Chaos in a Hybrid Food Chain Model

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Abstract

   In this paper, chaotic and periodic dynamics in a hybrid food chain system with Holling type IV and Lotka-Volterra responses are discussed. The system is observed to be dissipative. The global stability of the equilibrium points is analyzed using Routh-Hurwitz criterion and Lyapunov direct method. Chaos phenomena is characterized by attractors and bifurcation diagram. The effect of the controlling parameter of the model is investigated theoretically and numerically.

Keywords Bifurcation diagram, Chaos, Global stability, Holling type IV, Lotka-Volterra.

الفوضى في نموذج السلسلة الغذائية الهجينة

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الخلاصة

   في هذه البحث، تمت مناقشة السلوك الديناميكي الفوضوي والدوري في نظام السلسلة الغذائية الهجينة مع مؤثر هولينج من النوع الرابع ولوتكا–فولتيرا في النظام البيئي. لوحظ أن النظام محدود بانتظام. تم تحليل الاستقرارية لنقاط التوازن باستخدام معيار روث–هوارتز وطريقة ليابونوف المباشرة. تم تشخيص ظاهرة الفوضى عن طريق رسوم الاتراكتر والبافريكشن. تمت دراسة تأثير معامل التحكم في ديناميكية نظام السلسلة الغذائية تحليليا وعدديا.

Introduction

   Nonlinear phenomena play important rules in design systems of engineering and structures. The works of Lotka [1] and Volterra [2] are the most major works in mathematical models and dynamical systems. Researchers in ecology and engineering reported the essence of chaos [3-12]. In the analysis of the mechanical model, the geometrical non-linearity, represented by the pendulum, may lead to chaotic dynamics [4, 13]. Continuous three species food chain models from dissipative dynamics are accounted for to have complicated patterns, such chaos limit cycle, and stable cycle [14-17]. Two

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types of hybrid functional responses of a food chain model were investigated [18] and the numerical works proved the existence of chaotic dynamics. Upadhyay *et al*. [19] proposed and studied a tri-trophic food chain model with Sokol-Howell functional response, incorporating the multiple gestation time delays. They explored the existence of various dynamical structures, such as Hopf-bifurcation, periodic solutions, and chaotic dynamics. Fayeldi *et al*. [20] applied Euler method to discretize a SIR epidemic model with non-monotone incidence rate. They showed analytically and numerically that the discrete model is dynamically consistent with its continuous model. Liu and Huang [21] investigated the dynamical behavior of a predator-prey system with Holling type IV functional response, in which both species are subject to harvesting. They illustrated that using the harvesting effort as a control parameter can change the behaviors of the system, which may be useful for the biological management.

The primary motivation behind this work is to examine the dynamics of a three species model. A food chain model with Lotka-Volterra and Holling type IV functional responses is proposed and studied both theoretically and numerically. The food chain model is modified to the food chain model studied in [9], so that it involves Lotka-Volterra type of functional response in the second and third levels, instead of Holling type-II functional response that was used in the previous work. The idea of such modification comes from the fact of availability of food at the second level in the environment. The second reason for such modification is that we want to reduce the intensity of nonlinearity in the system and then study the effects of such reduction on the existence of chaos.

The Mathematical Model

Consider a three species food chain model consisting of the prey, their density at time t is denoted by $x(t)$, the intermediate predator, their density at time t is denoted by $y(t)$, and the top predator, their density at time t is denoted by $z(t)$. It is assumed that the intermediate predator preys upon its sole food at the lower level according to a modified Holling type IV functional response, and the top predator preys upon the intermediate predator at the second level according to Lotka-Volterra functional response. The dynamics of the above food chain model can be represented by the following system:

$$\frac{dx}{dt} = ax - bx^2 - \frac{w_1 xy}{h + x^2} = F_1(x, y, z)$$
$$\frac{dy}{dt} = \frac{w_2 xy}{h + x^2} - d_1 y - w_3 yz = F_2(x, y, z) \qquad (1)$$
$$\frac{dz}{dt} = w_4 yz - d_2 z = F_3(x, y, z),$$

with $x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$. Obviously, system (1) is continuous and has continuous partial derivatives on the positive octant $R_+^3 = \{(x, y, z) \in x(0) \geq 0, y(0) \geq 0, z(0) \geq 0\}$, and hence the solution of the system (1) exists and is unique.

Here, the positive constants a, b, d_j, h; $j = 1,2$ and w_k; $k = 1,2,3,4$ denote the following: a is the growth rate of the prey x, b represents the intraspecific competition of prey x, w_k represent the maximum values attainable by each per capita rate, h is the half-saturation constant, and d_j represent the death rates of the intermediate and the top predators, respectively. Moreover, it is easy to verify that system (1) is uniformly bounded.

Stability Analysis

This section deals with the stability analysis of the non-negative equilibrium points of system (1). There are at most four non-negative equilibrium points, which are denoted respectively as E_0, E_1, E_2 and the interior equilibrium point E_3.

The trivial equilibrium point $E_0 = (0,0,0)$ and the axial equilibrium point $E_1 = \left(\frac{a}{b}, 0, 0\right)$ always exists. While the planer equilibrium point is $E_2 = (\bar{x}, \bar{y}, 0)$ where

$$\bar{x} = \frac{w_2}{2d_1} \mp \frac{\sqrt{w_2^2 - 4d_1^2 h}}{2d_1}; \quad \bar{y} = \frac{(a - b\bar{x})(h + \bar{x}^2)}{w_1}. \qquad (2)$$

Clearly, there are two positive planar equilibrium points denoted by $E_{21} = (\bar{x}_1, \bar{y}_1, 0)$ and $E_{22} = (\bar{x}_2, \bar{y}_2, 0)$ if and only if the following condition holds

$$w_2^2 - 4d_1^2 h > 0, \qquad (3a)$$

while system (1) has a unique planar equilibrium point $E_2 = (\bar{x}, \bar{y}, 0)$ provided that

$$w_2{}^2 = 4d_1{}^2 h. \tag{3b}$$

The positive equilibrium point $E_3 = (x^*, y^*, z^*)$ exists in the interior of the first octant if and only if the following conditions hold

$$\frac{d_2}{w_4} \leq \frac{ah}{w_1}, \tag{4a}$$

$$d_1 < \frac{w_2 x^*}{h + x^{*2}}, \tag{4b}$$

where

$$y^* = \frac{d_2}{w_4}; z^* = \frac{1}{w_3}\left(\frac{w_2 x^*}{h + x^{*2}} - d_1\right). \tag{5}$$

While x^* is a positive root of the following equation

$$bx^3 - ax^2 + bhx + \left(w_1 \frac{d_2}{w_4} - ah\right) = 0. \tag{6}$$

According to Descartes's rule of signs, Eq. (6) has either one positive root (given by E_3) and two complex conjugate roots or three positive roots. Straightforward computation using Mathematica software shows that Eq. (6) has only one positive root with two complex conjugate roots.

Now, the local dynamical behavior of the food chain system (1) near the above equilibrium points is investigated. It is observed that the eigenvalues of the variational matrix of system (1) near the trivial equilibrium point (E_0) are determined as

$$\gamma_{01} = a > 0; \gamma_{02} = -d_1, \gamma_{03} = -d_2, \tag{7}$$

while the eigenvalues near the axial equilibrium point (E_1) are given by

$$\gamma_{11} = -a < 0; \gamma_{12} = \frac{w_2 ab}{hb^2 + a^2} - d_1, \gamma_{13} = -d_2. \tag{8}$$

Hence, the trivial equilibrium point is a saddle point, while the axial equilibrium point is locally asymptotically stable provided that the following condition holds

$$\frac{w_2 ab}{hb^2 + a^2} < d_1. \tag{9}$$

Moreover, the eigenvalues of system (1) near the planar equilibrium point E_2 can be written as:

$$\gamma_{21} = \frac{T}{2} - \frac{\sqrt{T^2 - 4D}}{2}; \gamma_{22} = \frac{T}{2} + \frac{\sqrt{T^2 - 4D}}{2}, \gamma_{23} = w_4 \bar{y} - d_2, \tag{10}$$

where $T = \bar{x}\left[-b + \frac{2w_1 \bar{x}\bar{y}}{(h + \bar{x}^2)^2}\right]$ and $D = \frac{w_1 w_2 (h - \bar{x}^2)\bar{x}\bar{y}}{(h + \bar{x}^2)^3}$. Thus, according to these eigenvalues, if the condition (3a) holds then both the planar equilibrium points are saddle points due to the negativity of D that leads to the existence of positive eigenvalues. However, if the condition (3b) holds, then the unique planar equilibrium point becomes non-hyperbolic point due to the existence of zero eigenvalue and hence the linearization does not give the real stability type.

Now, for the positive equilibrium point, the variational matrix of system (1) around $E_3 = (x^*, y^*, z^*)$ is computed as:

$$V(x^*, y^*, z^*) = (v_{ij})_{3\times3} \tag{11}$$

where

$$v_{11} = -bx^* + \frac{w_1 x^{*2} y^*}{B^{*2}}, \quad v_{12} = -\frac{w_1 x^*}{B^*}, \quad v_{13} = 0, \quad v_{21} = \frac{w_2 y^*(h - x^{*2})}{B^{*2}},$$

$$v_{22} = \frac{w_2 x^*}{B^*} - b - w_3 z^*, \quad v_{23} = -w_3 y^*, \quad v_{31} = 0, \quad v_{32} = w_4 z^*, \quad v_{33} = 0.$$

here $B^* = h + x^{*2}$. Then the characteristic equation of the variational matrix (11) can be written as

$$\lambda^3 + K_1 \lambda^2 + K_2 \lambda + K_3 = 0 \tag{12}$$

where

$$K_1 = bx^* - \frac{2w_1 x^{*2} y^*}{B^{*2}}$$

$$K_2 = \frac{w_1 w_2 x^* y^*(h - x^{*2})}{B^{*3}} + w_3 w_4 y^* z^*$$

$$K_3 = \left(bx^* - \frac{2w_1x^{*2}y^*}{B^{*2}}\right)(w_3y^*)(w_4z^*)$$

while

$$K_1K_2 - K_3 = \left(bx^* - \frac{2w_1x^{*2}y^*}{B^{*2}}\right)\left(\frac{w_1x^*}{B^*}\right)\left(\frac{w_2y^*(h - x^{*2})}{B^{*2}}\right)$$

Clearly, $K_1 > 0, K_3 > 0$ and $K_1K_2 - K_3 > 0$ if and only if the following conditions are fulfilled

$$\frac{2w_1x^*y^*}{B^{*2}} < b \tag{13a}$$

$$x^{*2} < h \tag{13b}$$

Therefore, according to the Routh-Hurwitz criterion [22], the locally asymptotically stable $E_3 = (x^*, y^*, z^*)$ takes place. However, the region of the global stability of the positive equilibrium point is investigated in the following theorem.

Theorem 1. Assume that $E_3 = (x^*, y^*, z^*)$ of system (1) is locally asymptotically stable and the following condition holds

$$\frac{w_1y^*(x+x^*)}{BB^*} < b \tag{14a}$$

$$xy^* + x^*y < xy + x^*y^* \tag{14b}$$

$$\frac{w_2}{B} < \frac{w_1}{B} + \frac{w_2(x+x^*)}{BB^*} \tag{14c}$$

where $B = (h + x^2)$. Then, the basin of attraction in the $Int.(R_+^3)$ of E_3 satisfies the above conditions.

Proof. . Consider the positive definite function

$$L(x, y, z) = \int_{x^*}^x \frac{(X-x^*)}{X} dX + \int_{y^*}^y \frac{(Y-y^*)}{Y} dY + \frac{w_3}{w_4} \int_{z^*}^z \frac{(Z-z^*)}{Z} dZ,$$

Taking the derivative of L with respect to t along the solutions of system (1), we get after some algebraic manipulation that

$$\frac{dL}{dt} = -\left[b - \frac{w_1y^*(x+x^*)}{BB^*}\right](x-x^*)^2 - \left[\frac{w_1}{B} - \frac{w_2}{B} + \frac{w_2(x+x^*)}{BB^*}\right](x-x^*)(y-y^*)$$

Clearly, under the given conditions that $\frac{dL}{dt}$ is negative semi definite and then by using Lyapunov method, the positive equilibrium point is stable. Moreover, it is easy to verify that, by using LaSalle's invariance principle , E_3 is the only invariant set for which the solution is approached asymptotically and hence it becomes an attracting point. Therefore, E_3 is an asymptotically stable point for any initial point that belongs to the region that is satisfying the above conditions.

Numerical exploration

The global dynamics of system (1) is studied numerically using six order Runge-Kutta method. Many cases of system (1) are obtained for the following set of fixed parameters values

$$a = 0.7, b = 0.91, h = 0.15, w_2 = 5.95, w_3 = 0.9, w_4 = 0.6, d_1 = 0.0005, d_2 = 0.01 \tag{15}$$

Bifurcation diagram and the typical 3D attractors of system (1) are plotted with their projections and time series in the boundary planes. The fundamental target is to investigate the possibility of chaotic behavior in system (1) by relying upon the controlling parameter and keeping different parameters of (15) fixed.
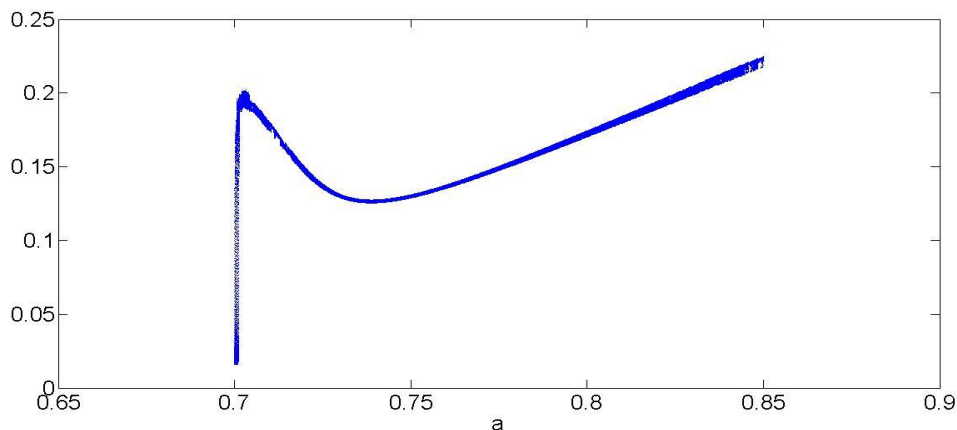

Figure 1- Bifurcation diagram of system (1) for data set (15) that showing the successive maxima of y as a function of $a \in (0.65, 0.90)$ and $w_1 = 6.1$.
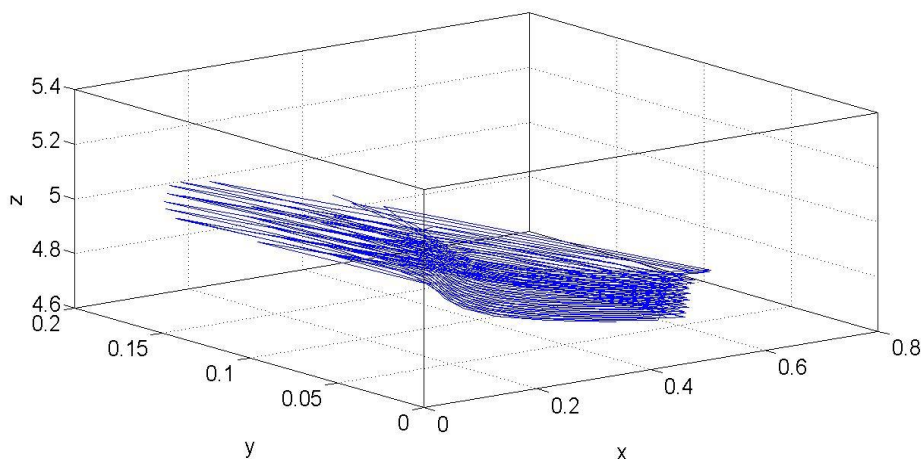
Figure 2- 3D chaotic attractor of system (1) for data set (15) with $w_1 = 6.1$

Now, in the first case when $w_1 = 6.1$ with the data set (15), it is observed that system (1) is chaotic and approaches to periodic attractor, as shown in Figure-2 and Figure-3a-b. In the second case, by decreasing the value of w_1 to 4.1, system (1) approaches to a limit cycles attractor, as shown in Figure-4a-b. The third case deals with keeping on decreasing the values of w_1 and d_2 to 0.001 for data (15), where we observe the change in the dynamics of system (1) from limit cycles attractor to stable point, as blotted in Figure-5a-b.
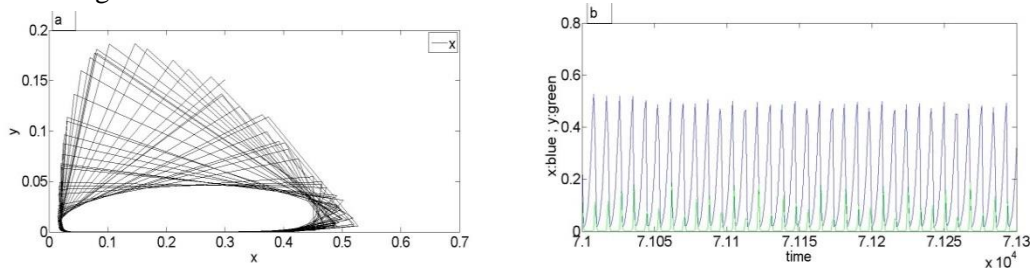

Figure 3- a) 2D projection in the $xy-$ plane of the chaotic attractor given by Figure-2, b) Time series of Figure-3a
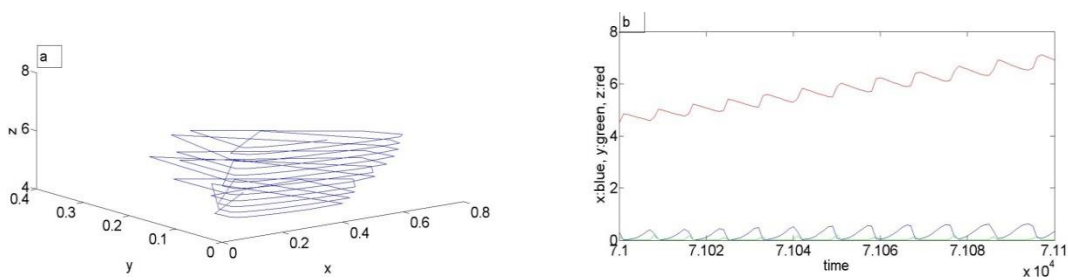

Figure 4- a) 3D of system (1) periodic attractor for data (15) with $w_1 = 4.1$, b) Time series of Figure-4a.
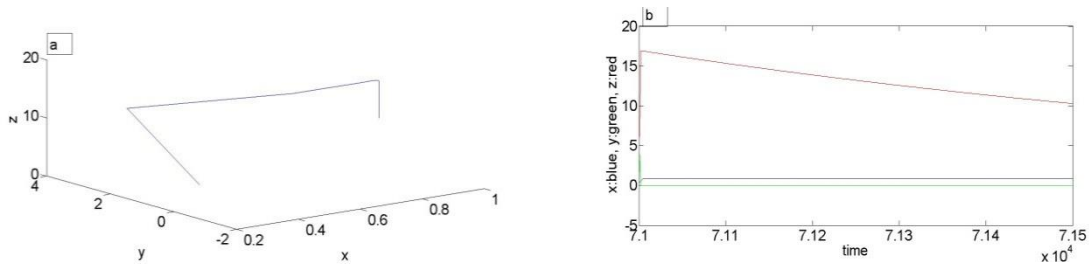
Figure 5- a) 3D of system (1) stable point for data (15) with $w_1 = d_2 = 0.001$,
b) Time series of Figure-5a

Conclusions
   A food chain system with modified Holling type IV and Lotka-Volterra functional responses is proposed and discussed. The global stability of the system is investigated using Routh-Hurwitz criterion and the second method of Lyapunov. Bifurcation diagram and attractors are blotted to study the behavior of the hybrid continuous time model. The parameter w_1 *is* controlling the dynamics of this food chain model. Moreover, replacing Holling type-II functional response in [9] by Lotka-Volterra response in system (1) above reduces the effect of nonlinearity and makes the dynamics less chaotic. In fact, the system approaches clearly a periodic behavior.

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