Stability and Anti-Chaos Control of Discrete Quadratic Maps

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Abstract

　A dynamical system describes the consequence of the current state of an event or particle in future. The models expressed by functions in the dynamical systems are more often deterministic, but these functions might also be stochastic in some cases. The prediction of the system's behavior in future is studied with the analytical solution of the implicit relations (Differential, Difference equations) and simulations. A discrete-time first order system of equations with quadratic nonlinearity is considered for study in this work. Classical approach of stability analysis using Jury's condition is employed to analyze the system's stability. The chaotic nature of the dynamical system is illustrated by the bifurcation theory. The enhancement of chaos is performed using Cosine Chaotification Technique (CCT). Simulations are carried out for different parameter values.

Keywords: Discrete Dynamical System, quadratic maps, Cosine Chaotification Technique, chaos, bifurcation

التحكم في الاستقرار ومضاد الفوضى في النظام التربيعية المنفصلة

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الخلاصه

يصف النظام الديناميكي نتيجة الحالة الحالية لحدث أو جسيم في المستقبل. النماذج التي يتم التعبير عنها من خلال الدوالفي الأنظمة الديناميكية غالبٗ ا ما تكون حتمية ولكن هذه الوظائف قد تكون عشوائية في بعض الحالات أيضًا. تتم دراسة التنبؤ بسلوك النظام في المستقبل بالحل التحليلي للعلاقة الضمنية (المعادلات التفاضلية ، معادلات الفروق) والمحاكاة. في هذا العمل تمت دراسة المعادلات من الدرجة الاولى المنفصلة مع اللاخطية التربيعية. يتم استخدام النهج الكلاسيكي تم استعمال شروط لتحليل استقرار النظام Jury. تتضح الطبيعة الفوضوية للنظام الديناميكي من خلال نظرية التشعب. يتم إجراء تحسين الفوضى باستخدام تقنية تشوه جيب التمام (CCT). يتم تنفيذ عمليات المحاكاة لقيم المعلمات المختلفة.

1. Introduction

　　The dynamical systems with a given initial point can be solved with time progressing in small steps to determine the future position of the event under consideration. Finding a trajectory or orbit that describes the system required more complicated mathematical techniques before the arrival of computers and only handful of dynamical systems were dealt with. Emergence of the technological

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advancement played a vital role in simplifying the process of finding orbits. Mathematical formulation of the tractable events in nature helps in answering various questions that are posed on the dynamics of the event by carrying out theoretical and numerical analyses. The mathematical modeling of real life can be classified broadly as continuous time models (differential equations) and discrete time models (difference equations, maps). The behavior of discrete dynamical systems is very complex to analyze their behavior. It needs more study to understand how the dynamics of the system can be working. Many researchers used different methods for analyzing the behavior of discrete dynamical systems [5, 7, 12, 18]. The nature of the dynamical systems can be studied by quantitative or qualitative approaches. The quantitative approaches give clear understanding of the systems under consideration. But it is not always possible to follow quantitative approaches. In the case of nonlinear systems it is more suitable to follow a qualitative approach, since finding the analytical solutions are not possible for every model constructed. Since most of the real life models are nonlinear in nature, a qualitative approach proves to confer a crucial study of dynamical behaviors of the system. For this study, it is necessary, but highly nontrivial, to detect the fixed point of the discrete dynamical systems and analyze the stability and bifurcation of each fixed point. The discrete dynamical systems have been studied in several areas of physics, biology, neural networks, and many other [3, 9, 13, 15, 17, 20]. The stability analysis and chaos of the discrete ecological systems were studied by various authors [8, 10, 11, 17, 14]. The chaotic study on ecological models was of greater interest to mathematicians and scientists all over the world [1, 6, 14, 19]. Qamar Din *et al*. established the strategy of establishing the chaos control for a discrete predator-prey system [4]. In this study, we investigate the qualitative behavior of this system:

$$x_{n+1} = Ax_n + By_n \qquad (1)$$
$$y_{n+1} = Cx_n^2 + Kx_ny_n + Ey_n^2$$

where $A \neq 1, B \neq 0, C, K, E > 0$ are real parameters.

The system was investigated earlier [16], where the authors used an algebraic approach for stability and bifurcation methods for analyzing bifurcations and chaos. They dealt with the parameter conditions for establishing the two kinds of bifurcations.

In this study, the analysis of stability of the dynamical system (1) for a non-trivial fixed point is carried out using the Jury's condition. The chaotic nature of the system is described with the bifurcation diagrams and the change in the behavior is discussed with the phase portraits. The paper is formatted with stability conditions in section 2, while examples are provided in section 3. The bifurcation theory is described in section 4 and the anti-chaos control is implemented in section 5, followed with conclusions.

2.   Stability Conditions of System (1)

This section presents the fixed points of system (1) and the stability conditions that are obtained from the eigenvalues of the Jacobian matrix at the fixed point.

The fixed points of system (1) are $F_0 = (0,0)$ and

$$F_1 = \left(\frac{(1-A)B}{A^2E - ABK + B^2C - 2AE + BK + E}, \frac{(A-1)^2}{A^2E - ABK + B^2C - 2AE + BK + E} \right).$$

This fixed point exists only when $A^2E + B^2C + BK + E \neq ABK + 2AE$. We use the following lemma to analyze the stability of fixed points of system (1), which can be evaluated by the relations between roots and coefficients of a quadratic equation.

Lemma 1.[12]. Let $P(\lambda) = \lambda^2 - W\lambda + V$. Suppose that $P(1) > 0$ and λ_1 and λ_2 are two roots of $P(\lambda) = 0$. Then

(i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $P(-1) > 0$ and $V < 1$;
(ii) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $P(-1) < 0$;
(iii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $P(-1) > 0$ and $V > 1$;
(iv) $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $P(-1) = 0$ and $W \neq 0, 2$;
(v) λ_1 and λ_2 are complex and $|\lambda_1| = |\lambda_2| = 1$ if and only if $W^2 - 4V < 0$ and $V = 1$.

The nontrivial fixed point (F_I) is considered for the analysis of stability of system (1), which can be determined by the absolute value of the roots of the equation obtained from the Jacobean matrix at (F_I). Jacobean matrix at F_I is

$$J(F_I) = \begin{bmatrix} A & B \\ \Omega_1 & \Omega_2 \end{bmatrix} \qquad (2)$$

where

$$\Omega_1 = \frac{(A - 1)[K(A - 1) - 2BC]}{A^2E - ABK + B^2C - 2AE + BK + E}$$

and

$$\Omega_2 = \frac{(A - 1)[-KB + 2E(A - 1)]}{A^2E - ABK + B^2C - 2AE + BK + E}$$

The characteristic equation of $J(x, y)$ is written as $\lambda^2 - Trace[J(F_I)] + Det[J(F_I)] = 0$. Here

$$Trace[J(F_I)] = A + \frac{(A - 1)[-KB + 2E(A - 1)]}{A^2E - ABK + B^2C - 2AE + BK + E}$$

and

$$Det[J(F_I)] = \frac{(A - 1)(2A^2E - 2ABK + 2B^2C - 2AE + BK)}{A^2E - ABK + B^2C - 2AE + BK + E}$$

The eigenvalues are given by

$$\lambda_{1,2} = \frac{1}{2}\left[\frac{(A^3E - A^2BK + AB^2C - 3AE + BK + 2E) \pm \sqrt{\Delta}}{A^2E - ABK + B^2C - 2AE + BK + E}\right]$$

where

$$\Delta = A^6E^2 - 2A^5BEK + 2A^4B^2CE + A^4B^2K^2 - 2A^3B^3CK + A^2B^4C^2 - 8A^5E^2 + 16A^4BEK - 16A^3B^2CE - 8A^3B^2K^2 + 16A^2B^3CK - 8AB^4C^2 + 26A^4E^2 - 44A^3BEK + 34A^2B^2CE + 18A^2B^2K^2 - 26AB^3CK + 8B^4C^2 - 44A^3E^2 + 56A^2BEK - 28AB^2CE - 16AB^2K^2 + 12B^3CK + 41A^2E^2 - 34ABEK + 8B^2CE + 5B^2K^2 - 20AE^2 + 8BEK + 4E^2.$$

We recall some definitions of topological kinds for a fixed point F_I. F_I is called a sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. A sink is locally asymptotic stable. (x, y) is called a source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$. A source is locally unstable. F_I is called a saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$). Also, F_I is called non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$. Using Jury's criterion [5, 12], we obtain the condition for local stability of the fixed point F_I.

Proposition 2. The steady state F_I of system (1) is

(i)   Sink point if $\psi < E < \psi_1$

(ii)   Saddle if $E < min\{\psi, \psi_1\}$.

(iii)   Source if $E > max\{\psi, \psi_1\}$.

(iv)   Non-hyperbolic if one of the following conditions holds

(1) $E = \psi$.

(2) $E = \psi_1$.

where

$$\psi = \frac{3A^2KB - 3AB^2C - 2ABK + CB^2 - KB}{3A^3 - 3A^2 - 3A + 3}$$

and

$$\psi_1 = \frac{2A^2BK - 2AB^2C - 4ABK + 3B^2C + 2BK}{2A^3 - 5A^2 + 4A - 1}$$

Proof: According to Lemma 1, the characteristic polynomial is given by

$$P(\lambda) = \lambda^2 - Trace[J(F_1)] + Det[J(F_1)]$$

(i)   In viewing condition (i) of Lemma1, the fixed point is a sink ($|\lambda_1| < 1, |\lambda_2| < 1$) iff $P(-1) > 0, Det[J(F_I)] < 1$.

First, we shall derive the condition for stability using $P(-1) > 0$.

$$P(-1) = 3[A^3E - A^2E - A^2BK + AB^2C + E - AE] - B^2C + KB + 2ABK > 0$$

yields

$$E > \frac{3A^2KB - 3AB^2C - 2ABK + CB^2 - KB}{3A^3 - 3A^2 - 3A + 3} = \psi.$$

$Det[J(F_I)] < 1$, then

$$E < \frac{2A^2BK - 2AB^2C - 4ABK + 3B^2C + 2BK}{2A^3 - 5A^2 + 4A - 1} = \psi_1.$$

By combining both conditions, we have

$$\frac{3A^2KB - 3AB^2C - 2ABK + CB^2 - KB}{3A^3 - 3A^2 - 3A + 3} < E < \frac{2A^2BK - 2AB^2C - 4ABK + 3B^2C + 2BK}{2A^3 - 5A^2 + 4A - 1}$$

Hence, the fixed point F_I is a sink if $\psi < E < \psi_1$.

(ii)   From condition (ii) of Lemma 1,   the fixed point is a saddle $(|\lambda_1| < 1, |\lambda_2| > 1 \ (or \ |\lambda_1| > 1, |\lambda_2| < 1))$ iff $P(-1) < 0, Det[J(F_I)] < 1$.

Similar to the above proof, the fixed point F_I is a saddle if $E < \min\{\psi, \psi_1\}$.

(iii)   Utilizing condition (iii) of Lemma 1, the fixed point is a source $(|\lambda_1| > 1, |\lambda_2| > 1)$ iff $P(-1) > 0, Det[J(F_I)] > 1$.

Thus, the fixed point F_I is a source if $E > \max\{\psi, \psi_1\}$.

(iv)   From conditions (iv) and (v) of Lemma 1, the fixed point F_I is non-hyperbolic if one of the following holds:

(1) $E = \psi$

(2) $E = \psi_1$

3.   Numerical Results for Stability

This section exhibits the time line and phase trajectories for (1) around the interior fixed point F_I .

Example 1. The phase portrait for system (1) with time plots are given in Figures $1(I - II)$ for $A = 1.07, B = -0.091, C = 0.4, E = 0.1,$ and $K = 2$, where the initial position is at $x = 0.4$ and $y = 0.3$, the interior point is $F_I = (0.385071, 0.296208)$. Thus $1 + Trace[J(F_I)] + Det[J(F_I)] = 3.8687 > 0$ and $Det[J(F_I)] = 0.969383 < 1$, which satisfies the Jury's criteria. Also, $|\lambda_{1,2}| = 0.984572 < 1$. Hence, the system attains stability.
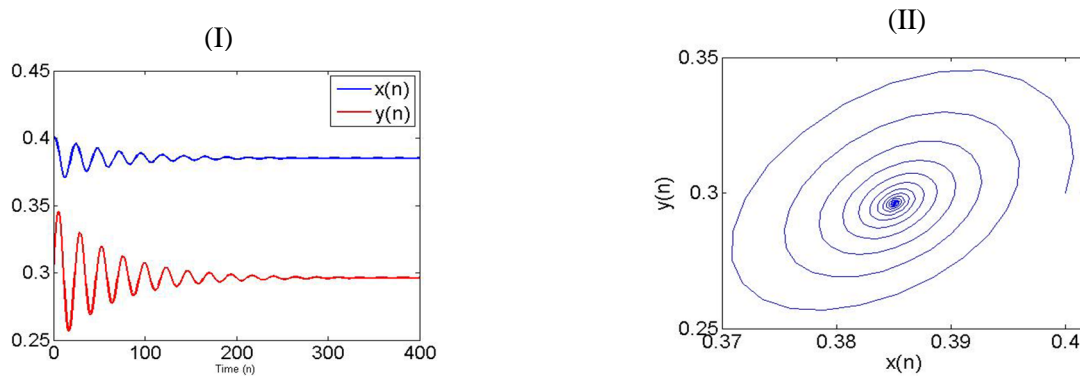
(I)

(II)

Figure 1 : (I) Time plot showing the stability of system (1), (II) Spiral phase trajectory of system (1) towards F_1 .

Example 2. Closed orbits for system (1) are presented in Figures $2(I - II)$ for the following parameter values $A = 1.07, B = -0.0356, C = 0.54, E = 0.01, K = 2$ with initial position at $(x = 0.4, \ y = 0.9)$. Using the values, we get $1 + Trace[J(F_I)] + Det[J(F_I)] = 3.987739 > 0$ and $Det[J(F_I)] = 1.028869 > 1$. Numerically, Jury's criterion is satisfied. Also, the eigenvalues are $|\lambda_{1,2}| = 1.014332 > 1$. Hence, the system is unstable.
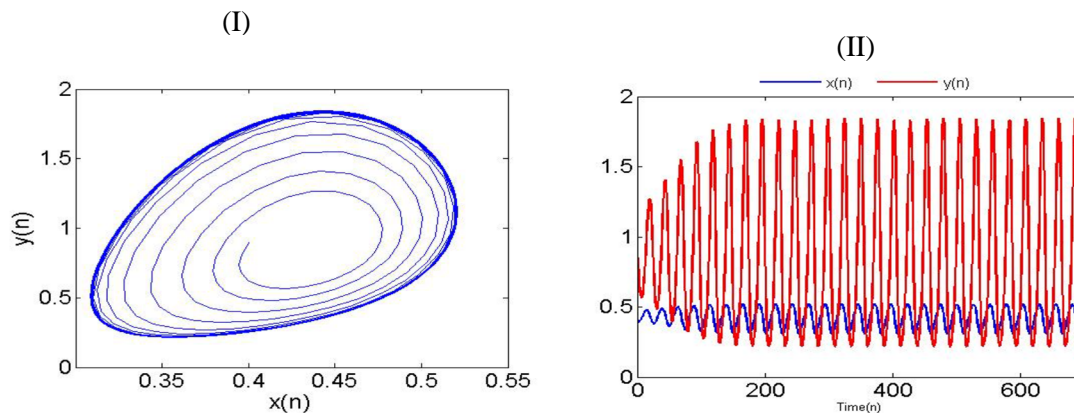
(I)

(II)

Figure 2: (I) Oscillatory behavior of system (1), (II) Phase trajectory of the system showing a periodic orbit near F_1.

4.   **Neimark Sacker Bifurcation**

Bifurcation is a sudden change in the nature of the equilibrium and periodic states of the system. The study of the trajectories and their classification is crucial in understanding the behavior of dynamical systems. The trajectories of any dynamical system may not always be simple and periodic. The analysis of the different aspects of trajectories leads to the study of qualitative behaviors. In the case of a simple dynamical system, knowing the trajectories is more often sufficient, but in most dynamical systems, realization of individual trajectories is very complicated. The parametric influence on the trajectories is what makes the study interesting and attracting. The change in parameters of the system may result in abrupt changes of the trajectories from periodical motion to rather erratic and random movements. Such different states of changes in parameters are captured using the bifurcation diagrams. The bifurcation analysis of system (1) is using the traditional bifurcation technique. When the condition (iv.2) of Proposition (2) holds, a pair of conjugate complex eigenvalues of $J(F_I)$ are obtained. The condition (iv.2) can be written as:

$$NS_{F_I} = \{(E, A, B, C, K): E = \psi_1, A \neq 1; B \neq 0, C, K > 0\}. \tag{3}$$

By varying E in the neighborhood of NS_{F_I}, Neimark-Sacker Bifurcation will appear. The value of the bifurcation parameter (E) is varied in the range $[0.2, 0.54]$ and values of other parameters $A = 1.05, B = -.05, C = 0.6, K = 2$ are fixed with the initial state at $(x, y) = (0.3, 0.3)$. The Neimark-Sacker bifurcation emerges from the fixed point $(0.343749, 0.343749)$ at $E = 0.3090909091$. It shows the correctness of Proposition (2). At $E = 0.3090909091$, the eigenvalues are $|\lambda_{1,2}| = 0.9750 + 0.2222i$, with $|\lambda_{1,2}| = 1$. Here the Neimark –Sacker bifurcation for system (1) in $(E - x)$ plane and $(E - y)$ plane are given in Figure 3(I-II). Lyapunov exponent in Figure 3(III) describes chaos in system (1). The negative value of Lyapunov denotes the stable region for the system, while the positive values represent its chaotic region. The bifurcation point is understood with the value of Lyapunov exponent being zero. Figures 4 and 5 illustrate the different phase trajectories obtained from the bifurcation diagrams presented in Figure 3, which clearly portray the transformation of the system from stability to chaos. Initially, the straight line in the diagram represent the stable nature of the system. In Figure 4, the first three portraits, (I), (II), (III), at $E = 0.2, 0.25, 0.3$, respectively, and fixed parameter values $A = 1.05, B = -0.05, C = 0.6, K = 2$, present spiral trajectories moving inwards to the fixed points. This inward spiral motion confirms the stability of the system for the values of parameters. A stable closed orbit is formed with the trajectory starting from the initial state and moving inwards toward the fixed point for $E = 0.31$, as in portrait (IV) of Figure 4. For values of $E > 0.33$, with the other values remaining fixed, the system becomes unstable. These orbits that are moving inwards become completely closed for some values of $E > 0.33$, after which the orbits start moving away from the fixed points. Unstable orbits are very clearly expressed by portraits in Figure 5.
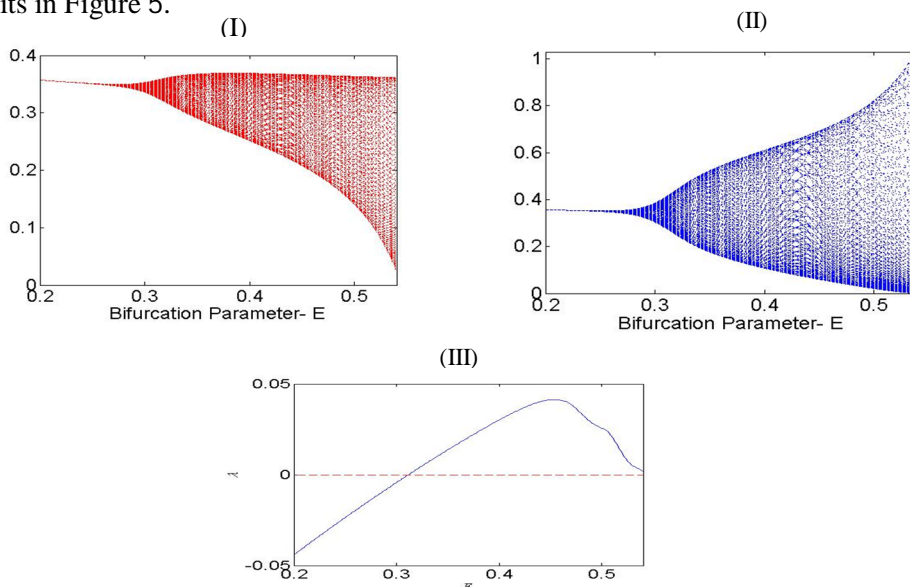

Figure 3: (I) Bifurcation diagram of the system in E- x; (I) Bifurcation diagram of the system in E -y; (I) Lyapunov Exponent of the system
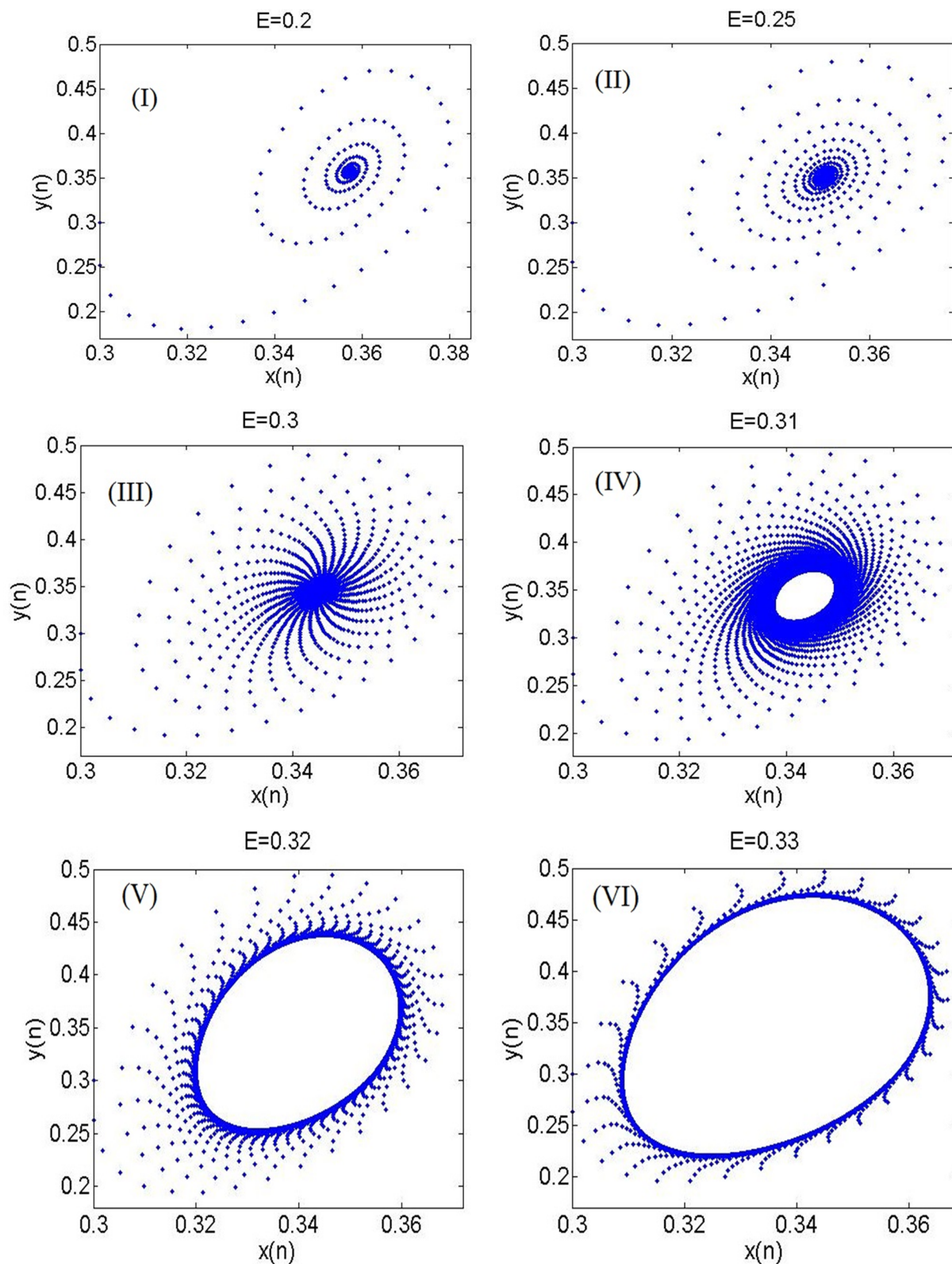
Figure 4: Phase trajectories for various values of $E \in [0.2, 0.33]$ to illustrate the complexity in Bifurcation diagrams given in Figure 2 .
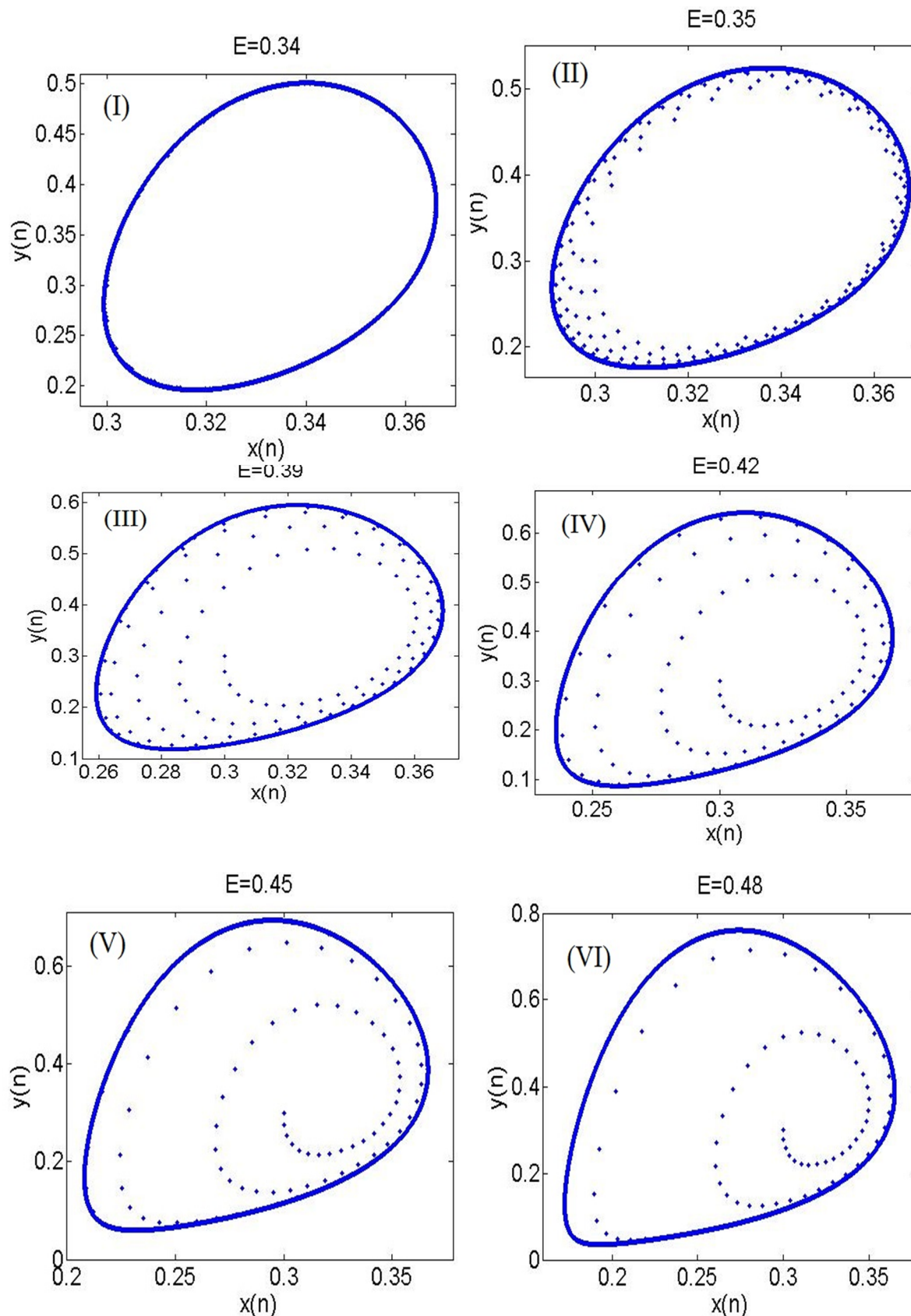
Figure 5: Phase trajectories for various values of $E \in [0.34, 0.48]$ to illustrate the complexity in Bifurcation diagrams given in Figure 2 .

5.   Enhancing Chaos of Quadratic Map (1)

The CCT [22] is employed in this section to enhance the chaotic behavior of the considered quadratic map (1). The chaos theory has proved to be a challenging and exciting field till date. Initially, chaos was considered to damage the systems and affect the efficiency of its performance, which led to the emergence of the techniques to control chaos. Such technique has an increasing interest due to its application in engineering, population dynamics, biological systems such as human heart and brain functioning, mixing problems such as medical drugs, CNN (Cellular Neural Networks), economics, industries, and military. It was later confirmed that the existence of chaos in systems is equally important as that of controlling chaos [21, 22]. The anti-control of chaos (chaotifcation) has soon gained enough attention of the researchers over the years. Like chaos control, anti-chaos control has also a wide range of applications. For example, in the mixing of fluids, strong chaotic behavior is expected for better mixing.
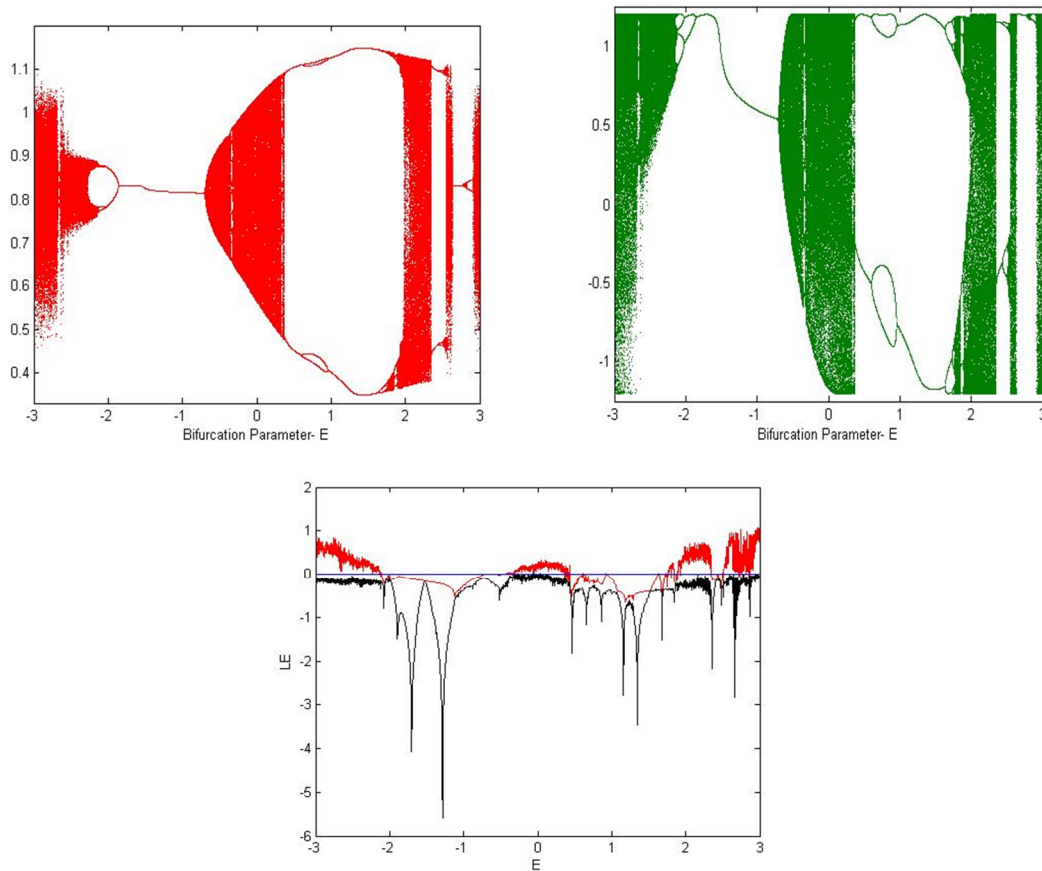
Figure 6: Chaotic behavior of system (4) with $\beta = 1.2$ and $E \in [-3, 3]$

The enhanced system of quadratic maps, obtained by applying CCT to (1), is given by

$$x_{n+1} = \beta \left(\cos(A\, x_n + B\, y_n) \right)$$
$$y_{n+1} = \beta \left(\cos(A\, x_n^2 + K\, x_n\, y_n + E\, y_n^2) \right) \qquad (4)$$

where $\beta > 0$, $A \neq 1$, $B \neq 0$, $C, K, E > 0$ are real parameters.

We shall now analyze the enhancement of the chaos of quadratic map (1) for different values of β. Let the parameters value be fixed as $A = 1.05$, $B = -0.06$, $C = 0.65$, $K = 2$ with initial conditions $(x, y) = (0.2, 0.3)$. The chaotic behavior of (4) is analyzed with varying the parameter E and $\beta = \{1, 2, 3, 4\}$.

The bifurcation diagrams of parameter (E) along the x-plane and y-plane, which are obtained using parameter values given above and with $\beta = 1.2$, $E \in [-3, 3]$, are presented in Figure (6). The bifurcation diagrams in Figure-6 explain the presence of the non chaotic regions. The aforementioned non chaotic region is further reduced for $\beta = 2$, $E \in [-1; 1]$ in Figure (7) and completely eradicated for $\beta = 2$, $E \in [-1, 1]$.
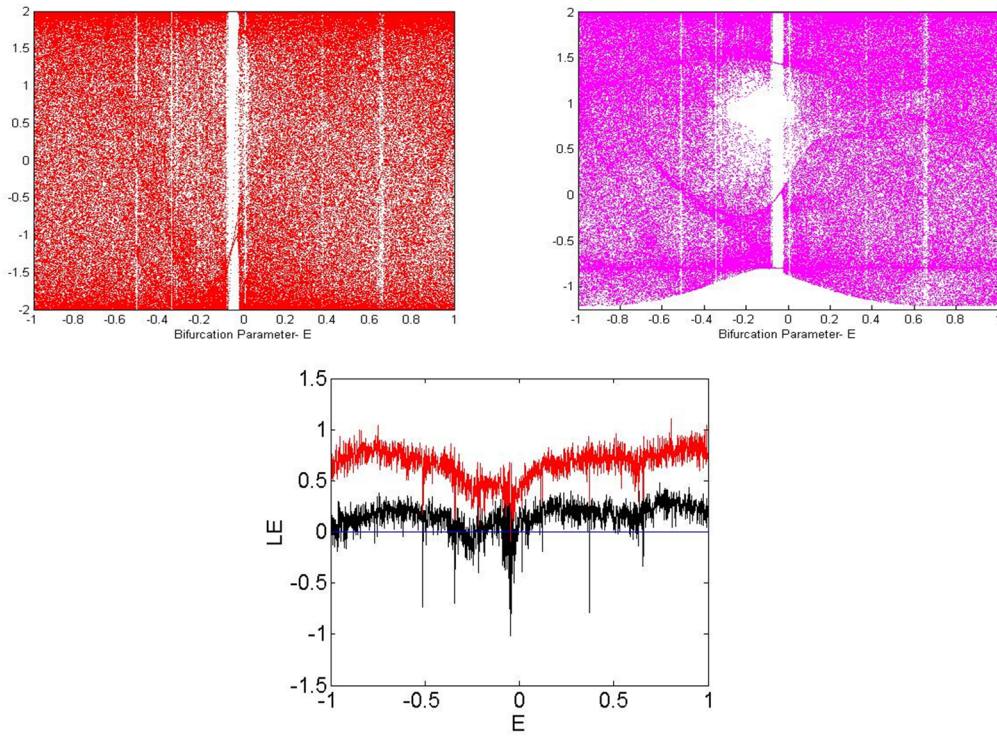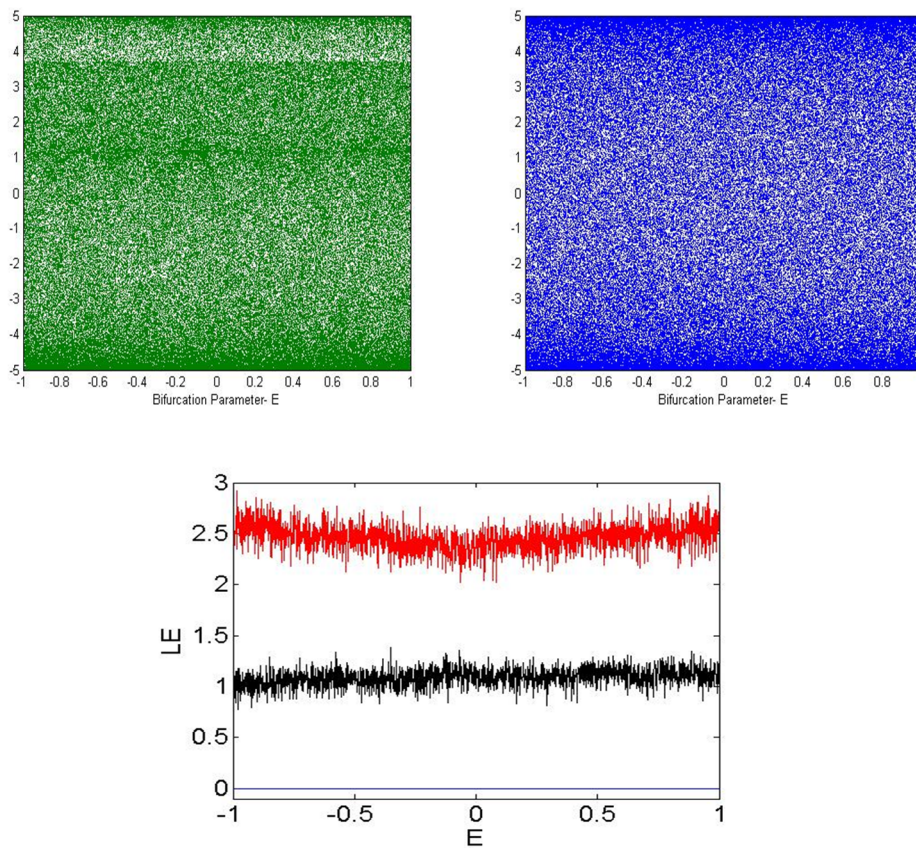
Figure 7: Chaotic behavior of system (4) with $\boldsymbol{\beta = 2}$, $\mathbf{E} \in [-1, 1]$

Figure 8 : Chaotic behavior of system (4) with $\boldsymbol{\beta = 5}$, $\mathbf{E} \in [-1, 1]$

The Lyapunov exponents are used to explain the chaotic dynamics of the systems. Here, a comparison of Lyapunov exponents in Figures (6), (7), and (8) illustrates the transition of the system from chaos to hyperchaos. In order to further confirm the transition of system (4), phase portraits are presented in Figure (9). For the considered parameter values, the quadratic map (1) is stable in the spiral inwards form, as shown in Figure (9A). Using system (4), the phase portrait occupies a smaller region which increases with the increase in the value of β. For $\beta = 5$, the enhanced map completely occupies the phase space $x, y \in [-5, 5]$. This variance in region is presented in Figure (9B), (9C), and (9D).
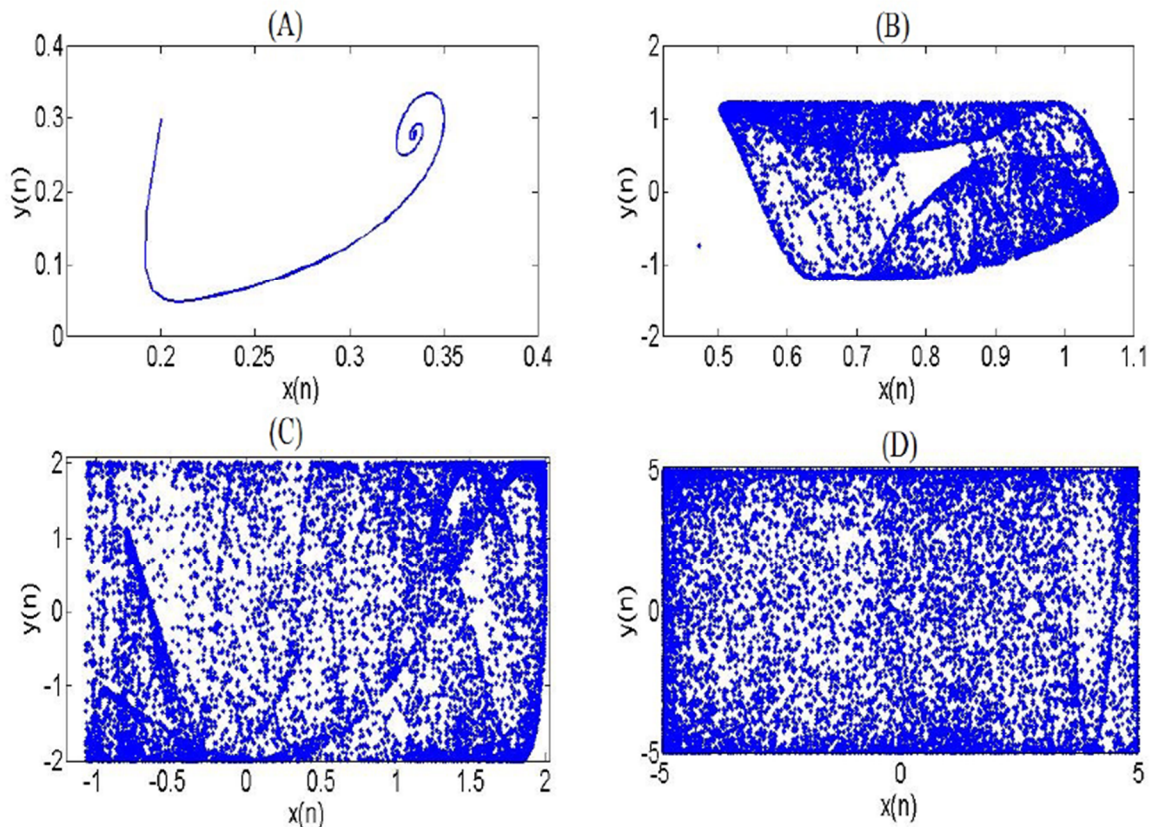

Figure 9:-Phase trajectories: (A) System (1) (Discrete map), (B) Enhanced Map (4) with $\beta = 1.2$, (C) Enhanced Map (4) with $\beta = 2$, (D) Enhanced Map (4) with $\beta = 5$.

6.   Conclusions
        The stability and bifurcation analyses of a discrete dynamical system with quadratic nonlinearities are carried out in this work. The stability conditions for the fixed interior of system (1) are obtained using the Jury's conditions. The traditional bifurcation technique is employed for bifurcation analysis. Numerical simulations are carried out for different parameter values, strengthening the theoretical results. The chaos of the quadratic maps is enhanced using CCT and the behaviors are studied using bifurcations, Lyapunov exponents, and phase portraits.

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