Aubad Iraqi Journal of Science, 2021, Vol. 62, No. 1, pp: 289-294
DOI: 10.24996/ijs.2021.62.1.27

ISSN: 0067-2904

A_4-Graph of Finite Simple Groups

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Received: 12/9/2020 Accepted: 27/10/2020

Abstract
Let G be a finite group and X be a conjugacy class of order 3 in G. In this paper, we introduce a new type of graphs, namely A_4-graph of G, as a simple graph denoted by $\mathcal{A}_4(G,X)$ which has X as a vertex set. Two vertices, x and y, are adjacent if and only if $x \neq y$ and $x y^{-1}=y x^{-1}$. General properties of the A_4-graph as well as the structure of $\mathcal{A}_4(G,X)$ when $G \cong {}^3D_4(2)$ will be studied.

Keywords: Finite simple group, exceptional groups ,diameter.

البيان –A4 للزمر البسيطة المنتهية

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الخلاصة
لتكن G زمرة منتهية و X صف من الرتبة الثالثة في G . في هذه البحث نقوم بتقديم نوع جديد من البيانات يسمى بيان A_4– للزمرة G يعرف بانه بيان بسيط يرمز له بالرمز $\mathcal{A}_4(G,X)$ بحيث انه نقطتين مختلفتين x,y في البيان ترتبط بحافة اذا وفقط اذا حققت الشرط $x y^{-1}=y x^{-1}$. الخواص العامة للبيان A_4– بالإضافة الى ذلك قد تم دراسة هيكل $\mathcal{A}_4(G,X)$ عندما $G \cong {}^3D_4(2)$.

Introduction

Analyzing the group structures using graph structures, on which the group acts upon , can be an effective method which gives rise to many interesting results. Currently, this style of studying the algebraic properties of groups is the most common. There is a remarkable number of researches in this area, see for example [1, 2, 3]. Assume that G is a finite groups and X is a conjugacy class of order 3 in G. In this work, we present the A_4-graph of G as a simple graph denoted by $\mathcal{A}_4(G,X)$. The vertices set of A_4-graph is X, and $x,y \in X$ are joined by an edge if and only if $x \neq y$ and $x y^{-1}=y x^{-1}$. Firstly, we note about the A_4-graph, if x is adjacent to y, then the subgroup is generated by x and y, $\langle x,y \rangle \cong A_4$. For this reason, we named the graph as A_4-graph. Throughout this paper, we let G be a finite groups and X is a G- conjugacy class of order 3..

The aim of this work is to present the general properties of the A_4-graph and describe certain features of $\mathcal{A}_4(G,X)$, when G is an exceptional Lie type group of characteristic two ${}^3D_4(2)$ information about this group, which can be found with details in [4].

For $x \in X$, we define the i^{th} disc of x, $\Delta_i(x)$, ($i \in N$) to be

$$\Delta_i(x)=\{y \in X \mid d(x,y)=i\}$$

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where d(,) is the usual distance metric on the graph $\mathcal{A}_4$(G,X). Certainly, G is acting by conjugation on X imbedding G in the group of graph automorphisms of $\mathcal{A}_4$(G,X). Obviously, G is a transitive on the vertices of $\mathcal{A}_4$(G,X). We now choose $t \in X$ to be a fixed representative of the class X. The aim is to describe the disc structure of vertex t in $\mathcal{A}_4$(G,X). The diameter of $\mathcal{A}_4$(G,X) will be denoted by Diam $\mathcal{A}_4$(G,X) and is defined as

$$\text{Diam } \mathcal{A}_4(G,X) = \max_{x \in X}\{i \mid \Delta_i(x) \neq \emptyset \text{ and } \Delta_{i+1}(x) = \emptyset\}$$

For deep details about concepts of graph theorem, we may refer to [5]. Finally we shall rely upon the Atlas for the names of conjugacy classes of G [6].

1- General Properties of $\mathcal{A}_4$(G,X)

Definition 1.1: Let G be a finite group. For G-conjugacy classes X of order 3, we assign a simple graph which is called A_4-graph and denoted by $\mathcal{A}_4$(G,X), with vertices set being the set X, and two vertices $x,y \in X$ are adjacent if and only if $x \neq y$ and $x y^{-1} = y x^{-1}$.

The next examples are to illustrate the structure of A_4-graph for certain finite groups.

Examples 1.2

(1) Let $G \cong S_5$ be a symmetric group of degree 5 and t=(3,4,5), then we have : $X = t^G$=[(3,4,5), (3,5,4), (2,3,4), (2,3,5), (2,4,3), (2,4,5), (2,5,3), (2,5,4), (1,2,3), (1,2,4), (1,2,5), (1,3,2), (1,3,4), (1,3,5), (1,4,2), (1,4,3), (1,4,5), (1,5,2), (1,5,3), (1,5,4)]. The graph $\mathcal{A}_4$(G,X) is connected with Diam $\mathcal{A}_4$(G,X) =3. The disc structures of the graph $\mathcal{A}_4$(G,X) are:

$\Delta_0(t)=t$, $\Delta_1(t)$={ (2,3,5), (2,4,3), (2,5,4), (1,3,5), (1,4,3), (1,5,4)}, $\Delta_2(t)$={ (2,3,4), (2,4,5), (2,5,3), (1,2,3), (1,2,4), (1,2,5), (1,3,2), (1,3,4), (1,4,2), (1,4,5), (1,5,2), (1,5,3) },$\Delta_3(t)$= {(3,5,4)}.

This can be achieved computationally by using the gap package **YAGS** [7] as we describe in the next procedure which proceeds as follows:

Procedure 1

1. Define the group **G** and ***t***.
2. Compute the **G**-Conjugacy classes **X=*t*^G**.
3. Compute A_4(**G,X**) by using the code **GraphByRelation**.
4. Draw the graph by using the code **Draw**.
5. Compute the diameter of the graph by using the code **Diameter**.
6. Set $\Delta_0(t)$=t. **For** i in {1,2,3} **Do**
7. **For** y_1 in $\Delta_{i-1}(t)$ **Do**
8. **For** y_2 in $X \setminus \Delta_0(t) \cup \Delta_1(t) \cup \ldots \cup \Delta_{i-1}(t)$ **Do**
9. **If** $y_1 * y_2^{-1} = y_2 * y_1^{-1}$ **Then**
10. **Add**($\Delta_i(t)$,y_2)

Now, to simplify the graph drawing, we replace the vertex by its position in the set X . For example, we label 1 instead of the first elements in set X, which is (3,4,5), and 2 for the second element (3,5,4), and so on.

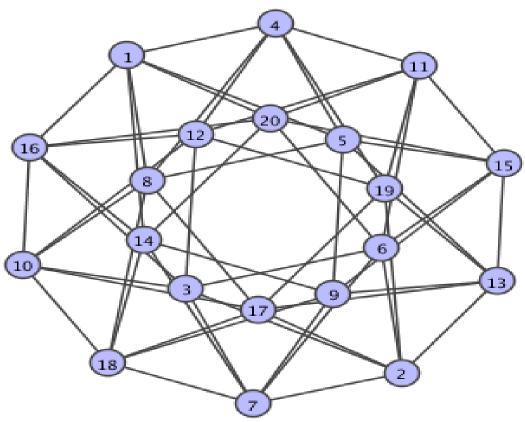

Figure 1-The structure of $\mathcal{A}_4(S_5,(1,2,3)^{S5})$.

(2) Dihedral group $D_6 =< a,b \mid a^3=b^2=1, bab=a^{-1} >$ has only one class of elements of order 3. Then, the class consists of the set $X=\{a,a^{-1}\}$. Clearly, $a^2 \neq a^{-2}$, then the $\mathcal{A}_4(D_6,X)$ is disconnected with single vertex connected components.

Now, we will establish some general properties for $\mathcal{A}_4(G,X)$.

Proposition 1.3: Let G be a finite group and X be a conjugacy class of order 3 in G. Then, graph $\mathcal{A}_4(G,X)$ has the following properties:

1- $\mathcal{A}_4(G,X)$ is a simple undirected graph.

2- $\mathcal{A}_4(G,X)$ is a regular graph.

Proof

1- From the definition of the graph $\mathcal{A}_4(G,X)$, we have two vertices $x,y \in X$ are adjacent if they are distinct and $x y^{-1}=y x^{-1}$. The first condition implies that the graph has no loops. If an edge (x,y) exists between the two vertices x and y, then $x \neq y$ and $x y^{-1}=y x^{-1}$. But this also implies $y x^{-1} = x y^{-1}$ and that the edge (y,x) also exists, with a unique edge incident (x,y) and (y,x) (we may assume the graph without multiple edges). This shows that the graph is without multiple edges and undirected.

2- To show the regularity of the graph, let $x,y \in X$ be two vertices of $\mathcal{A}_4(G,X)$, then there is one to one correspondence between $\Delta_1(x)$ and $\Delta_1(y)$ that can be seen by looking to the map $\Im:\Delta_1(x) \rightarrow \Delta_1(y)$, which is defined as $\Im(a)=a^g$, for all $a \in \Delta_1(x)$ and $g \in G$, which satisfies $x^g=y$ or $g x g^{-1}=y$ (note that g always exists because x,y are conjugates in G). The map $\Im$ is well-defined, since if $a \in \Delta_1(x)$ then $a^g y^{-1}= a^g (x^g)^{-1}= g a x^{-1} g^{-1}= g x a^{-1} g^{-1}= y (a^g)^{-1}$. This implies that $a^g \in \Delta_1(y)$. The map $\Im$ is obviously one to one and onto.

Note that the second property means that the choice of the fix $t \in X$ will be arbitrary. This is because of the regularity of the graph; each vertex has the same number of neighbors, so if we take $s \in X$ and $s \neq t$ then $|\Delta_1(t)|= |\Delta_1(s)|$, and for any path in the graph that contains t, we can conjugate by g such that $t^g=s$, then we obtain a path that contains s with the same length.

We should also note that $C_G(t)$ is acting by conjugation on X. Also, if x is adjacent to y then for any $w \in C_G(x)$ we have x adjacent to y^w. Thus, we can easily prove the following lemma:

Lemma 1.4: Δ_i(t) of the $\mathcal{A}_4(G,X)$ is a union of certain C_G(t)-orbits.

Proof

Suppose that $a \in \Delta_i$(t) and w commute with t. We aim to show that $a^w \in \Delta_i$(t). Since $a \in \Delta_i$(t), thus the path which contains a and t is of a length that is at most equal to i. Then, it is clear that if we conjugate this path by w we obtain a new path from a^w and t is of a length that is at most equal to i. Thus, $a^w \in \Delta_i$(t), as requested.

For any group G and two subgroups H and K of G, the double coset of K in G is defined as the set $HGK=\{h\, g\, k \mid h \in H, g \in G \text{ and } k \in K\}$. The number of C_G(t)-orbits is called the permutation ranks of C_G(t) on X. The next result shows the way of obtaining the size of C_G(t)-orbits.

Proposition 1.5:[8]. Suppose that G is a finite group and X is a conjugacy class of G. Then, the number of the C_G(t)-orbits is equal to the number of (C_G(t), C_G(t))-double cosets.

The above result does not only tell the permutation ranks. It also provides a representative for C_G(t)-orbits.

For a G-conjugacy class C, define the set:

$$X_C=\{x \in X \mid tx \in X\}.$$

One can see that if $X_C \neq \emptyset$ then it is equal to a union of certain $C_G(t)$-orbits of X. The way of X_C breaks into C_G(t)-orbits. It will be essential to determine which discs of t contain the vertices in X_C. Also, knowing the size of X_C can be beneficial by leading to class structure constants. Class structure constants are the sizes of the sets:

$$\{(g_1, g_2) \in C_1 \times C_2 \mid g_1 g_2 = g\}$$

where C_1, C_2, C_3 are G-conjugacy classes and g is a fixed element of C_3. Now, these constants can be calculated directly from the complex character table of G, which are recorded in the Atlas and are available electronically in the standard libraries of the computer algebra package Gap [9]. If we take $C_1 = C$, $C_2 = X = C_3$ and $g = t$, then in this case

$$|X_C| = \frac{|G|}{|C_G(t)||C_G(h)|}\sum_{i=1}^{s}\frac{\chi_i(h)\chi_i(t)\overline{\chi_i(t)}}{\chi_i(1)}$$

where h is a representative from C and $\chi_1, \chi_2, \ldots, \chi_s$ are the complex irreducible characters of G. The next proposition gives the criteria to decide the order of tx if $x \in \Delta_1(t)$, which can be found by the below result:

Proposition 1.6

1- Let x , y be any two distinct vertices in $\mathcal{A}_4$(G,X). Then if x is adjacent to y then the subgroup generated by x and y, $\langle x,y\rangle$, isomorphic to A_4, the alternating group of degree 4.

2- For x in $\Delta_1(t)$, we have that tx has the order 3.

Proof

1- It is well known that the standard presentation of $A_4 = \langle z,w | z^3=w^3, zwz=w^{-1}\rangle$. Now, x and y have the order 3 and $x\,y^{-1}=y\,x^{-1}$. Then, we have $x\,y^{-1}x=y$. If we set x=z and $y^{-1}=w$, we obtain that $\langle x,y\rangle \cong A_4$.

2- Since $x \in \Delta_1(t)$ then $tx^{-1}=xt^{-1}$, which leads to $x=tx^{-1}t$. As x,t have order 3, then we have $(tx)^3= tx\ tx\ tx= t\ tx^{-1}t\ \ t\ \ t\ x^{-1}t\ \ t\ \ tx^{-1}t = 1$. This illustrates that tx has order 3.

2. Disc structures of $\mathcal{A}_4$(G,X), G$\cong$ $^3D_4(2)$

The exceptional group $^3D_4(2)$ has the factor order $2^{12}.3^4.7^2.13$ and two classes of order 3, namely 3A and 3B. The class 3A has a centralizer structure that is isomorphic to $(((C_3 \times C_3) : C_3) : Q_8) : C_3$, while the class 3B has a centralizer structure that is isomorphic to $C_3 \times PSL(2,8)$.

These results of the next theorem were obtained computationally with the aid of Gap and the OnLine Atlas. In the context of these computations, we allocate the C_G(t)-orbits on X. Representatives, in Gap format, for each of these orbits are to be obtained as downloadable files in [11], as they may be of value in other investigations of such group. In Section 3, we also give information on the action of $C_G(t)$ on X. Specially , we provide the $C_G(t)$-orbit sizes for each $X_C \neq \emptyset$.

The main result of the paper is as follows.

Theorem 2.2: Let G be isomorphic to $^3D_4(2)$. Then

1- The sizes of the discs $\Delta_i(t)$ are listed in **Table 1** and the G-conjugacy classes of tx for $x\in \Delta_i(t)$; $i \in \mathbb{N}$ are given in **Table 2**.

2- If (G,X)= ($^3D_4(2)$,3A)= ($^3D_4(2)$,3B), then Dim $\mathcal{A}_4$(G,X).= 5.

Proof

First, we fix an arbitrary element t in the class 3A or 3B and we set Δ_0(t)= t. Then, we calculate the C_G(t)-orbits by using the Double Cosets of C_G(t) in G, as we describe in **section 1** and by Gap. Now, we break C_G(t)-orbits into X_C sets using the class representative from the OnLine Atlas. This can be seen in the below table.

Table 1-The Discs for $\mathcal{A}_4$(G,X)., G$\cong$$^3D_4(2)$.

X=t^G	|X|	$\Delta_1(t)$	$\Delta_2(t)$	$\Delta_3(t)$	$\Delta_4(t)$	$\Delta_5(t)$
3A	139776	27	648	13491	105463	20146
3B	326144	243	39852	285255	792	1

The above table proves that Dim $\mathcal{A}_4$(G,X)=5 for (G,X)=($^3D_4(2)$,3A)= ($^3D_4(2)$,3B).

For $i\neq 0$, $\Delta_i(t)$ is equal to the set of each element in X_C which adjacencies with some elements in $\Delta_{i-1}(t)$. We employ this property to obtain the following table below.

Table 2-The conjugacy class of products tx for $x \in \Delta i(t)$.

X=t^G	$\Delta_1(t)$	$\Delta_2(t)$	$\Delta_3(t)$	$\Delta_4(t)$	$\Delta_5(t)$
3A	3A(27)	4A(216), 7D(216^2)	3A(216^2,504),4A(216), 6A(1512^2),7D(216, 1512^4),12A(1512^2)	1A,3A(378),4A(378),4B,6A (1512^2),7AC,7D (1512^4),8B,9AC (504,1512^3), 12A(1512^2), 13AC,21AC (1512^2),28AC (1512^2)	3A(56,378),3B,4C,6B, 9AC(504), 21AC(504, 1512),28AC(1512^2)

3B	3B(81^3)	6A(81^4),8A,9AC(216), 14AC(648), 18AC(648^4), 21AC(648^6),28AC(324^2, 648^4)	3A,3B(216^2),4B,4C, 6A(648^6),6B,7AC,7D,8B, 9AC(324^9,648^9), 12A,13AC, 14AC(324^9,648^7), 18AC(216^6,324^6,1648^{14}),21AC(648^{19}), 28AC(324^2,648^{12})	2B,3B(72^2), 4A,9AC(108)	1A

As mentioned, we are using the class names in Atlas, although we have made some adjustments. First, we suppress the "slave" notation to write the class name of $^3D_4(2)$. Second, for the purpose of simplification, we compress the letter part of the class name, since we aim to union these classes and their characters in alphabetical sequence. As in the example shown in **Table 2**, for G≅$^3D_4(2)$ and X = 3A, 21AC is short-hand for 21A∪ 21B∪21C .

3. C_G(t)-Orbits On X

As mentioned above, we provide tables that include the sizes of the CG(t)-orbits, where C_G(t) acts upon a non-empty set X_C, with C is a G-conjugacy class. In the next tables, we employ an exponential notation to state the multiplicity of a certain size. For example, in the table for $\mathcal{A}_4$($^3D_4$(2),3A), the entry 216^2,378 next to 4A is implying that X_{4A} is the union of three CG(t)-orbits, two of which have the size of 216 and one has the size of 378. While, in the table for $\mathcal{A}_4$($^3D_4$(2),3B), the entry 324^4,648^{14} next to 28AC indicates that each of X_{28A} , X_{28B} and X_{28C} is the union of eighteen C_G(t)-orbits, four of which have the size of 324 and fourteen have the size of 648. We give details of the permutation ranks in our next table.

Table 3-Class sizes and Permutation Rank for $\mathcal{A}_4$($^3D_4$(2),X).

A4-Graph	**|X=t^G|**	**Permutation Rank**
$\mathcal{A}_4$($^3D_4$(2),3A)	118	139776
$\mathcal{A}_4$($^3D_4$(2),3B)	600	326144

In order to calculate the CG(t)-Orbits of $\mathcal{A}_4$($^3D_4$(2),3A) and $\mathcal{A}_4$($^3D_4$(2),3B), we present the following Procedure:

Procedure 2

1. Choose $t\in$3A or 3B.
2. Compute Centralizer in G of *t*, $C_G(t)$.
3. Compute Double Cosets of $C_G(t)$ in G (C_G(t)-orbits, which can be obtained from **Proposition 1.5.**
4. Break C_G(t)-orbits into X_C sets using the class representative from the OnLine Atlas.
5. Use the class structure constants to compute the size of X_C.

3.1 **C_G(t)-Orbits of $\mathcal{A}_4$($^3D_4$(2),3A)**

Table 4-C_G(t)-Orbits of $\mathcal{A}_4$($^3D_4$(2),3A)

1A	1	3A	27,56,216^2,378^2,504	3B	56
4A	216^2,378	4B	378	4C	756^2
6A	1512^4	6B	756^2	7AC	504
7D	216^3,1512^8	8B	1512^8	9AC	504^2,1512^3
12A	1512^4	13AC	1512^9	21AC	504,1512^4
28AC	1512^4				

3.2 C_G(t)-Orbits of $\mathcal{A}_4$($^3D_4$(2),3B)

Table 5-C_G(t)-Orbits of $\mathcal{A}_4$($^3D_4$(2),3B)

1A	1	2B	81^3	3A	216^5
3B	72^2,81^3,216^2	4A	81	4B	81^3,216^6

4C	648^{4}	6A	$81^{4},648^{6}$	6B	648^{10}
7AC	$108,324^{2}$	7D	648^{9}	8AB	648^{12}
9AC	$108,216,324^{9},648^{9}$	12A	$216^{12},648^{12}$	13AC	648^{49}
14AC	$324^{9},648^{8}$	18AC	$216^{6},324^{6},648^{18}$	21AC	648^{25}
28AC	$324^{4},648^{16}$				

It is worth noting in the case of $\mathcal{A}_4(^3D_4(2),3B)$ that the distance between *t* and *x* is almost decided by the G-class to which contains *tx*.

Conclusions

This paper shows the relation between two important branches of mathematics, which are the graph theory and the group theory. During this work, a new graph was introduced, namely the A4-graph. This graph was employed to study the structure of certain finite simple groups. Valuable results were obtained; for example, the general properties of A_4-graphs were given along with the analysis of $\mathcal{A}_4$ **(**G,X**)**, G≅ $^3D_4(2)$.

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