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Efficient Modification of the Decomposition Method for Solving a System of PDEs

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Abstract

This paper presents an analysis solution for systems of partial differential equations using a new modification of the decomposition method to overcome the computational difficulties. Convergence of series solution was discussed with two illustrated examples, and the method showed a high-precision, being a fast approach to solve the non-linear system of PDEs with initial conditions. There is no need to convert the nonlinear terms into the linear ones due to the Adomian polynomials. The method does not require any discretization or assumption for a small parameter to be present in the problem. The steps of the suggested method are easily implemented, with high accuracy and rapid convergence to the exact solution, compared with other methods that can be used to solve systems of PDEs.

Keywords: System of PDEs, Decomposition Technique, Convergence Analysis.

تعديل كفوء لطريقة التفكيك لحل منظومة معادلات تفاضلية جزئية

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الخلاصة

يعرض البحث الحل التحليلي لمنظومة المعادلات التفاضلية الجزئية باستخدام تعديل جديد لطريقة التفكيك للتغلب على صعوبة الحسابات. تم مناقشة التقارب لسلسلة الحل مع مثالين توضيحيين و الطريقة اثبتت دقتها العالية و سرعة تقارب الحل لمنظومة المعادلات التفاضلية الجزئية الخطية و غير الخطية ذات الشروط الابتدائية دون الحاجة لتحويل الحدود الغير خطية الى خطية لوجود متعددة حدود ادومين ولا يقتضي اي تقطيع او فرضية للمعلمة الصغيرة لتوضيح المسالة. خطوات الطريقة المقترحة سهلة التنفيذ عالية الدقة و سريعة التقارب للحل المضبوط مقارنة مع الطرق الاخرى المستخدمة في حل منظومات المعادلات التفاضلية الجزئية.

1. Introduction

The systems of partial differential equations (PDEs) have been used to describe many important models in real life, such as contamination, distribution of shallow water, heat, wave's contamination, and the chemical reaction – distribution model [1-4]. The general ideas and key characteristics of these systems are generally applicable [5]. In recent years, many authors have focused on solving the non-linear systems of PDEs using various methods, such as Homotopy analysis method (HAM) [6], variational iteration method (VIM) [7], differential transform method (DTM) [8], Homotopy

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perturbation method (HPM) [9-10], Adomain decomposition method (ADM) [11-13], coupled Laplace decomposition method [14], and semi analytic technique [15]. Recently, the decomposition method and its modifications have been used in a wider scope to solve different types of PDEs. In 2001, Wazwaz and Al-sayed [16] presented a modification of the ADM for non-linear operator, which replaced the process of dividing f (non operator function) into two parts by an infinite series of components. Another modification is the restarted ADM [17]. In 2005, Wazwaz [18] found another modification to the ADM to overcome the difficulties that arise when the equation consists of singular points. This modification is useful for similar models with singularities. Luo [19] proposed another modification based on separating the ADM into two steps and, hence, it is termed the two steps ADM (TSAMD). The purpose behind the proposed scheme is to identify the exact solution more readily and eliminate some calculations. Herein we suggest a new modification for solving the non-linear systems of PDEs with initial conditions to overcome the computational difficulties.

2. Description of the Suggested Modification

The procedure of the suggested modification (MDM) to solve the non-linear system of PDEs is presented here. Firstly, we write the nonlinear system of PDEs as follows:

$$L_t u + L_X u + N_1(u,v) = h_1(X,t) \tag{1}$$

$L_t v + L_X v + N_2(u,v) = h_2(X,t)$

Subject to ICs:

$$u\,(X,0) = f(X)\,; \quad v(X,0) = g\,(X) \tag{2}$$

where X is a space variable that belongs in the $R, L:$ linear differential operator $\left(L_t = \frac{\partial}{\partial t}\right)$, $\left(L_X = \frac{\partial}{\partial x}\right)$, N_1 and N_2 are nonlinear operators, and $h_1(X,t), h_2(X,t)$ are the nonhomogeneous part.

By taking $L_t^{-1} = \int_0^t (.)dt$, to the system, we get

$u(X,t) = f(X) + L_t^{-1}(h_1) - L_t^{-1}L_X u - L_t^{-1}[N_1(u,v)]$

$$v(x,t) = g(X) + L_t^{-1}(h_2) - L_t^{-1}L_X v - L_t^{-1}[N_2(u,v)] \tag{3}$$

The linear unknown functions $u(X,t)$ and $v(X,t)$ can be decomposed by infinite series of components, illustrated in equation (4), and $h_1,\ h_2$ should be decomposed infinite series illustrated in equation (5).

$$u(X,t) = \sum_{m=0}^{\infty} a_m(X)t^m \ , v(X,t) = \sum_{m=0}^{\infty} b_m(X)t^m \tag{4}$$

$$h_1 = \sum_{m=0}^{\infty} r_m(X)t^m \ \ and\ h_2 = \sum_{m=0}^{\infty} s_m(X)t^m \tag{5}$$

$N_1(u,v), N_2(u,v)$ are nonlinear terms that can be represented by an infinite series of polynomials, as follows

$$N_1(u,v) = \sum_{m=0}^{\infty} A_m(X)t^m = A_0 + A_1 t + A_2 t^2 + \cdots$$

$$N_2(u,v) = \sum_{m=0}^{\infty} B_m(X)t^m = B_0 + B_1 t + B_2 t^2 + \cdots \tag{6}$$

where $A_m$and B_m are Adomian polynomials

$$A_m = \frac{1}{m!}\frac{d^m}{d\lambda^m}\left[N\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0}, \quad m = 0,1,2,\ldots\ .$$

By substituting (4) and (5) in the system (3), we get

$$\sum_{m=0}^{\infty} a_m(X)t^m = f(x) + L_t^{-1}\left(\sum_{m=0}^{\infty} r_m(X)t^m\right) - L_t^{-1}L_X\left(\sum_{m=0}^{\infty} a_m(X)t^m\right) - L_t^{-1}\left(\sum_{m=0}^{\infty} A_m(X)t^m\right)$$

$$\sum_{m=0}^{\infty} b_m(X)t^m = g(x) + L_t^{-1}\left(\sum_{m=0}^{\infty} s_m(X)t^m\right) - L_t^{-1}L_x\left(\sum_{m=0}^{\infty} b_m(X)t^m\right) - L_t^{-1}\left(\sum_{m=0}^{\infty} B_m(X)t^m\right)$$

Now, we integrate the right side to get:

$$\sum_{m=0}^{\infty} a_m(X)t^m = f(x) + \left(\sum_{m=0}^{\infty} r_m(X)\frac{t^{m+1}}{m+1}\right) - L_x\left(\sum_{m=0}^{\infty} a_m(X)\frac{t^{m+1}}{m+1}\right) - \left(\sum_{m=0}^{\infty} A_m(X)t^m\frac{t^{m+1}}{m+1}\right)$$

$$\sum_{m=0}^{\infty} b_m(X)t^m = g(X) + \left(\sum_{m=0}^{\infty} s_m(X)\frac{t^{m+1}}{m+1}\right) - L_X\left(\sum_{m=0}^{\infty} b_m(X)\frac{t^{m+1}}{m+1}\right) - \left(\sum_{m=0}^{\infty} B_m(X)\frac{t^{m+1}}{m+1}\right)$$

Let $m = m-1$ in the right side of the above system, we have

$$\sum_{m=0}^{\infty} a_m(X)t^m = f(X) + \left(\sum_{m=1}^{\infty} r_{m-1}(X)\frac{t^m}{m}\right) - L_x\left(\sum_{m=1}^{\infty} a_{m-1}(X)\frac{t^m}{m}\right) - \left(\sum_{m=1}^{\infty} A_{m-1}(X)\frac{t^m}{m}\right)$$

$$\sum_{m=0}^{\infty} b_m(X)t^m = g(x) + \left(\sum_{m=1}^{\infty} s_{m-1}(X)\frac{t^m}{m}\right) - L_x\left(\sum_{m=1}^{\infty} b_{m-1}(X)\frac{t^m}{m}\right) - \left(\sum_{m=1}^{\infty} B_{m-1}(X)\frac{t^m}{m}\right)$$

We have the following recursive relation:

$a_0(X) = f(X)$

$$\text{Let } u_0 = a_0, u_1 = a_1 t, u_2 = a_2 t^2, \ldots \tag{7}$$

Let $w_0 = b_0, w_1 = b_1 t, w_2 = b_2 t^2, \ldots$

After substituting in Adomian polynomials, we apply the relation (6):

$$a_m(X) = \frac{1}{m}\left[r_{m-1}(X) - \frac{\partial}{\partial x}\left(a_{m-1}(X)\right) - A_{m-1}(X)\right] \tag{8}$$

In the same manner,

$b_0(X) = g(X)$

$$b_m(x) = \frac{1}{m}\left[s_{m-1}(X) - \frac{\partial}{\partial x}\left(b_{m-1}(X)\right) - B_{m-1}(X)\right] \tag{9}$$

The coefficient $a_m(X)$, $b_m(X)$ in eqs. (8) and (9) is substituted in eq. (4) to get the solution $u(X,t)$ and $v(X,t)$.

3. Illustrative Problems

In this section, the suggested modification (MDM) is used to solve the nonlinear system of PDEs.

Problem 1

Consider the following 2D nonlinear system of Burgers equation [20-21]:

$u_t + uu_x + wu_y = u_{xx} + u_{yy}$

$w_t + uw_x + ww_y = w_{xx} + w_{yy}$

subject to IC: $u(x,y,0) = x + y$, $w(x,y,0) = \mathrm{x} - \mathrm{y}$, $(\mathrm{x},\mathrm{y},\mathrm{t}) \in R^2 \times [0, \frac{1}{\sqrt{2}})$.

Solution

By taking $L_t^{-1} = \int_0^t (.)dt$ to the system, we obtain

$$u(x,y,t) = u(x,y,0) + \frac{\partial^2}{\partial x^2}L_t^{-1}[u] + \frac{\partial^2}{\partial y^2}L_t^{-1}[u] - L_t^{-1}[uu_x] - L_t^{-1}[wu_x]$$

$$w(x,y,t) = w(x,y,0) + \frac{\partial^2}{\partial x^2}L_t^{-1}[w] + \frac{\partial^2}{\partial y^2}L_t^{-1}[w] - L_t^{-1}[uw_x] - L_t^{-1}[wu_x]$$

$$u(x,y,t) = x + y + \frac{\partial^2}{\partial x^2}L_t^{-1}[u] + \frac{\partial^2}{\partial y^2}L_t^{-1}[u] - L_t^{-1}[uu_x] - L_t^{-1}[wu_x]$$

$$w(x,y,t) = x - y + \frac{\partial^2}{\partial x^2}L_t^{-1}[w] + \frac{\partial^2}{\partial y^2}L_t^{-1}[w] - L_t^{-1}[uw_x] - L_t^{-1}[wu_x]$$

$$let\ u(x,t) = \sum_{m=0}^{\infty} a_m(x)t^m\ , w(x,t) = \sum_{m=0}^{\infty} b_m(x)t^m$$

$$\sum_{m=0}^{\infty} a_m(x)t^m = x + y + \frac{\partial^2}{\partial x^2}\left(\sum_{m=0}^{\infty} a_m(x,y)\frac{t^{m+1}}{m+1}\right) + \frac{\partial^2}{\partial y^2}\left(\sum_{m=0}^{\infty} a_m(x,y)\frac{t^{m+1}}{m+1}\right) - \sum_{m=0}^{\infty} A_m(x,y)\frac{t^{m+1}}{m+1} - \sum_{m=0}^{\infty} B_m(x,y)\frac{t^{m+1}}{m+1}$$

$$\sum_{m=0}^{\infty} b_m(x)t^m = x - y + \frac{\partial^2}{\partial x^2}\left(\sum_{m=0}^{\infty} b_m(x,y)\frac{t^{m+1}}{m+1}\right) + \frac{\partial^2}{\partial y^2}\left(\sum_{m=0}^{\infty} b_m(x,y)\frac{t^{m+1}}{m+1}\right) - \sum_{m=0}^{\infty} C_m(x)\frac{t^{m+1}}{m+1} - \sum_{m=0}^{\infty} D_m(x,y)\frac{t^{m+1}}{m+1}$$

where

$$u(x,y,t) = \sum_{m=0}^{\infty} a_m(x,y)t^m, w(x,y,t) = \sum_{m=0}^{\infty} b_m(x,y)t^m$$

$$u\frac{\partial u}{\partial x} = \sum_{m=0}^{\infty} A_m(x,y)t^m = A_0 + A_1 t + A_2 t^2 + \cdots$$

$$w\frac{\partial u}{\partial y} = \sum_{m=0}^{\infty} B_m(x,y)t^m = B_0 + B_1 t + B_2 t^2 + \cdots$$

$$u\frac{\partial w}{\partial x} = \sum_{m=0}^{\infty} C_m(x,y)t^m = C_0 + C_1 t + C_2 t^2 + \cdots$$

$$w\frac{\partial w}{\partial y} = \sum_{m=0}^{\infty} D_m(x,y)t^m = D_0 + D_1 t + D_2 t^2 + \cdots$$

Let $m = m - 1$, then, on the right side of the above system, we have

$$\sum_{m=0}^{\infty} a_m(x)t^m = x + y + \frac{\partial^2}{\partial x^2}\left(\sum_{m=1}^{\infty} a_{m-1}(x,y)\frac{t^m}{m}\right) + \frac{\partial^2}{\partial y^2}\left(\sum_{m=1}^{\infty} a_{m-1}(x,y)\frac{t^m}{m}\right) - \sum_{m=1}^{\infty} A_{m-1}(x,y)\frac{t^m}{m} - \sum_{m=1}^{\infty} B_{m-1}(x,y)\frac{t^m}{m}$$

$$\sum_{m=0}^{\infty} b_m(x)t^m = x - y + \frac{\partial^2}{\partial x^2}\left(\sum_{m=1}^{\infty} b_{m-1}(x,y)\frac{t^m}{m}\right) + \frac{\partial^2}{\partial y^2}\left(\sum_{m=1}^{\infty} b_{m-1}(x,y)\frac{t^m}{m}\right) - \sum_{m=1}^{\infty} C_{m-1}(x)\frac{t^m}{m} - \sum_{m=1}^{\infty} D_{m-1}(x,y)\frac{t^m}{m}$$

where $a_0 = x + y$ and $b_0 = x - y$

$$a_m(x,y) = \frac{1}{m}\left[\frac{\partial^2}{\partial x^2}\big(a_{m-1}(x,y)\big) + \frac{\partial^2}{\partial y^2}\big(a_{m-1}(x,y)\big) - A_{m-1}(x,y) - B_{m-1}(x,y)\right]$$

Also,

$$b_m(x,y) = \frac{1}{m}\left[\frac{\partial^2}{\partial x^2}\big(b_{m-1}(x,y)\big) + \frac{\partial^2}{\partial y^2}\big(b_{m-1}(x,y)\big) - C_{m-1}(x,y) - D_{m-1}(x,y)\right]$$

$$a_1(x,y) = \frac{1}{1}\left[\frac{\partial^2}{\partial x^2}\big(a_0(x,y)\big) + \frac{\partial^2}{\partial y^2}\big(a_0(x,y)\big) - A_0(x,y) - B_0(x,y)\right]$$

Let $u_0 = a_0, u_1 = a_1 t, u_2 = a_2 t^2, \ldots$

$$a_0 a_{0x} + (a_1 a_{0x} + a_0 a_{1x})t + (a_2 a_{0x} + a_1 a_{1x} + a_0 a_{2x})t^2 + \cdots = A_0 + A_1 t + A_2 t^2 + \cdots$$

Let $w_0 = b_0, w_1 = b_1 t, w_2 = b_2 t^2, \ldots$

$$b_0 a_{0y} + \big(b_1 a_{0y} + b_0 a_{1y}\big)t + \big(b_2 a_{0y} + b_1 a_{1y} + b_0 a_{2y}\big)t^2 = B_0 + B_1 t + B_2 t^2 + \cdots$$

$$a_0 b_{0x} + (a_1 b_{0x} + a_0 b_{1x})t + (a_2 b_{0x} + a_1 b_{1x} + a_0 b_{2x})t^2 + \cdots = C_0 + C_1 t + C_2 t^2 + \cdots$$

$$b_0 b_{0y} + \big(b_1 b_{0y} + b_0 b_{1y}\big)t + \big(b_2 b_{0y} + b_1 b_{1y} + b_0 b_{2y}\big)t^2 + \cdots = D_0 + D_1 t + D_2 t^2 + \cdots$$

$A_0(x,y) = x + y$, $B_0(x,y) = x - y$

$a_1(x,y) = [0 + 0 - (x + y) - (x - y)] = [-x - y - x + y] = -2x$

and so,

$$b_1(x,y) = \frac{1}{1}\left[\frac{\partial^2}{\partial x^2}\big(b_0(x,y)\big) + \frac{\partial^2}{\partial y^2}\big(b_0(x,y)\big) - C_0(x,y) - D_0(x,y)\right]$$

$C_0(x,y) = x + y\,, D_0(x,y) = -x + y$
$b_1(x,y) = [0 + 0 - (x + y) - (-x + y)]$
$b_1(x,y) = [-x - y + x - y] = -2y$
$a_2(x,y) = \frac{1}{2}\left[\frac{\partial^2}{\partial x^2}\big(a_1(x,y)\big) + \frac{\partial^2}{\partial y^2}\big(a_1(x,y)\big) - A_1(x,y) - B_1(x,y)\right]$
$A_1(x,y) = -4x - 2y\,, \qquad B_1(x,y) = -2y$
$a_2(x,y) = \frac{1}{2}[0 + 0 - (-4x - 2y) - (-2y)]$
$a_2(x,y) = \frac{1}{2}[4x + 2y + 2y] = \frac{1}{2}[4x + 4y] = 2x + 2y$
$b_2(x,y) = \frac{1}{2}\left[\frac{\partial^2}{\partial x^2}\big(b_1(x,y)\big) + \frac{\partial^2}{\partial y^2}\big(b_1(x,y)\big) - C_1(x,y) - D_1(x,y)\right]$
$C_1(x,y) = -2x\,, D_1(x,y) = 4y - 2x$
$b_2(x,y) = \frac{1}{2}[0 + 0 - (-2x) - (4y - 2x)]$
$b_2(x,y) = \frac{1}{2}[2x - 4y + 2x] = \frac{1}{2}[4x - 4y] = 2x - 2y$
$a_3(x,y) = \frac{1}{3}\left[\frac{\partial^2}{\partial x^2}\big(a_2(x,y)\big) + \frac{\partial^2}{\partial y^2}\big(a_2(x,y)\big) - A_2(x,y) - B_2(x,y)\right]$
$A_2(x,y) = 8x + 4y\,, \qquad B_2(x,y) = 4x - 4y$
$a_3(x,y) = \frac{1}{3}[0 + 0 - (8x + 4y) - (4x - 4y)]$
$a_3(x,y) = \frac{1}{3}[-8x - 4y - 4x + 4y] = \frac{1}{3}[-12x] = -4x$
$b_3(x,y) = \frac{1}{3}\left[\frac{\partial^2}{\partial x^2}\big(b_2(x,y)\big) + \frac{\partial^2}{\partial y^2}\big(b_2(x,y)\big) - C_2(x,y) - D_2(x,y)\right]$
$C_2(x,y) = 4x + 4y\,, D_2(x,y) = -4x + 8y$
$b_3(x,y) = \frac{1}{3}[0 + 0 - (4x + 4y) - (-4x + 6y)]$
$b_3(x,y) = \frac{1}{3}[-4x - 4y + 4x - 8y] = \frac{1}{3}[-12y] = -4y$
$a_4(x,y) = \frac{1}{4}\left[\frac{\partial^2}{\partial x^2}\big(a_3(x,y)\big) + \frac{\partial^2}{\partial y^2}\big(a_3(x,y)\big) - A_3(x,y) - B_3(x,y)\right]$
$A_3(x,y) = -16x - 8y\,, B_3(x,y) = -8y$
$a_4(x,y) = \frac{1}{4}[0 + 0 - (-16x - 8y) - (-8y)]$
$a_4(x,y) = \frac{1}{4}[16x + 8y + 8y] = \frac{1}{4}[16x + 16y] = 4x + 4y$
$b_4(x,y) = \frac{1}{4}\left[\frac{\partial^2}{\partial x^2}\big(b_3(x,y)\big) + \frac{\partial^2}{\partial y^2}\big(b_3(x,y)\big) - C_3(x,y) - D_3(x,y)\right]$
$C_3(x,y) = -8x\,, D_3(x,y) = 16y - 8x$
$b_4(x,y) = \frac{1}{4}[0 + 0 - (-8x) - (16y - 8x)]$
$b_4(x,y) = \frac{1}{4}[8x - 16y + 8x] = \frac{1}{4}[16x - 16y] = 4x - 4y$
$u(x,y,t) = \sum_{m=0}^{\infty} a_m(x,y)t^m = a_0 + a_1 t + a_2 t^2 + \cdots$
$u(x,y,t) = x + y - 2xt + (2x + 2y)t^2 - 4xt^3 + (4x + 4y)t^4 + \cdots$
$u(x,y,t) = x + y + 2(x + y)t^2 + 4(x + y)t^4 + \cdots - 2xt - 4xt^3 - 8xt^5 - \cdots$
$u(x,y,t) = (x + y)(1 + 2t^2 + 4t^4 + \cdots) - 2xt(1 + 2t^2 + 4t^4 + \cdots)$
$u(x,y,t) = (x + y)\left(\frac{1}{1-2t^2}\right) - 2xt\left(\frac{1}{1-2t^2}\right) = \frac{x+y-2xt}{1-2t^2}$
which is closed to the exact solution:

$$u(x,y,t)=\frac{x+y-2xt}{1-2t^2}$$

and

$$w(x,y,t)=\sum_{m=0}^{\infty}b_m(x,y)t^m=b_0+b_1t+b_2t^2+b_3t^3+b_4t^4+\cdots$$

$$w(x,y,t)=x-y-2yt+(2x-2y)t^2-4yt^3+(4x-4y)t^4+\cdots$$

$$w(x,y,t)=(x-y+2(x-y)t^2+4(x-y)t^4+8(x-y)t^6+\cdots)+(-2yt-4yt^3-8yt^5-\cdots)$$

$$w(x,y,t)=(x-y)(1+2t^2+4t^4+8t^6+\cdots)-2yt(1+2t^2+4t^4+8t^6+\cdots)$$

$$w(x,y,t)=(x-y)\left(\frac{1}{1-2t^2}\right)-(2yt)\left(\frac{1}{1-2t^2}\right)=\frac{x-y-2yt}{1-2t^2}$$

that is closed to the exact solution:

$$w(x,y,t)=\frac{x-y-2yt}{1-2t^2}$$

This problem was solved in [20-22] by using ADM and its modification. However, only a series solution, but not the exact solution, was obtained.

Problem 2

Consider a system of 3^{rd} order nonlinear PDE [23]

$u_t+vv_x=0$

$v_t+v_{xxx}+uv_x+u_xv=0$

subject to IC: $u(x,0)=2\,sech^2(x)$, $v(x,0)=2\,sech(x)$

Solution

By taking $L_t^{-1}=\int_0^t(.)dt$ to the system, we obtain

$$u(x,t)=u(x,0)-L_t^{-1}[vv_x]$$

$$v(x,t)=v(x,0)-\frac{\partial^3}{\partial x^3}L_t^{-1}[v]-L_t^{-1}[uv_x]-L_t^{-1}[u_xv]$$

$$u(x,t)=2\,sech^2(x)-L_t^{-1}[vv_x]$$

$$v(x,t)=2sech(x)-\frac{\partial^3}{\partial x^3}L_t^{-1}[v]-L_t^{-1}[uv_x]-L_t^{-1}[u_xv]$$

Let $u(x,t)=\sum_{m=0}^{\infty}a_m(x)t^m$, $v(x,t)=\sum_{m=0}^{\infty}b_m(x)t^m$

$$\sum_{m=0}^{\infty}a_m(x)t^m=2sech^2(x)-(\sum_{m=0}^{\infty}A_m(x)\frac{t^{m+1}}{m+1})$$

$$\sum_{m=0}^{\infty}b_m(x)t^m=2sech(x)-\frac{\partial^3}{\partial x^3}\left(\sum_{m=0}^{\infty}b_m(x)\frac{t^{m+1}}{m+1}\right)-\sum_{m=0}^{\infty}B_m(x)\frac{t^{m+1}}{m+1}-\sum_{m=0}^{\infty}C_m(x)\frac{t^{m+1}}{m+1}$$

where

$$vv_x=\sum_{m=0}^{\infty}A_m(x)t^m=A_0+A_1t+A_2t^2+\cdots$$

$$uv_x=\sum_{m=0}^{\infty}B_m(x)t^m=B_0+B_1t+B_2t^2+\cdots$$

$$u_xv=\sum_{m=0}^{\infty}C_m(x)t^m=C_0+C_1t+C_2t^2+\cdots$$

Let $m=m-1$ in the right side of the above system, then we have

$$\sum_{m=0}^{\infty}a_m(x)t^m=2sech^2\,(x)-(\sum_{m=1}^{\infty}A_{m-1}(x)\frac{t^m}{m})$$

$$\sum_{m=0}^{\infty}b_m(x)t^m=2sech(x)-\frac{\partial^3}{\partial x^3}\left(\sum_{m=1}^{\infty}b_{m-1}(x)\frac{t^m}{m}\right)-\sum_{m=1}^{\infty}B_{m-1}(x)\frac{t^m}{m}-\sum_{m=1}^{\infty}C_{m-1}(x)\frac{t^m}{m}$$

$$a_0(x)=2\,sech^2(x)$$

$$a_m(x) = \frac{1}{m}[-(A_{m-1}(x))]$$

Also,

$$b_0(x) = 2sech(x)$$

$$b_m(x) = \frac{1}{m}\left[-\frac{\partial^3}{\partial x^3}\left(b_{m-1}(x)\right) - B_{m-1}(x) - C_{m-1}(x)\right]$$

Let $u_0 = a_0, u_1 = a_1 t, u_2 = a_2 t^2, \ldots$

Let $v_0 = b_0, v_1 = b_1 t, v_2 = b_2 t^2, \ldots$

$$b_0 b_{0x} + (b_1 b_{0x} + b_0 b_{1x})t + (b_2 b_{0x} + b_1 b_{1x} + b_0 b_{2x})t^2 + \cdots = A_0 + A_1 t + A_2 t^2 + \cdots$$

$$a_0 b_{0x} + (a_1 b_{0x} + a_0 b_{1x})t + (a_2 b_{0x} + a_1 b_{1x} + a_0 b_{2x})t^2 + \cdots = B_0 + B_1 t + B_2 t^2 + \cdots$$

$$a_{0x} b_0 + (a_{1x} b_0 + a_{0x} b_1)t + (a_{2x} b_0 + a_{1x} b_1 + a_{0x} b_2)t^2 + \cdots = C_0 + C_1 t + C_2 t^2 + \cdots$$

$$a_1(x) = \frac{1}{1}[-(A_0(x))]$$

Then

$$a_1(x) = \frac{1}{1}[4tanh(x)sech^2(x)] = 4tanh(x)sech^2(x)$$

$$b_1(x) = \frac{1}{1}\left[-\frac{\partial^3}{\partial x^3}\left(b_0(x)\right) - B_0(x) - C_0(x)\right]$$

$$b_1(x) = \left[-\frac{\partial^3}{\partial x^3}\left(2sech(x)\right) + 4tanh(x)sech^3(x) + 8tanh(x)sech^3(x)\right]$$

$$b_1(x) = [-10tanh(x)sech^3(x) + 2tanh^3(x)sech(x) + 4tanh(x)sech^3(x) + 8tanh(x)sech^3(x)]$$

$$b_1(x) = [2tanh(x)sech^3(x) + 2tanh^3(x)sech(x)]$$

$$b_1(x) = \left[2tanh(x)sech(x)\left(sech^2(x) + tanh^2(x)\right)\right]$$

$$b_1(x) = 2tanh(x)sech(x)$$

$$a_2(x) = \frac{1}{2}[-(A_1(x))]$$

$$a_2(x) = \frac{1}{2}[8tanh^2(x)sech^2(x) - 4sech^4(x)]$$

$$a_2(x) = \frac{1}{2}\left[4sech^2(x)\left(2tanh^2(x) - sech^2(x)\right)\right]$$

$$a_2(x) = \frac{1}{2}\left[4sech^2(x)\left(\frac{cosh(2x) - 1}{cosh^2(x)} - \frac{1}{cosh^2(x)}\right)\right]$$

$$a_2(x) = \frac{1}{2}[4sech^2(x)(-2 + \cosh(2x))sech^2(x)]$$

$$a_2(x) = \frac{1}{2}[4sech^4(x)(-2 + cosh(2x))]$$

$$a_2(x) = -2sech^4(x)(2 - cosh(2x))$$

$$b_2(x) = \frac{1}{2}\left[-\frac{\partial^3}{\partial x^3}\left(b_1(x)\right) - B_1(x) - C_1(x)\right]$$

$$b_2(x) = \frac{1}{2}\left[-\frac{\partial^3}{\partial x^3}\left(2tanh(x)sech(x)\right) + 12tanh^2(x)sech^3(x) - 4sech^5(x) + 24tanh^2(x)sech^3(x) - 8sech^5(x)\right]$$

$$b_2(x) = \frac{1}{2}[10sech^5(x) - 36tanh^2(x)sech^3(x) + 2tanh^4(x)\mathrm{sech}(x) + 12tanh^2(x)sech^3(x) - 4sech^5(x) + 24tanh^2(x)sech^3(x) - 8sech^5(x)]$$

$$b_2(x) = \frac{1}{2}[-2sech^5(x) + 2tanh^4(x)sech(x)]$$

$$b_2(x) = \frac{1}{2}\left[2sech(x)\left(-sech^4(x) + tanh^4(x)\right)\right]$$

$$b_2(x) = \frac{1}{2}\left[2sech(x)\left(tanh^4(x) - sech^4(x)\right)\right]$$

$$b_2(x) = \frac{1}{2}\left[2sech(x)\left(tanh^2(x) - sech^2(x)\right)\right]$$
$$b_2(x) = \frac{1}{2}\left[2sech(x)\left(\frac{\frac{1}{2}(cosh(2x)-1)}{cosh^2(x)} - \frac{1}{cosh^2(x)}\right)\right]$$
$$b_2(x) = \frac{1}{2}\left[2sech^3(x)\left(\frac{1}{2}cosh(2x) - \frac{3}{2}\right)\right]$$
$$b_2(x) = \frac{1}{2}[sech^3(x)(cosh(2x) - 3)]$$
$$b_2(x) = \frac{1}{2}[sech^3(x)(3 - cosh(2x))]$$
$$b_2(x) = -\frac{1}{2}(3 - cosh(2x))sech^3(x)$$

and so on

$$u(x,t) = \sum_{m=0}^{\infty} a_m(x)t^m = a_0 + a_1 t + a_2 t^2 + \cdots$$
$$u\,(x\,,t) = 2sech^2(x) + \left(4ctanh(x)sech^2(x)\right)t - \left(2sech^4(x)(2 - cosh(2x))\right)t^2 + \cdots$$
$$v\,(x\,,t) = \sum_{m=0}^{\infty} b_m(x)t^m = b_0 + b_1 t + b_2 t^2 + \cdots$$
$$v\,(\,x,t) = 2sech(x) + \left(2tanh(x)sech(x)\right)t - \frac{1}{2}(3 - cosh(2x))sech^3(x)t^2 + \cdots$$

that is closed to the exact solution:

$u(x,t) = 2sech^2(x-t)$, $v(x,t) = 2sech(x-t)$

Problem 2 is solved in [24] by using ADM and its modification, but only the series solution, not the exact solution was obtained.

4. Convergence Analysis of the Series Solution

In this section, the convergence analysis of the series solution for the non-linear systems of PDEs is discussed. The sufficient requirement for convergence of the suggested modification is addressed. We show that the series solution for the systems of PDEs is converge to the exact solution.

Definition 1 [20]

A Banach space is a complete, normed, Vector space.

All norms on a Vector space of finite dimensions are equivalent. Every finite-dimensional standard space is a Banach space, over **R** or **C.**

Definition 2 [21]

Let X be a set and let $f: x \to x$ be a function that maps x into itself. Such a function is often called an operator. A fixed point of f is an element$x \in X$, for which $f(x) = x$.

Definition 3 [25]

Let (X,d) be a metric space. A mapping $T: X \to X$ is a contraction mapping, or contraction, if there exists a constant c, with $0 \le c < 1$, such that

$d\left(T(x),T(y)\right) \le cd(x,y), x,y \in X$

Definition 4 [26]

Let (X,d) be a complete metric space. A mapping $T: X \to X$ is a nonlinear contraction mapping, or nonlinear contraction, if there exists a constant c, with $0 \le c < 1$, such that

$d\left(T(x),T(y)\right) \le c\left(d(x,y)\right)d(x,y)\,, x,y \in X.$

Theorem 5 (Banach's fixed-point theorem)

A contractive function T on a Banach space S has a Unique fixed point X^* in R^2 [27].

Theorem 6 (Sufficient Condition for Convergence)

If X and Y are Banach spaces and $N: X \to Y$ is a contractive nonlinear mapping, that is

$\forall w, w^* \in X; \| N(w) - N(w^*) \| \le \gamma \| w - w^* \|, 0 < \gamma < 1,$

then, according to Banach's fixed-point theorem, N has a unique fixed-point U, that is $N(u) = \mathrm{U}$.

Assume that the sequence generated by the suggested method can be written as

$w_n = N(w_{n-1}), w_{n-1} = \sum_{i=0}^{n-1} w_i\,, n = 1,2,3, \ldots$

Suppose that $W_0 = w_0 \in B_r(w)$ where $B_r(w) = \{w^* \in X \mid\mid w^* - w \mid\mid < r\}$, then we have
i.$w_n \in B_r(w)$
ii. $\lim_{n\to\infty} W_n = w$
Proof
(i) By the inductive approach, for $n = 1$, we have
$\mid\mid W_1 - w \mid\mid = \mid\mid N(W_0) - N(w) \mid\mid \leq \gamma \mid\mid w_0 - w \mid\mid$
Assume that $\mid\mid W_{n-1} - w \mid\mid \leq \gamma \mid\mid w_{n-1} - w \mid\mid$
$\leq \gamma^2 \mid\mid w_{n-2} - w \mid\mid$
$\leq \gamma^3 \mid\mid w_{n-3} - w \mid\mid$
$\leq \gamma^{n-1} \mid\mid w_0 - w \mid\mid$
As in the induction hypothesis, then
$\mid\mid W_n - w \mid\mid = \mid\mid N(W_{n-1}) - N(w) \mid\mid \leq \gamma \mid\mid w_{n-1} - w \mid\mid \leq \gamma^n \mid\mid w_0 - w \mid\mid$
Using (i), we have
$\mid\mid W_n - w \mid\mid \leq \gamma^n \mid\mid w_0 - w \mid\mid \leq \gamma^n r < r \Rightarrow W_n \in B_r(w)$
Because of $\mid\mid W_n - w \mid\mid \leq \gamma^n \mid\mid w_0 - w \mid\mid$ and
$\lim_{n\to\infty} \gamma^n = 0, \lim_{n\to\infty} \mid\mid W_n - w \mid\mid = 0$
that is
$\lim_{n\to\infty} W_n = w$

5. Conclusions

In this article, a new modification of the decomposition method is suggested to solve the non-linear system. We obtained an exact analytical solution, where ADM or other modifications are used to solved the same examples but cannot achieve an exact analytical solution. Moreover, u_0 in ADM and its modification is $u_0 = f(x) + tg(x)$, versus $u_0 = a_0 = f(x), a_1 = g(x)$ in MDM, which is the main reason for simplifying the steps of solution. Moreover, in MDM, the nonlinear terms are easier to compute than in ADM or its modifications. The convergence concept of the decomposition series was thoroughly investigated to confirm the rapid convergence of the resulting series. Hence, this approach is very efficient, with easy implementation and rapid convergence to the exact solutions.

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