DOI: 10.24996/ijs.2021.62.10.22

Iraqi Journal of Science

ISSN: 0067-2904

Oscillation Criteria of Solutions of Third Order Neutral Integro-Differential Equations

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Received: 22/9/2020 Accepted: 27/2/2021

Abstract
Some necessary and sufficient conditions are obtained that guarantee the oscillation of all solutions of two types of neutral integro-differential equations of third order. The integral is used in the sense of Riemann-Stieltjes. Some examples were included to illustrate the obtained results

Keywords: Third Order Neutral Differential Equations; Integro-Differential Equations; Oscillation Criteria, Riemann-Stieltjes Integral.

معيار التذبذب لحلول المعادلات التفاضلية– التكاملية المحايدة من الرتبة الثالثة

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الخلاصة
تم الحصول على بعض الشروط الضرورية والكافية لضمان تذبذب كل الحلول لنوعين من المعادلات التفاضلية–التكاملية المحايدة من الرتبة الثالثة. التكامل المستخدم هو تكامل ريمان ستيلتجز. قدمنا بعض الامثلة لتوضيح النتائج التي تم الحصول عليها.

1. Introduction

In this paper, the third order neutral integro differential equations are studied. Consider the following integro-differential equations of the form

$$[x(t)-p(t)x(\tau(t))]''' = \delta\int_0^t x(t-s)dr(t,s), \quad t\geq 0, \delta=\pm 1 \qquad (1.1)$$

where the integral is in the sense of Riemann-Stieltjes, and assume that the following hypotheses are fulfilled:

(H1) $p,\tau \in [[0,\infty)\ ,\ R^+]$, τ is increasing and $\lim_{t\to\infty}\tau(t)=\infty$.

(H2) $r(t,s)$ is increasing with respect to s for $s\in[0,t]$.

(H3) $g(t)=r(t,t)-r(t,0), g\in C[[0,\infty),(0,\infty)]$.

A function $x(t)$ is a solution of eq.(1.1) if $x(t)-p(t)x(\tau(t))$ is three times continuously differentiable on $[t_x,\infty), t_x=\min\{t_0,\ \tau(t_0),\ \sigma(t_0)\}, t_0\geq 0$ and $x(t)$ satisfies

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eq.(1.1) on $[t_x, \infty)$. A solution $x(t)$ is said to be oscillatory if it has arbitrarily large zeros on $[t_x, \infty)$, otherwise it is said to be nonoscillatory. Eq.(1.1) is said to be oscillatory if all of its solutions are oscillatory. There has been much research concerning oscillatory and nonoscillatory behaviors of solutions to different classes of third order nonlinear neutral differential equations; we refer the reader to [1,2,4, 6]. In earlier works [1, 2], the authors obtained some necessary and sufficient conditions to ensure the oscillation of the first order neutral integrodifferential equations. In others [3-5], the authors studied delay integrodifferential equations and established some conditions for oscillation. In this paper, some necessary and sufficient conditions have been obtained to ensure the oscillation of eq.(1.1), where $\delta = 1$ or $\delta = -1$. The next lemma is useful in proving the main results of the paper.

Lemma 1. ([6], Lemma 2.2)

i- In addition to the conditions

(a) $p \in C\big([t_0, \infty), (0, \infty)\big)$;

(b) $\tau \in C([t_0, \infty), R); \tau$ is strictly increasing and $\tau(t) \le t,\ t \ge t_0$,

suppose that $0 < p(t) \le 1$ for $t \ge t_0$. Let $x(t)$ be a continuous nonoscillatory solution of the functional inequality

$$x(t)[x(t) - p(t)x(\tau(t))] < 0$$

defined in a neighborhood of infinity. Then $x(t)$ is bounded.

ii- In addition to the conditions

(a) $p \in C\big([t_0, \infty), (0, \infty)\big)$;

(b) $\tau \in C([t_0, \infty), R); \tau$ is strictly increasing and $\tau(t) \ge t,\ t \ge t_0$,

suppose that $p(t) \ge 1$ for $t \ge t_0$. Let $x(t)$ be a continuous nonoscillatory solution of the functional inequality

$$x(t)[x(t) - p(t)x(\tau(t))] > 0$$

defined in a neighborhood of infinity. Then $x(t)$ is bounded.

2. Main Results

In this section , we present four results for the oscillation of all solutions of eq.(1.1). First, we begin to study eq.(1.1) with $\delta = +1$.

Theorem 2.1. Assume that $(\mathrm{H1}) - (\mathrm{H3})$ hold, $p(t) \ge 1, \tau(t) \ge t$ and

$$\limsup_{t\to\infty} \int_0^t g(s)\, ds = \infty, \tag{2.1}$$

$$\limsup_{t\to\infty} \int_T^t \int_T^v \frac{dr(v,s)}{p\big(\tau^{-1}(t-s)\big)} dv = \infty, \qquad T \ge t_0. \tag{2.2}$$

Then, every solution of eq.(1.1) oscillates on $[t_0, \infty)$.

Proof. Suppose that $x(t)$ is an eventually positive solution of eq. (1.1) . Let

$$z(t) = x(t) - p(t)x\big(\tau(t)\big) \tag{2.3}$$

then by eq.(1.1), we get

$$z'''(t) = \int_0^t x(t-s) dr(t,s) \ge 0, \qquad t \ge t_0, \tag{2.4}$$

hence $z''(t), z'(t), z(t)$ are monotone functions. We claim that $z''(t) < 0,\ t \ge t_1 \ge t_0$. Otherwise, if $z''(t) \ge 0,\ t \ge t_1 \ge t_0$, yields $z'(t) > 0, z(t) > 0, t \ge t_2 \ge t_1$ and $\lim_{t\to\infty} z(t) = \infty$ implies that $\lim_{t\to\infty} x(t) = \infty$. On the other side, by lemma 1-ii, $x(t)$ is bounded, which is a contradiction. Hence $z''(t) < 0,\ t \ge t_1 \ge t_0$, then there are two possibilities to consider:

(1) $z'(t) < 0, t \ge t_2 \ge t_1$; (2) $z'(t) > 0,\ t \ge t_2 \ge t_1$.

If (1) holds, that is $z'(t) < 0,\ t \ge t_2$ then $z(t) < 0$ and $\lim_{t\to\infty} z(t) = -\infty$.

By (2.3), we obtain $z(t) \ge -p(t)x(\tau(t))$

$$x(t) \geq -\frac{1}{p(\tau^{-1}(t))} z(\tau^{-1}(t)). \quad (2.5)$$

By substituting (2.5) in (2.4), we obtain

$$z'''(t) \geq -\int_0^t \frac{z(\tau^{-1}(t-s))}{p(\tau^{-1}(t-s))} dr(t,s), \quad (2.6)$$

$$\geq -z(\tau^{-1}(t)) \int_{t_2}^t \frac{1}{p(\tau^{-1}(t-s))} dr(t,s).$$

By integrating the last inequality from t_2 to t, we get

$$z''(t) - z''(t_2) \geq -z(\tau^{-1}(t_2))\int_{t_2}^t \int_{t_2}^v \frac{dr(v,s)}{p(\tau^{-1}(v-s))} dv. \quad (2.7)$$

As $t \to \infty$, then (2.7) leads to $z''(t) \to \infty$, which is a contradiction since $z''(t)$ is negatively increasing.
(2) $z'(t) > 0,\ t \geq t_2$. In this case, there are two possibilities to investigate:
(a) $z(t) > 0,\ t \geq t_3 \geq t_2$; (b) $z(t) < 0,\ t \geq t_3 \geq t_2$.
If (a) holds, $z(t) > 0$, then $x(t) \geq z(t)$, by $(1.1)_+$ we obtain

$$z'''(t) \geq \int_0^t z(t-s)dr(t,s) \geq z(t)g(t). \quad (2.8)$$

By integrating (2.8) from t_3 to t, it follows that

$$z''(t) - z''(t_3) \geq z(t_3)\int_{t_3}^t g(s)ds.$$

As $t \to \infty$, the last inequality implies that $\lim_{t\to\infty} z''(t) = \infty$, which is a contradiction.
Finally, if (b) holds, then $z(t) < 0,\ z'(t) > 0,\ z''(t) < 0,\ t \geq t_3$.
In this case, again, (2.6) is fulfilled. It follows from (2.6) that

$$z'''(t) \geq -z(\tau^{-1}(0)) \int_{t_3}^t \frac{1}{p(\tau^{-1}(t-s))} dr(t,s), \qquad z(\tau^{-1}(0)) \leq z(\tau^{-1}(t_3)) < 0$$

Integrating the last inequality from t_3 to t yields

$$z''(t) - z''(t_3) \geq -z(\tau^{-1}(0))\int_{t_3}^t \int_{t_3}^v \frac{1}{p(\tau^{-1}(v-s))} dr(v,s)dv.$$

As $t \to \infty$, it follows that $\lim_{t\to\infty} z''^{(t)} = \infty$, which is a contradiction. Hence, every solution of eq.(1.1) oscillates on $[t_0, \infty)$.

Theorem 2.2. Assume that (H1) $-$ (H3) hold, $0 < p(t) \leq 1, \tau(t) \leq t$ and (2.1), (2.2) hold, then every bounded solution of eq.(1.1) oscillates on $[t_0, \infty)$.

Proof. Suppose that $x(t)$ is an eventually positive bounded solution of eq.(1.1), then $z'''(t) \geq 0, t \geq t_0$ and $z''(t), z'(t), z(t)$ are monotone functions if $z''(t) > 0,\ t \geq t_1 \geq t_0$, then $z'(t) > 0$ and $z(t) \to \infty$, which implies that $\lim_{t\to\infty} x(t) = \infty$, which is a contradiction, since $x(t)$ is bounded. If $z''(t) < 0,\ t \geq t_1 \geq t_0$, we claim that $z'(t) > 0$, for $t \geq t_2 \geq t_1$. Otherwise if $z'(t) < 0, t \geq t_2 \geq t_1$, then $z(t) < 0$ and $\lim_{t\to\infty} z(t) = -\infty$ implies to $\lim_{t\to\infty} x(t) = \infty$, which is a contradiction.
Hence, $z'(t) > 0,\ t \geq t_2 \geq t_1$ and there are two possibilities to investigate:
(a) $z(t) > 0, z'(t) > 0, z''(t) < 0,\ t \geq t_3 \geq t_2$; (b) $z(t) < 0, z'(t) > 0,\ z''(t) < 0,\ t \geq t_3 \geq t_2$. The proof of the cases (a) and (b) is similar to (2)-(a) and (2)-(b) in theorem 2.1.
In the following results, we study the eq.(1.1) when $\delta = -1$.

Theorem 2.3 Assume that (H1) $-$ (H3) hold, $0 < p(t) \leq 1, \tau(t) \leq t$ and (2.1), (2.2) hold, then every solution of eq.(1.1) oscillates on $[t_0, \infty)$.

Proof. Suppose that $x(t)$ is an eventually positive solution of eq.(1.1). Then by eq.(1.1), we get

$$z'''(t) = -\int_0^t x(t-s)d(r,s) \leq 0, \qquad t \geq t_0, \tag{2.9}$$

Hence, $z''(t), z'(t), z(t)$ are monotone functions. We claim that $z''(t) > 0,\ t \geq t_1 \geq t_0$. Otherwise, if $z''(t) \leq 0,\ t \geq t_1 \geq t_0$, this yields that $z'(t) < 0, z(t) < 0, t \geq t_2 \geq t_1$ and $\lim_{t\to\infty} z(t) = -\infty$, which implies that $\lim_{t\to\infty} x(t) = \infty$. On the other side, by lemma 1-i, $x(t)$ is bounded, which is a contradiction. Hence, $z''(t) > 0,\ t \geq t_1 \geq t_0$, then there are two possibilities to consider:

(1) $z'(t) > 0, t \geq t_2 \geq t_1$; (2) $z'(t) < 0,\ t \geq t_2 \geq t_1$.

If (1) holds , then $z(t) > 0$ and $\lim_{t\to\infty} z(t) = \infty$.

Then, $x(t) \geq z(t)$. By eq.(1.1), we obtain

$$z'''(t) \leq -\int_0^t z(t-s)dr(t,s) \leq -z(t)g(t). \tag{2.10}$$

By integrating (2.10) from t_2 to t, it follows that

$$z''(t) - z''(t_2) \leq -z(t_2)\int_{t_2}^t g(s)ds.$$

As $t \to \infty$, the last inequality implies that $\lim_{t\to\infty} z''(t) = -\infty$, which is a contradiction.

(2) $z'(t) < 0,\ t \geq t_2$. In this case, there are two possibilities to investigate:

(a) $z(t) < 0,\ t \geq t_3 \geq t_2$; (b) $z(t) > 0,\ t \geq t_3 \geq t_2$

Let (a) holds, then by (2.3) we obtain $z(t) \geq -p(t)x(\tau(t))$ and (2.5) holds.

By substituting (2.5) in eq.(1.1), we obtain

$$z'''(t) \leq \int_0^t \frac{z\left(\tau^{-1}(t-s)\right)}{p\left(\tau^{-1}(t-s)\right)}dr(t,s), \tag{2.11}$$

$$\leq z\left(\tau^{-1}(t)\right)\int_{t_2}^t \frac{1}{p\left(\tau^{-1}(t-s)\right)}dr(t,s).$$

By integrating the last inequality from t_2 to t, we get

$$z''(t) - z''(t_2) \leq z(\tau^{-1}(t_2))\int_{t_2}^t\int_{t_2}^v \frac{dr(v,s)}{p\left(\tau^{-1}(v-s)\right)}dv. \tag{2.12}$$

As $t \to \infty$, then (2.12) leads to $\lim_{t\to\infty} z''(t) = -\infty$. This is a contradiction, since $z''(t)$ is positively decreasing.

If (b) holds, that is $z(t) > 0, z'(t) < 0,\ t \geq t_3 \geq t_2$, then $x(t) \geq z(t)$. By (1.1) , we obtain

$$z'''(t) \leq -\int_0^t z(t-s)dr(t,s) \leq -z(0)g(t) \leq -z(t_3)g(t), \qquad t_3 \geq 0.$$

By integrating the last inequality from t_3 to t, it follows that

$$z''(t) - z''(t_3) \leq -z(t_3)\int_{t_3}^t g(s)ds.$$

As $t \to \infty$, the last inequality implies that $\lim_{t\to\infty} z''(t) = \infty$, which is a contradiction.

Finally if (b) holds, then we have $z(t) < 0,\ z'(t) < 0,\ z''(t) > 0,\ t \geq t_3$.

By substituting (2.5) in eq.(1.1), we obtain

$$z'''(t) \leq \int_0^t \frac{z\left(\tau^{-1}(t-s)\right)}{p\left(\tau^{-1}(t-s)\right)}dr(t,s),$$

$$\leq z\left(\tau^{-1}(t)\right)\int_{t_3}^t \frac{1}{p\left(\tau^{-1}(t-s)\right)}dr(t,s).$$

Integrating the last inequality from t_3 to t yields

$$z''(t) - z''(t_3) \leq z(\tau^{-1}(t_3)) \int_{t_3}^{t} \int_{t_3}^{v} \frac{1}{p\big(\tau^{-1}(v-s)\big)} dr(v,s) dv.$$

As $t \to \infty$, it follows that $\lim_{t\to\infty} z''(t) = \infty$, which is a contradiction.

Theorem 2.4. Assume that (H1) − (H3) hold, $p(t) \geq 1, \tau(t) \geq t$ and (2.1), (2.2) hold, then every bounded solution of eq.(1.1) oscillates on $[t_0, \infty)$.

Proof. Suppose that $x(t)$ is an eventually positive bounded solution of eq.(1.1), then $z'''(t) \leq 0, t \geq t_0$ and $z''(t), z'(t), z(t)$ are monotone functions. If $z''(t) < 0,\ t \geq t_1 \geq t_0$ then $z'(t) < 0, z(t) < 0$ and $z(t) \to -\infty$, implies that $\lim_{t\to\infty} x(t) = \infty$. This is a contradiction, since $x(t)$ is bounded. If $z''(t) > 0,\ t \geq t_1 \geq t_0$, we claim that $z'(t) < 0$, for $t \geq t_2 \geq t_1$. Otherwise, if $z'(t) > 0, t \geq t_2 \geq t_1$ then $z(t) > 0$ and $\lim_{t\to\infty} z(t) = \infty$. implies $\lim_{t\to\infty} x(t) = \infty$, which is a contradiction.

Hence, $z'(t) < 0,\ t \geq t_2 \geq t_1$, and there are two possibilities to investigate:

(a) $z(t) > 0, z'(t) > 0, z''(t) < 0,\ t \geq t_3 \geq t_2$; (b) $z(t) < 0, z'(t) > 0,\ z''(t) < 0,\ t \geq t_3 \geq t_2$. Proof of cases (a) and (b) is similar to cases (2)-(a) and (2)-(b) in theorem 3.

3. Applications

In this section, some examples are given to illustrate the obtained results.

Example 3.1. Consider the neutral integro-differential equation

$$[x(t) - 2x(t+\pi)]''' - \int_0^t x(t-s)\, dr(t,s) = 0, \qquad t_0 \geq 0, \tag{3.1}$$

where $p(t) = 2, r(t,s) = 3(t+s), \tau(t) = t + \pi, g(t) = 3(t+t) - 3t = 3t$

$$\limsup_{t\to\infty} \int_0^t g(v)\, dv = 3 \lim_{t\to\infty} \int_0^t v\, dv = \infty,$$

$$\limsup_{t\to\infty} \int_T^t \int_T^v \frac{dr(v,s)}{p\big(\tau^{-1}(v-s)\big)} dv = \frac{3}{2} \lim_{t\to\infty} \int_T^t (v-T)\, ds = \infty, \qquad T \geq t_0.$$

All conditions of theorem 1 are met. Thus, according to theorem 1, every solution of (3.1) oscillates; for instance, $x(t) = \cos t$ is such a solution.

Example 3.2. Consider the neutral integro-differential equation

$$\left[x(t) - \frac{1}{4}x(t-\pi)\right]''' - \int_0^t x(t-s) dr(t,s) = 0 \qquad (*)$$

where $p(t) = \frac{1}{4}$, $r(t,s) = (t+s)$, $\tau(t) = t - \pi$. To verify that conditions (2.1) and (2.2) hold, we have

$$\limsup_{t\to\infty} \int_0^t g(v)\, dv = \lim_{t\to\infty} \int_0^t v\, dv = \infty.$$

$$\limsup_{t\to\infty} \int_T^t \int_T^v \frac{dr(v,s)}{p\big(\tau^{-1}(v-s)\big)} dv = 4 \lim_{t\to\infty} \int_T^t (v-T)\, ds = \infty, \qquad T \geq t_0.$$

According to theorem 2, every solution of (3.2) oscillates; for instance, $x(t) = \cos t$ is such a solution.

Conclusions

In this paper, two types of neutral integrodifferential equations of third order were studied, where the integration is in the sense of Riemann-Stieltjes. Some necessary and sufficient conditions were obtained to ensure the oscillation of all solutions of these equations. The results included some illustrative examples.

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