_n^2 b_n^2 \leq 1, \tag{12}$$

The inequality (12) would obviously imply (9) if

$$\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\gamma]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\gamma} \leq \left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\rho\rho]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\rho}\right]^2$$
$$= u^2,$$

then

$$\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\gamma]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\gamma} \leq u^2,$$

Or

$$\frac{1-\gamma}{1+\beta} \geq \frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\rho]\,\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{u^2-(1-\lambda+n\lambda)\,\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}$$

The right hand is a decreasing function of n and it is at maximum if $n=2$.
Now

$$\frac{1-\gamma}{1+\beta} \geq \frac{(1-\lambda+n\lambda)(n-1)(1-\rho)^2}{\left[(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\gamma]\right]^2\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)-(1-\lambda+n\lambda)(1-\rho)^2}$$

By simplifying the last inequality, we get

$$\frac{1-\gamma}{1+\beta} \geq \frac{(1-\rho)^2}{(1-\lambda+n\lambda)(2+\beta-\alpha)(1-\mu)(\theta+1)-(1-\rho)^2},$$

or

$$\gamma \leq 1-\frac{(1-\rho)^2(1+\beta)}{(1-\lambda+n\lambda)(2+\beta-\rho)(1-\mu)(\theta+1)-(1-\rho)^2}.$$

This completes the proof of theorem. □

Next, we obtain the inclusive properties of the class $\mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$.

Theorem 6. Let $\beta \geq 0, 0 \leq \rho < 1, 0 \leq \lambda \leq 1, \gamma \geq 0, 0 \leq \mu < 1$ and $0 \leq \theta \leq 1$. Then $\mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta) \subseteq \mathbb{A}(0,\beta,\rho,\gamma,q,\sigma,\delta)$, where

$$\gamma \leq 1-\frac{(n-1)(1-\rho)(1+\beta)\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\gamma)](\theta+1)\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)-(1-\rho)\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}, \tag{13}$$
$$n \geq 2.$$

Proof. Let $f \in \mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$, then in view of Theorem 1, we have

$$\sum_{n=2}^{\infty}\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\rho+\beta)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\rho}a_n\le 1,$$

We wish to find the value γ, such that

$$\sum_{n=2}^{\infty}\frac{[n(1+\beta)-(\beta+\gamma)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\gamma}a_n\le 1 \qquad (14)$$

The inequality (13) would obviously imply (14) if

$$\frac{[n(1+\beta)-(\beta+\gamma)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\gamma}\le\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\rho+\beta)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\rho}=u.$$

Therefore,

$$\frac{[n(1+\beta)-(\beta+\gamma)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\gamma}\le u, \qquad (15)$$

Now, (15) gives the simplification

$$\frac{1-\gamma}{1+\beta}\ge\frac{(n-1)\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{u-\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)},\qquad (n\ge 2). \qquad (16)$$

The right-hand side of (16) decreases as *n* increases and, hence, it is maximum for $n=2$.
So, (16) is satisfied provided that

$$\frac{1-\gamma}{1+\beta}\ge\frac{(n-1)(1-\rho)\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\rho)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)-(1-\rho)K(n,\mu,\theta)}=d$$

Obviously, $d<1$, and

$$\gamma\le 1-\frac{(n-1)(1-\rho)(1+\beta)\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\rho)]\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)-(1-\rho)\Omega_{\lambda,q}^{\gamma,m}(\sigma,\delta)},$$

Theorem 7. Let $\beta\ge 0, 0\le\rho<1, \lambda_1\ge\lambda_2\ge 0, 0\le\theta\le 1, 0\le\mu\le 1$. Then, $\mathbb{A}(\lambda_1,\beta,\rho,\gamma,q,\sigma,\delta)\subseteq\mathbb{A}(\lambda_2,\beta,\rho,\gamma,q,\sigma,\delta)$.
The proof of Theorem 7 follows also from Theorem 6.
Now, we determine a set of inclusion relations involving $(n,\tau)-$ neighborhoods. We define the $(n,\tau)-$ neighborhoods of a function $f\in R$ by

$$N_{n,\tau}(f)=\left\{g\in R: g(z)=z+\sum_{n=2}^{\infty}b_nz^n \ \ and \ \sum_{n=2}^{\infty}n|a_n-b_n|\le\tau, 0\le\tau<1\right\}, \qquad (17)$$

Also, we need the following definition.
Definition 2. The function *f*, defined by (1), is said to be a member of the class $\mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$ if there exists a function $g\in\mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$, such that

$$\left|\frac{f(z)}{g(z)}-1\right|\le 1-\ell, (z\in U, 0\le\ell<1).$$

Theorem 8. Let $g\in\mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$ and

$$\ell=1-\frac{\tau(1+\lambda)(2+\beta-\rho)(1-\mu)(\theta+1)a_2}{2\{(1+\lambda)(2+\beta-\rho)(1-\mu)(\theta+1)a_2-(1-\rho)\}}, \qquad (18)$$

then, $N_{n,\tau}(g)\subset\mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta,\ell)$.

Proof. Let $g\in N_{n,\tau}(g)$. Then, we have from (17) that

$$\sum_{n=2}^{\infty}n|a_n-b_n|\le\frac{\tau}{2}$$

Also, since $g\in\mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$, we have from Theorem 1 that

$$\sum_{n=2}^{\infty}b_n\le\frac{(1-\rho)}{(1+\lambda)(2+\beta-\lambda)(1-\mu)(\theta+1)a_2}$$

So that

$$\left|\frac{f(z)}{g(z)}-1\right| < \frac{\sum_{n=2}^{\infty}|a_n-b_n|}{1-\sum_{n=2}^{\infty}b_n} \leq \frac{\tau}{2}\cdot\frac{(1+\lambda)(2+\beta-\lambda)(1-\mu)(\theta+1)a_2}{(1+\lambda)(2+\beta-\lambda)(1-\mu)(\theta+1)a_2-(1-\rho)} = 1-\ell.$$

Thus, by definition, $f \in \mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta,\ell)$, for ℓ given by Equation 18.

Theorem 9. Let c be a real number such that $c > -1$. If $f \in \mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$, then the function F_c, defined by

$$F_c(z) = \frac{c+1}{z^c}\int_0^z s^{c-1}f(s)ds, \tag{19}$$

also belongs to $\mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then

$$\begin{aligned} F_c(z) &= \frac{c+1}{z^c}\int_0^z s^{c-1}(s+\sum_{n=2}^{\infty}a_n s^n)ds \\ &= \frac{c+1}{z^c}\int_0^z (s^c+\sum_{n=2}^{\infty}s^{c-1+n}a_n)ds \\ &= \frac{c+1}{z^c}\left[\frac{s^{c+1}}{c+1} - \sum_{n=2}^{\infty}\frac{s^{c+n}}{c+n}\,a_n\right]_0^z \\ &= z+\sum_{n=2}^{\infty}\frac{c+1}{c+n}a_n z^n, \end{aligned}$$

Hence,

$$F_c(z) = z+\sum_{n=2}^{\infty}\frac{c+1}{c+n}a_n z^n.$$

Therefore,

$$\sum_{n=2}^{\infty}\frac{(c+1)(1-\lambda+n\lambda)[n(1+\beta)-(\rho+\beta)]K(n,\mu,\theta)}{(c+n)}a_n \leq (1-\lambda+n\lambda)[n(1+\beta)-(\rho+\beta)]K(n,\mu,\theta)a_n \leq 1-\rho$$

Hence, $F_c \in \mathbb{A}(\lambda,\beta,\rho,\gamma,q,\sigma,\delta)$.

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