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Numerical Solution of Energy Equation in Porous Channels under Effects of Radiation Field

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Abstract

In this paper, we built a mathematical model for convection and thermal radiation heat transfer of fluid flowing through a vertical channel with porous medium under effects of horizontal magnetic field (MF) at the channel. This model represents a 2-dimensional system of non-linear partial differential equations. Then, we solved this system numerically by finite difference methods using Alternating Direction Implicit (ADI) Scheme in two phases (steady state and unsteady state). Moreover, we found the distribution and behaviour of the heat temperature inside the channel and studied the effects of Brinkman number, Reynolds number, and Boltzmann number on the heat temperature behaviour. We solved the system by building a computer program using MATLAB.

Keywords: Heat transfer, Porous medium, Brinkman number, Pouger number, Boltzmann number, Reynolds number.

الحل العددي لمعادلة الطاقة في القنوات المسامية تحت تأثير مجال الإشعاع

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الخلاصة

لقد تم توظيف هذا البحث لبناء نموذج رياضياتي لانتقال الحرارة بالحمل الحراري والاشعاع الحراري لمائع يجري في قناة مسامية أفقية وتحت تأثير مجال مغناطيسي(**MF**) عمودي على مستوى القناة ,والذي يمثل نظاما من المعادلات التفاضلية الجزئية غير الخطية في بعدين. ومن ثم معالجة المعادلات التفاضلية الناتجة باستخدام الطريقة الضمنية للاتجاهات المتعاقبة (**ADI**)(**Alternative Direction Implicit Method**) وفي كلا الحالتين المعتمدة على الزمن (**unsteady state**) والحالة اللازمنية (**steady state**)، اذ تم ايجاد توزيع وسلوك درجات الحرارة داخل القناة، ثم دراسة تأثير كل من عدد برينكمان (**Brinkman number**) وعدد رينولدز (**Reynolds number**) بالاضافة الى عدد بولتزمان (**Boltzmann number**) في سلوك درجات الحرارة وذلك عن طريق برنامج حاسوبي باستخدام لغة (**MATLAB**).

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1. Introduction

Fluid is a substance that cannot resist a shear force or stress without moving. Fluid flow may be classified in different manners, such as turbulent, laminar, real, ideal, steady, unsteady, uniform, compressible, incompressible, etc. [1].

The heat transfer in electrically conducting fluid in circulatory and channels pipes, subject to the effects of magnetic transverse fields, is conducted in magneto hydrodynamics (MHD), flow meters, and pumps, and have applications in filtration, nuclear reactors, geothermal systems, and others.

The natural convection in enclosures with localized heating from below creeping flow to the onset of laminar instability was studied [2].

The stability of convection in a container of arbitrary shape when heated from below was also investigated [3]. The authors analyzed the stability of two forms of convection, one with the top flow and the other with the down flow, in a bounded domain and a layer with only one stable form and mentioned the chaotic conditions.

Another work investigated the convection in a rotating cylindrical annulus under the effect of MF [4]. The authors investigated the effects of radial and azimuthal components of MF on the convection columns in a fluid filled gap between two cylinders rotating rigidly about their common vertical axis numerically; the inner cylinder is cooled and the outer one is heated, such that the buoyancy force driving the convection is provided by the centrifugal force.

The effects of radiation in a magneto fluid-dynamic channel flow was shown [5]. The plane Hartmann flow was extended to account for thermally radiative effects with variable absorption coefficient and non-uniform temperature of channel walls. Furthermore, some aspects of stability were examined.

The convection beginning in an infinite rigid horizontal channel that has uniformly lower heat source was determined by using a 2-dimensional Galerkin formulation of the 3-dimensional Oberbeck-Boussinesq equations. The authors extended the previous results to the higher truncation levels to involve patterns of convection.

The thermal convection problem in a fluid layer with a lower heat source was introduced [7] and solved numerically when strong vertical MF parameter the layer. When the values of Hartmann number between (200-400), the stability of the 2-dimensional convection rolls was studied.

The convection inside a rotating cylindrical annulus was investigated by using a system that contains three coupled amplitude equations [8]. The authors described many features of a good approximation and showed that the time integrations based on the Galerkin expansion display transitions to chaotic convection at a high Rayleigh number.

The fully developed free convection problem of two fluid MHD flow in a slanted channel were discussed [9]. It was observed that the flow could be dominated effectively by the appropriate adjustment of the values of height ratio, electrical conductivity, and viscosity of the two fluids.

Another study proposed a solution to the magnetohydrodynamic (MHD) problem by the analytically free convection flow of an electrically conducting fluid between two heated parallel plates in the presence of an induced MF [10]. It was noted that the skin-friction increases first, then gradually decreases with the increase of Hartmann number to $y=1$.

The effects of heat and slip transfer on the peristaltic flow of a 3th fluid in an inclined asymmetric channel were reported [11]. In the same year, the impacts of radiation on MHD flow of Maxwell fluid in a channel with a porous medium were examined by employing the homotopy analysis method (HAM) [12].

The 2-dimensional steady flow of electrically conducting incompressible power-low fluid passing an infinite porous flat plate subjected to suction or blowing was investigated [13]. The authors also analyzed the heat transfer flow in the case when the plate is held at a fixed temperature. In the same year, HAM was applied to obtain an analytical solution of partial differential equations. Numerical results and graphical representation strongly reconfirmed the efficiency of the proposed scheme [14].

Another work [15] studied the approximated solutions for heat transfer over a porous plate and steady MHD mixed convection boundary layer flow in the presence of thermal and velocity. The MF impact on a viscous incompressible fluid was found to increase the fluid velocity by reducing the drag on the flow, which causes a decrease in the temperature of the fluid.

An earlier investigation [16] presented the impacts of mass transfer, viscous suction, and dissipation on the flow of 2-dimensional steady hydromagnetic viscous fluid between two parallel plates in the presence of thermal radiation. The authors found that velocity, temperature, and concentration decrease with the increase of suction and Reynolds number. They also reported associations among different physical properties. In this work, the solution of the equation of heat transfer in a porous channel with the presences of MF and radiation was investigated. It was found that the parameters of Brinkmann number (Fs), Reynolds number (Re), and Boltzmann number(B_0)have significant effects on the equation solution.

Another study [17] presented a Crank-Nicolson finite difference method to solve the time fractional 2-dimensional sub-diffusion equation in the case where the Grunwald-Letnikov definition is used for the time fractional derivative. The stability and convergence of the proposed Crank-Nicolson scheme were also analysed and the numerical examples were presented to test whether the numerical scheme is accurate and feasible. In the same year, the modified implicit finite-difference approximation for the stability and convergence of the proposed scheme were analysed [18]. It was found that the scheme is unconditionally stable and the approximate solution converges to the exact solution.

The numerical solutions of the equations of motion, heat transfer, and diffusion in a porous medium with the presences of radiation and MF were studied [19]. It was found that the parameters of Gr, R, Sc and Pr have significant impacts on the solutions of these equations. The aim of this article is to study the numerical solution to the problem of convection and radiation heat transfer of a fluid in a porous channel under the influence of magnetic field (MF) using the alternative direction implicit method (ADI). We found that the parameters of Fs , Re, and B_0 had significant effects through increasing and decreasing the fluid temperature within the channel.

2. Conservation Equation

In the present study, the steady laminar flow of fluid between parallel horizontal walls, distanced 2h apart, is considered. The velocities υ, ν are zero at edges, while T_1 and T_2 represent the temperature of the lower and upper plates, respectively.

The governing equation is:

$$\frac{\partial T}{\partial t} + \upsilon \frac{\partial T}{\partial x} + \nu \frac{\partial T}{\partial y} = \frac{k}{\rho C_v}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) - \frac{1}{\rho C_v}\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}\right) \quad (2.1).$$

where υ and ν are the values of velocity in x and y, T, ρ, k, C_v refer to temperature, density, permeability, and specific heat, and q_x and q_y are radiation values in x and y directions, respectively.

with the following boundary conditions:

$$\left.\begin{matrix} \upsilon = \nu = 0,0 \\ \mathrm{T} = \mathrm{T}_1, \mathrm{T}_2 \end{matrix}\right\} \quad y = \mp h \tag{2.2}$$

3. Non-dimensional Energy Equation

To solve the governing equation (2.1) with the boundary conditions (2.2), we need to introduce the following non-dimensional quantities [5, 20]:

$$\left.\begin{matrix} t = \frac{h}{\upsilon_0} t' \ , \ \mathrm{T} = \mathrm{T}_1 \varphi \ , \ \vec{q} = \mathrm{T}_1^4 \sigma \vec{Q} \\ x = hx' \ , \ y = hy' \ , \ \upsilon = \upsilon_0 \upsilon' \ , \ \nu = \upsilon_0 \nu' \end{matrix}\right\} \tag{3.1}$$

By substituting these quantities into equation (2.1), the governing equation becomes:

$$\frac{\partial \varphi}{\partial t'} + \upsilon' \frac{\partial \varphi}{\partial x'} + \nu' \frac{\partial \varphi}{\partial y'} = \frac{\gamma}{PrRe}\left(\frac{\partial^2 \varphi}{\partial x'^2} + \frac{\partial^2 \varphi}{\partial y'^2}\right) - \frac{\gamma Ec}{B_0}\left(\frac{\partial Q_x}{\partial x'} + \frac{\partial Q_y}{\partial y'}\right) \tag{3.2}$$

But $\nabla . \vec{Q} = 16\omega\varphi - 12\omega$ [5]; therefore, multiplying equation (3.2) by $\frac{PrRe}{\gamma}$ yields the non-dimensional energy equation:

$$PrL\left(\frac{\partial \varphi}{\partial t'} + \upsilon' \frac{\partial \varphi}{\partial x'} + \nu' \frac{\partial \varphi}{\partial y'}\right) = \left(\frac{\partial^2 \varphi}{\partial x'^2} + \frac{\partial^2 \varphi}{\partial y'^2}\right) - 16FsN\omega\varphi + 12FsN\omega \tag{3.3}$$

where:

$Ec = \frac{\upsilon_0^2}{C_p \mathrm{T}_1}$, $Re = \frac{h\rho_1 \upsilon_0}{\mu}$, $\omega = \propto_0 h$, $B_0 = \frac{\rho_1 \upsilon_0^3}{\mathrm{T}_1^4 \sigma}$, $\gamma = \frac{C_p}{C_v}$, $Pr = \frac{\mu C_p}{k}$, $Fs = PrEc$, $L = \frac{Re}{\gamma}$, $N = \frac{Re}{B_0}$ are Eckert number, Reynolds number, Bouger number, Boltzmann number, Specific heat ratio, Prandtel number, Brinkman number, and the new physical quantities, respectively. The non-dimensional boundary conditions become:

$$\left.\begin{matrix} \upsilon' = \nu' = 0.0 & at \quad y' = \pm 1 \\ \varphi = 0, 1.0 & at \quad y' = \pm 1 \end{matrix}\right\} \tag{3.4}$$

4. Solution of Heat Equation

In this section, we derive a new second-order ADI method for the numerical solution of the parabolic equations (3.3)-(3.4).

4.1: On X-direction

Using the central difference formula we have:

$$\frac{\partial \varphi}{\partial t'} = \frac{\varphi'_{i,j} - \varphi_{i,j,n}}{\Delta t'/2} \tag{4.1.1}$$

$$\upsilon' \frac{\partial \varphi}{\partial x'} = \upsilon'_{i,j,n} \frac{\varphi'_{i+1,j} - \varphi'_{i-1,j}}{2\Delta x'} \tag{4.1.2}$$

$$\nu' \frac{\partial \varphi}{\partial y'} = \nu'_{i,j,n} \frac{\varphi_{i,j+1,n} - \varphi_{i,j-1,n}}{2\Delta y'} \qquad (4.1.3).$$

$$\frac{\partial^2 \varphi}{\partial x'^2} = \frac{\varphi'_{i+1,j} - 2\varphi'_{i,j} + \varphi'_{i-1,j}}{(\Delta x')^2} \qquad (4.1.4).$$

$$\frac{\partial^2 \varphi}{\partial y'^2} = \frac{\varphi_{i,j+1,n} - 2\varphi_{i,j,n} + \varphi_{i,j-1,n}}{(\Delta y')^2} \qquad (4.1.5).$$

$$16FsN\omega\varphi = 16FsN\omega\varphi_{i,j,n} \qquad (4.1.6).$$

In addition, suppose that $\overline{\omega} = PrL$ and $\overline{B} = 12FsN\omega$, then by substituting the equations (4.1.1)-(4.1.6) in equation (3.3), we obtain:

$$\overline{\omega}\left[\frac{\varphi'_{i,j} - \varphi_{i,j,n}}{\Delta t'/2} + \frac{\varphi'_{i+1,j} - \varphi'_{i-1,j}}{2\Delta x'} \upsilon'_{i,j,n} + \frac{\varphi_{i,j+1,n} - \varphi_{i,j-1,n}}{2\Delta y'} \nu'_{i,j,n}\right] = \frac{\varphi'_{i+1,j} - 2\varphi'_{i,j} + \varphi'_{i-1,j}}{(\Delta x')^2} + \cdots$$

$$\ldots + \frac{\varphi_{i,j+1,n} - 2\varphi_{i,j,n} + \varphi_{i,j-1,n}}{(\Delta y')^2} + 16FsN\omega\varphi_{i,j,n} + \overline{B}$$

Moreover, this implies that:

$$\overline{\omega}\left[\frac{\varphi'_{i,j} - \varphi_{i,j,n}}{\Delta t'/2}\right] = \frac{\varphi'_{i+1,j} - 2\varphi'_{i,j} + \varphi'_{i-1,j}}{(\Delta x')^2} + \frac{\varphi_{i,j+1,n} - 2\varphi_{i,j,n} + \varphi_{i,j-1,n}}{(\Delta y')^2} - \cdots$$

$$\ldots - \overline{\omega}\left[\frac{\varphi'_{i+1,j} - \varphi'_{i-1,j}}{2\Delta x'} \upsilon'_{i,j,n} + \frac{\varphi_{i,j+1,n} - \varphi_{i,j-1,n}}{2\Delta y'} \nu'_{i,j,n}\right] + 16FsN\omega\varphi_{i,j,n} + \overline{B} \qquad (4.1.7).$$

By multiplying equation (4.1.7) by $\frac{\Delta t'}{2\overline{\omega}}$, we have:

$$\varphi'_{i,j} - \varphi_{i,j,n} = \frac{\Delta t'}{2\overline{\omega}}\left[\frac{\varphi'_{i+1,j} - 2\varphi'_{i,j} + \varphi'_{i-1,j}}{(\Delta x')^2} + \frac{\varphi_{i,j+1,n} - 2\varphi_{i,j,n} + \varphi_{i,j-1,n}}{(\Delta y')^2}\right] - \cdots$$

$$\ldots - \frac{\Delta t'}{2}\left[\frac{\varphi'_{i+1,j} - \varphi'_{i-1,j}}{2\Delta x'} \upsilon'_{i,j,n} + \frac{\varphi_{i,j+1,n} - \varphi_{i,j-1,n}}{2\Delta y'} \nu'_{i,j,n}\right] + \frac{8\Delta t'}{\overline{\omega}} FsN\omega\varphi_{i,j,n} + \cdots$$

$$\ldots + \frac{\Delta t'}{2\overline{\omega}} \overline{B} \qquad (4.1.8).$$

Let $\Delta x' = \Delta y' = h$ and $\Delta t' = d$, then:

$$\varphi'_{i,j} - \varphi_{i,j,n} = \frac{\mathrm{d}}{2h^2\overline{\omega}}\left[\varphi'_{i+1,j} - 2\varphi'_{i,j} + \varphi'_{i-1,j} + \varphi_{i,j+1,n} - 2\varphi_{i,j,n} + \varphi_{i,j-1,n}\right] - \cdots$$

$$\ldots - \frac{d}{4h}\left[(\varphi'_{i+1,j} - \varphi'_{i-1,j})\upsilon'_{i,j,n} + (\varphi_{i,j+1,n} - \varphi_{i,j-1,n})\nu'_{i,j,n}\right] + \frac{8FsNd\omega}{\overline{\omega}} \varphi_{i,j,n} + \cdots$$

...
$$+\frac{d}{2\overline{\omega}}\overline{B} \qquad (4.1.9).$$

And let $\lambda=\frac{d}{h^2}$, then:

$$\varphi'_{i,j}-\varphi_{i,j,n}=\frac{\lambda}{2\overline{\omega}}\varphi'_{i+1,j}-\frac{\lambda}{\overline{\omega}}\varphi'_{i,j}+\frac{\lambda}{2\overline{\omega}}\varphi'_{i-1,j}+\frac{\lambda}{2\overline{\omega}}\varphi_{i,j+1,n}-\frac{\lambda}{\overline{\omega}}\varphi_{i,j,n}+\frac{\lambda}{2\overline{\omega}}\varphi_{i,j-1,n}-\cdots$$
$$\ldots-\frac{\lambda h}{4}\varphi'_{i+1,j}\upsilon'_{i,j,n}+\frac{\lambda h}{4}\varphi'_{i-1,j}\upsilon'_{i,j,n}-\frac{\lambda h}{4}\varphi_{i,j+1,n}\nu'_{i,j,n}+\frac{\lambda h}{4}\varphi_{i,j-1,n}\nu'_{i,j,n}+\frac{8\lambda h^2FsN\omega}{\overline{\omega}}\varphi_{i,j,n}+\cdots$$
...
$$+\frac{\lambda h^2}{2\overline{\omega}}\overline{B} \qquad (4.1.10).$$

This implies that:

$$\varphi'_{i,j}-\frac{\lambda}{2\overline{\omega}}\varphi'_{i+1,j}+\frac{\lambda}{\overline{\omega}}\varphi'_{i,j}-\frac{\lambda}{2\overline{\omega}}\varphi'_{i-1,j}+\frac{\lambda h}{4}\varphi'_{i+1,j}\upsilon'_{i,j,n}-\frac{\lambda h}{4}\varphi'_{i-1,j}\upsilon'_{i,j,n}$$
$$=\varphi_{i,j,n}+\frac{\lambda}{2\overline{\omega}}\varphi_{i,j+1,n}-\frac{\lambda}{\overline{\omega}}\varphi_{i,j,n}+\frac{\lambda}{2\overline{\omega}}\varphi_{i,j-1,n}-\frac{\lambda h}{4}\varphi_{i,j+1,n}\nu'_{i,j,n}+\frac{\lambda h}{4}\varphi_{i,j-1,n}\nu'_{i,j,n}+\frac{8\lambda h^2FsN\omega}{\overline{\omega}}\varphi_{i,j,n}+\frac{\lambda h^2}{2\overline{\omega}}\overline{B}$$

So,

$$-\left[\frac{\lambda}{2\overline{\omega}}+\frac{\lambda h}{4}\upsilon'_{i,j,n}\right]\varphi'_{i-1,j}+\left[1+\frac{\lambda}{\overline{\omega}}\right]\varphi'_{i,j}-\left[\frac{\lambda}{2\overline{\omega}}-\frac{\lambda h}{4}\upsilon'_{i,j,n}\right]\varphi'_{i+1,j}$$
$$=\left[\frac{\lambda}{2\overline{\omega}}+\frac{\lambda h}{4}\nu'_{i,j,n}\right]\varphi_{i,j-1,n}+\left[1-\frac{\lambda(1-8h^2FsN\omega)}{\overline{\omega}}\right]\varphi_{i,j,n}+\left[\frac{\lambda}{2\overline{\omega}}-\frac{\lambda h}{4}\nu'_{i,j,n}\right]\varphi_{i,j+1,n}$$
$$+\frac{\lambda h^2}{2\overline{\omega}}\overline{B} \qquad (4.1.11).$$

By multiplying (4.1.11) by $\frac{2\overline{\omega}}{\lambda}$, we obtain:

$$-\left[1+\frac{h\overline{\omega}}{2}\upsilon'_{i,j,n}\right]\varphi'_{i-1,j}+2\left[1+\frac{\overline{\omega}}{\lambda}\right]\varphi'_{i,j}-\left[1-\frac{h\overline{\omega}}{2}\upsilon'_{i,j,n}\right]\varphi'_{i+1,j}$$
$$=\left[1+\frac{h\overline{\omega}}{2}\nu'_{i,j,n}\right]\varphi_{i,j-1,n}+\cdots$$
$$\ldots+2\left[\frac{\overline{\omega}}{\lambda}-(1+8h^2FsN\omega)\right]\varphi_{i,j,n}+\left[1-\frac{h\overline{\omega}}{2}\nu'_{i,j,n}\right]\varphi_{i,j+1,n}$$
$$+h^2\overline{B} \qquad (4.1.12).$$

By multiplying (4.1.112) by $\frac{1}{1+\frac{h\overline{\omega}}{2}\upsilon'_{i,j,n}}$, we get:

$$-\varphi'_{i-1,j}+\frac{2\left[1+\frac{\overline{\omega}}{\lambda}\right]}{\left[1+\frac{h\overline{\omega}}{2}\upsilon'_{i,j,n}\right]}\varphi'_{i,j}-\frac{\left[1-\frac{h\overline{\omega}}{2}\upsilon'_{i,j,n}\right]}{\left[1+\frac{h\overline{\omega}}{2}\upsilon'_{i,j,n}\right]}\varphi'_{i+1,j}=\frac{\left[1+\frac{h\overline{\omega}}{2}\nu'_{i,j,n}\right]}{\left[1+\frac{h\overline{\omega}}{2}\upsilon'_{i,j,n}\right]}\varphi_{i,j-1,n}+\cdots$$

$$... + \frac{2\left[\frac{\overline{\omega}}{\lambda} - (1 + 8h^2 FsN\omega)\right]}{\left[1 + \frac{h\overline{\omega}}{2} \upsilon'_{i,j,n}\right]} \varphi_{i,j,n} + \frac{\left[1 - \frac{h\overline{\omega}}{2} \nu'_{i,j,n}\right]}{\left[1 + \frac{h\overline{\omega}}{2} \upsilon'_{i,j,n}\right]} \varphi_{i,j+1,n} + \frac{h^2 \overline{B}}{\left[1 + \frac{h\overline{\omega}}{2} \upsilon'_{i,j,n}\right]}$$

which implies that:

$$-\varphi'_{i-1,j} + 2(f_1/f_3)\varphi'_{i,j} - (f_4/f_3)\varphi'_{i+1,j} = (f_5/f_3)\varphi_{i,j-1,n} + 2(f_2/f_3)\varphi_{i,j,n} + \cdots$$
$$\ldots + (f_6/f_3)\varphi_{i,j+1,n}$$
$$+ \left(h^2\overline{\mathrm{B}}/f_3\right) \quad (4.1.13).$$

where:

$$f_1 = \left[1 + \frac{\overline{\omega}}{\lambda}\right], \quad f_2 = \left[\frac{\overline{\omega}}{\lambda} - (1 + 8h^2 FsN\omega)\right], \quad f_3 = \left[1 + \frac{h\overline{\omega}}{2} \upsilon'_{i,j,n}\right],$$
$$f_4 = \left[1 - \frac{h\overline{\omega}}{2} \upsilon'_{i,j,n}\right], \quad f_5 = \left[1 + \frac{h\overline{\omega}}{2} \nu'_{i,j,n}\right], \quad f_6 = \left[1 - \frac{h\overline{\omega}}{2} \nu'_{i,j,n}\right]$$

Equation (4.1.13) can be reduced to give:

$$A(i)\varphi'_{i-1} + 2B(i)\varphi'_{i,j} + C(i)\varphi'_{i+1,j} = D(i) \quad (4.1.14)$$

where $i = 1,2, \ldots\ldots\ldots, M$, and:

$A(i) = -1$
$B(i) = 2(f_1/f_3)$
$C(i) = -(f_4/f_3)$
$D(i) = (f_5/f_3)\varphi_{i,j-1,n} + 2(f_2/f_3)\varphi_{i,j,n} + (f_6/f_3)\varphi_{i,j+1,n} + \left(h^2\overline{\mathrm{B}}/f_3\right)$

From equation (4.1.13), a tri-diagonal system can be created, which is solved numerically using the Gauss elimination method and using MATLAB at the time-step $t + 1/2$ [21].

4.2: On Y-direction

Using the formulas of the central differences, we have:

$$\frac{\partial\varphi}{\partial t'} = \frac{\varphi_{i,j,n+1} - \varphi'_{i,j}}{\Delta t'/2} \quad (4.2.1).$$

$$\upsilon' \frac{\partial\varphi}{\partial x'} = \upsilon'_{i,j,n} \frac{\varphi'_{i+1,j} - \varphi'_{i-1,j}}{2\Delta x'} \quad (4.2.2).$$

$$\nu' \frac{\partial\varphi}{\partial y'} = \nu'_{i,j,n} \frac{\varphi_{i,j+1,n+1} - \varphi_{i,j-1,n+1}}{2\Delta y'} \quad (4.2.3).$$

$$\frac{\partial^2\varphi}{\partial x'^2} = \frac{\varphi'_{i+1,j} - 2\varphi'_{i,j} + \varphi'_{i-1,j}}{(\Delta x')^2} \quad (4.2.4).$$

$$\frac{\partial^2 \varphi}{\partial y'^2} = \frac{\varphi_{i,j+1,n+1} - 2\varphi_{i,j,n+1} + \varphi_{i,j-1,n+1}}{(\Delta y')^2} \quad (4.2.5).$$

$$16FsN\omega\varphi = 16FsN\omega\varphi_{i,j,n+1} \quad (4.2.6).$$

Hence,

$$\overline{\omega}\left[\frac{\varphi_{i,j,n+1} - \varphi'_{i,j}}{\frac{\Delta t'}{2}} + \frac{\varphi'_{i+1,j} - \varphi'_{i-1,j}}{2\Delta x'} v'_{i,j,n} + \frac{\varphi_{i,j+1,n+1} - \varphi_{i,j-1,n+1}}{2\Delta y'} \nu'_{i,j,n}\right] = \frac{\varphi'_{i+1,j} - 2\varphi'_{i,j} + \varphi'_{i-1,j}}{(\Delta x')^2} + \frac{\varphi_{i,j+1,n+1} - 2\varphi_{i,j,n+1} + \varphi_{i,j-1,n+1}}{(\Delta y')^2} + 16FsN\omega\varphi_{i,j,n+1} + \overline{B}$$

And this implies that:

$$\overline{\omega}\left[\frac{\varphi_{i,j,n+1} - \varphi'_{i,j}}{\Delta t'/2}\right] = \frac{\varphi'_{i+1,j} - 2\varphi'_{i,j} + \varphi'_{i-1,j}}{(\Delta x')^2} + \frac{\varphi_{i,j+1,n+1} - 2\varphi_{i,j,n+1} + \varphi_{i,j-1,n+1}}{(\Delta y')^2} - \cdots$$

$$\ldots - \overline{\omega}\left[\frac{\varphi'_{i+1,j} - \varphi'_{i-1,j}}{2\Delta x'} v'_{i,j,n} + \frac{\varphi_{i,j+1,n+1} - \varphi_{i,j-1,n+1}}{2\Delta y'} \nu'_{i,j,n}\right] + 16FsN\omega\varphi_{i,j,n+1} + \overline{B} \quad (4.2.7).$$

By multiplying equation (4.2.7) by $\frac{\Delta t'}{2\overline{\omega}}$, we have:

$$\varphi_{i,j,n+1} - \varphi'_{i,j} = \frac{\Delta t'}{2\overline{\omega}}\left[\frac{\varphi'_{i+1,j} - 2\varphi'_{i,j} + \varphi'_{i-1,j}}{(\Delta x')^2} + \frac{\varphi_{i,j+1,n+1} - 2\varphi_{i,j,n+1} + \varphi_{i,j-1,n+1}}{(\Delta y')^2}\right] - \cdots$$

$$\ldots - \frac{\Delta t'}{2}\left[\frac{\varphi'_{i+1,j} - \varphi'_{i-1,j}}{2\Delta x'} v'_{i,j,n} + \frac{\varphi_{i,j+1,n+1} - \varphi_{i,j-1,n+1}}{2\Delta y'} \nu'_{i,j,n}\right] + \frac{8\Delta t'}{\overline{\omega}} FsN\omega\varphi_{i,j,n+1} + \frac{\Delta t'}{2\overline{\omega}}\overline{B} \quad (4.2.8).$$

Let $\Delta x' = \Delta y' = h$ *and* $\Delta t' = d$, then:

$$\varphi_{i,j,n+1} - \varphi'_{i,j} = \frac{\mathrm{d}}{2h^2\overline{\omega}}\left[\varphi'_{i+1,j} - 2\varphi'_{i,j} + \varphi'_{i-1,j} + \varphi_{i,j+1,n+1} - 2\varphi_{i,j,n+1} + \varphi_{i,j-1,n+1}\right] - \cdots$$

$$\ldots - \frac{d}{4h}\left[(\varphi'_{i+1,j} - \varphi'_{i-1,j})v'_{i,j,n} + (\varphi_{i,j+1,n+1} - \varphi_{i,j-1,n+1})\nu'_{i,j,n}\right] + \frac{8FsNd\omega}{\overline{w}}\varphi_{i,j,n+1} + \frac{d}{2\overline{\omega}}\overline{B} \quad (4.2.9).$$

And let $\lambda = \frac{d}{h^2}$, then:

$$\varphi_{i,j,n+1} - \varphi'_{i,j} = \frac{\lambda}{2\overline{\omega}}\varphi'_{i+1,j} - \frac{\lambda}{\overline{\omega}}\varphi'_{i,j} + \frac{\lambda}{2\overline{\omega}}\varphi'_{i-1,j} + \frac{\lambda}{2\overline{\omega}}\varphi_{i,j+1,n+1} - \frac{\lambda}{\overline{\omega}}\varphi_{i,j,n+1} + \frac{\lambda}{2\overline{\omega}}\varphi_{i,j-1,n+1} - \cdots$$

$$\ldots - \frac{\lambda h}{4}\varphi'_{i+1,j}v'_{i,j,n} + \frac{\lambda h}{4}\varphi'_{i-1,j}v'_{i,j,n} - \frac{\lambda h}{4}\varphi_{i,j+1,n+1}\nu'_{i,j,n} + \frac{\lambda h}{4}\varphi_{i,j-1,n+1}\nu'_{i,j,n} + \cdots$$

$$\dots + \frac{8\lambda h^2 FsN\omega}{\overline{w}} \varphi_{i,j,n+1} + \frac{\lambda h^2}{2\overline{\omega}} \overline{B} \qquad (4.2.10).$$

This implies that:

$$\varphi_{i,j,n+1} - \frac{\lambda}{2\overline{\omega}} \varphi_{i,j+1,n+1} + \frac{\lambda}{\overline{\omega}} \varphi_{i,j,n+1} - \frac{\lambda}{2\overline{\omega}} \varphi_{i,j-1,n+1} + \frac{\lambda h}{4} \varphi_{i,j+1,n+1} \nu'_{i,j,n} - \frac{\lambda h}{4} \varphi_{i,j-1,n+1} \nu'_{i,j,n} - \cdots$$

$$\dots - \frac{8\lambda h^2 FsN\omega}{\overline{\omega}} \varphi_{i,j,n+1} = \varphi'_{i,j} + \frac{\lambda}{2\overline{\omega}} \varphi'_{i+1,j} - \frac{\lambda}{\overline{\omega}} \varphi'_{i,j} + \frac{\lambda}{2\overline{\omega}} \varphi'_{i-1,j} - \frac{\lambda h}{4} \varphi'_{i+1,j} \upsilon'_{i,j,n} + \cdots$$

$$\dots + \frac{\lambda h}{4} \varphi'_{i-1,j} \upsilon'_{i,j,n} + \frac{\lambda h^2}{2\overline{\omega}} \overline{B}$$

So,

$$-\left[\frac{\lambda}{2\overline{\omega}} + \frac{\lambda h}{4}\ \nu'_{i,j,n}\right] \varphi_{i,j-1,n+1} + \left[1 + \frac{\lambda(1 - 8h^2 FsN\omega)}{\overline{\omega}}\right] \varphi_{i,j,n+1} - \left[\frac{\lambda}{2\overline{\omega}} - \frac{\lambda h}{4}\ \nu'_{i,j,n}\right] \varphi_{i,j+1,n+1}$$

$$= \left[\frac{\lambda}{2\overline{\omega}} + \frac{\lambda h}{4}\ \upsilon'_{i,j,n}\right] \varphi'_{i-1,j} + \left[1 - \frac{\lambda}{\overline{\omega}}\right] \varphi'_{i,j} + \left[\frac{\lambda}{2\overline{\omega}} - \frac{\lambda h}{4}\ \upsilon'_{i,j,n}\right] \varphi'_{i+1,j} + \frac{\lambda h^2}{2\overline{\omega}} \overline{B} \qquad (4.2.11).$$

By multiplying (4.2.11) by $\frac{2\overline{\omega}}{\lambda}$, we obtain:

$$-\left[1 + \frac{h\overline{\omega}}{2}\ \nu'_{i,j,n}\right] \varphi_{i,j-1,n+1} + 2\left[\frac{\overline{\omega}}{\lambda} + (1 - 8h^2 FsN\omega)\right] \varphi_{i,j,n+1} - \left[1 - \frac{h\overline{\omega}}{2}\ \nu'_{i,j,n}\right] \varphi_{i,j+1,n+1}$$

$$= \left[1 + \frac{h\overline{\omega}}{2}\ \upsilon'_{i,j,n}\right] \varphi'_{i-1,j} + 2\left[\frac{\overline{\omega}}{\lambda} - 1\right] \varphi'_{i,j} + \left[1 - \frac{h\overline{\omega}}{2}\ \upsilon'_{i,j,n}\right] \varphi'_{i+1,j} + h^2 \overline{B} \qquad (4.2.12).$$

By multiplying (4.1.12) by $\frac{1}{1 + \frac{h\overline{\omega}}{2} \nu'_{i,j,n}}$, we get:

$$-\varphi_{i,j-1,n+1} + \frac{2\left[\frac{\overline{\omega}}{\lambda} + (1 - 8h^2 FsN\omega)\right]}{\left[1 + \frac{h\overline{\omega}}{2}\ \nu'_{i,j,n}\right]} \varphi_{i,j,n+1} - \frac{\left[1 - \frac{h\overline{\omega}}{2}\ \nu'_{i,j,n}\right]}{\left[1 + \frac{h\overline{\omega}}{2}\ \nu'_{i,j,n}\right]} \varphi_{i,j+1,n+1}$$

$$= \frac{\left[1 + \frac{h\overline{\omega}}{2}\ \upsilon'_{i,j,n}\right]}{\left[1 + \frac{h\overline{\omega}}{2}\ \nu'_{i,j,n}\right]} \varphi'_{i-1,j} + \frac{2\left[\frac{\overline{\omega}}{\lambda} - 1\right]}{\left[1 + \frac{h\overline{\omega}}{2}\ \nu'_{i,j,n}\right]} \varphi'_{i,j} + \frac{\left[1 - \frac{h\overline{\omega}}{2}\ \upsilon'_{i,j,n}\right]}{\left[1 + \frac{h\overline{\omega}}{2}\ \nu'_{i,j,n}\right]} \varphi'_{i+1,j} + \cdots$$

$$\dots + \frac{h^2 \overline{B}}{\left[1 + \frac{h\overline{\omega}}{2}\ \nu'_{i,j,n}\right]} \qquad (4.2.13).$$

which implies that:

$$-\varphi_{i,j-1,n+1} + \frac{2g_2}{g_5}\varphi_{i,j,n+1} - \frac{g_6}{g_5}\varphi_{i,j+1,n+1} = \frac{g_3}{g_5}\varphi'_{i-1,j} + \frac{2g_1}{g_5}\varphi'_{i,j} + \frac{g_4}{g_5}\varphi'_{i+1,j} + \frac{h^2\overline{B}}{g_5} \quad (4.2.14).$$

where:

$$g_1 = \left[\frac{\overline{\omega}}{\lambda} - 1\right], \quad g_2 = \left[\frac{\overline{\omega}}{\lambda} + (1 - 8h^2FsN\omega)\right], \quad g_3 = \left[1 + \frac{h\overline{\omega}}{2}\, \upsilon'_{i,j,n}\right],$$

$$g_4 = \left[1 - \frac{h\overline{\omega}}{2}\, \upsilon'_{i,j,n}\right], \quad g_5 = \left[1 + \frac{h\overline{\omega}}{2}\, \nu'_{i,j,n}\right], \quad g_6 = \left[1 - \frac{h\overline{\omega}}{2}\, \nu'_{i,j,n}\right]$$

Equation (4.2.14) can be reduced to give:

$$A1(i)\varphi_{i,j-1,n+1} + B1(i)\varphi_{i,j,n+1} + C1(i)\varphi_{i,j+1,n+1} = D1(i) \quad (4.2.15).$$

where $i = 1,2, \ldots\ldots\ldots, M$, and:

$A1(i) = -1$

$B1(i) = 2(g_2/g_5)$

$C1(i) = -(g_6/g_5)$

$D1(i) = (g_3/g_5)\varphi'_{i-1,j} + 2(g_1/g_5)\varphi'_{i,j} + (g_4/g_5)\varphi'_{i+1,j} + \left(h^2\overline{\mathrm{B}}/g_5\right)$

From equation (4.2.14), we create a tri-diagonal system which is solved numerically using the Gauss elimination method and using MATLAB at the time-step $t + 1$ [21].

With boundary conditions:

$$\upsilon' = cons. \quad \upsilon'_{0,j,n} = 0 \ , \ \upsilon'_{i,0,n} = 0$$
$$\nu' = cons. \quad \nu'_{0,j,n} = 0 \ , \ \nu'_{i,0,n} = 0 \quad (4.2.16).$$
$$\varphi_{i,0,n} = 0 \quad \varphi_{i,N,n} = 1.0$$

5. Results and Discussion

The numerical computations were performed using the described alternative direction implicit method on the energy equation to study the effects of physical parameter values that appear in this equation, such as Brinkman parameter Fs , Reynolds parameter Re, and Boltzmann parameter B_0, as shown in Figures 1-4.

In Figure 1, the xy plane represents the location of points on the lattice, that is, the coordinates of the point in the t-matrix resulting from the numerical solution of the energy equation, while the third dimension z represents the temperature values at those points. It is noticed that the increase in Fs causes an increase in temperature inside the boundary layer, as shown in Figure 2. From Figure 3, we observe that the increase in Re causes an increase in temperature inside the boundary layer. Finally, we found that the decrease in B_0 causes an increase in temperature inside the boundary layer, as shown in Figure 4.

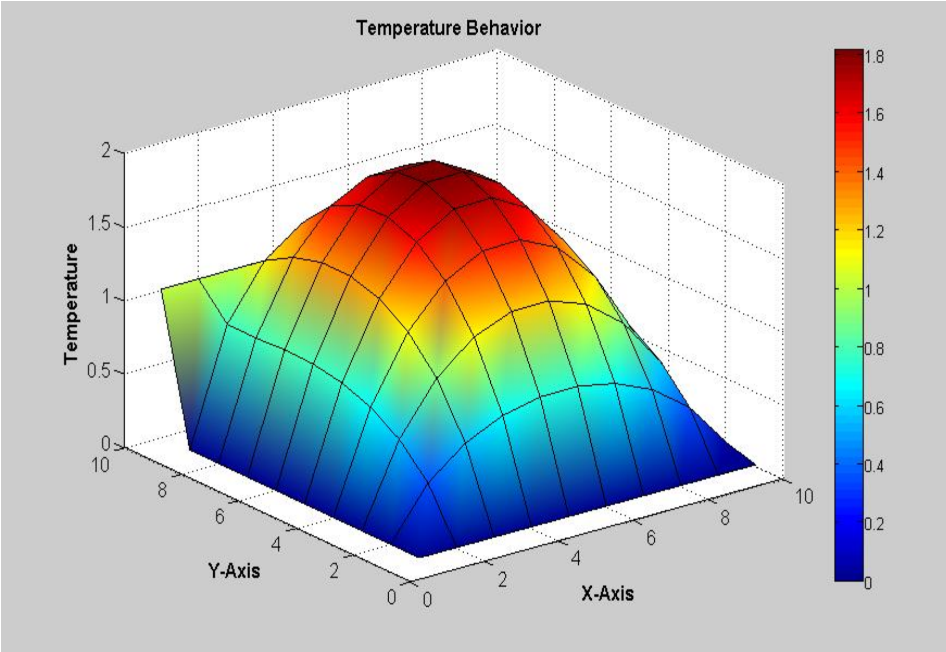

Figure 1- Temperature Behaviour

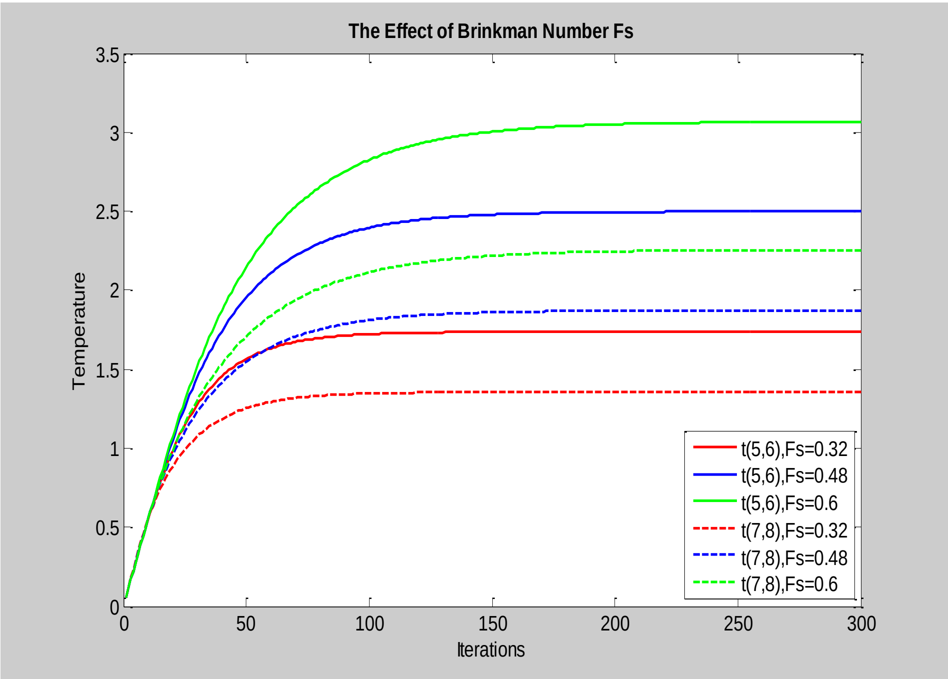

Figure 2- The effects of Brinkman Number Fs at two selected points with $Re = 1000$, $B_0 = 15$

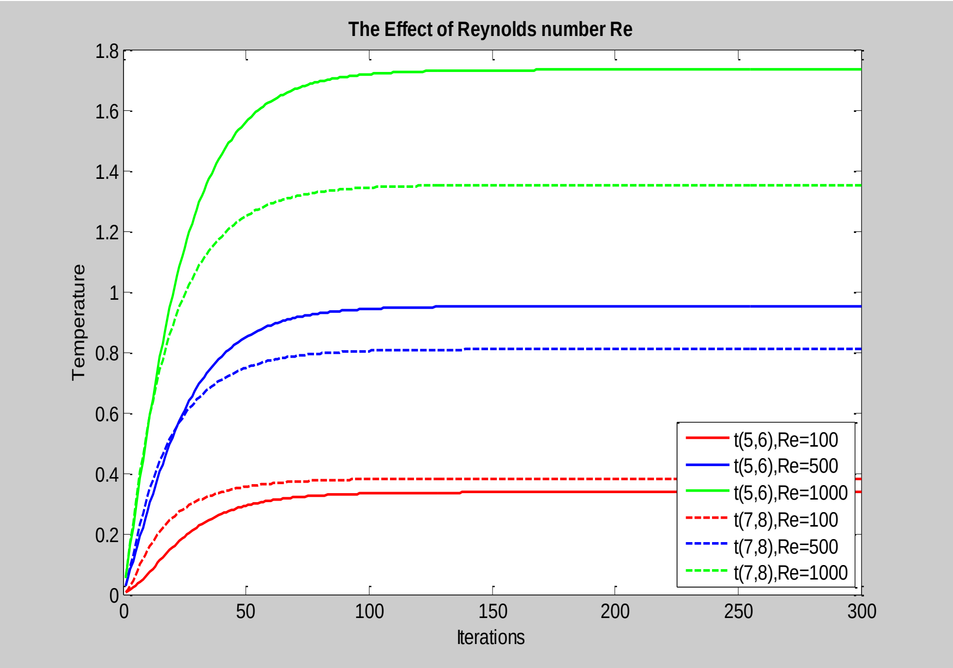

Figure 3- The effects of Reynolds Number Re at two selected points with $Fs = 0.32$, B_0=15

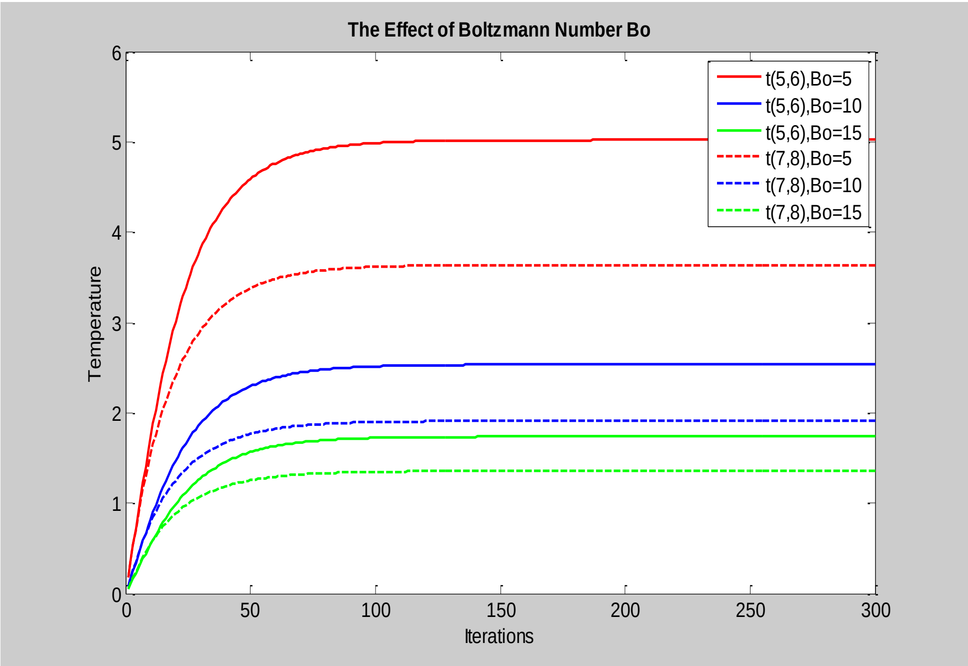

Figure 4- The effects of Boltzmann Number B_0 at two selected points with $Re = 1000$, $Fs = 0.32$

6. Conclusions

1. The present study is concerned with the energy equation, that represents a partial differential equation obtained from fluid flow in channels under the influence of magnetic field, perpendicular to the channel, with the presence of radiation. The study shows the acquisition of the steady state from the unsteady state. We reached several results; the temperature inside the boundary layer increases when the Brinkman number increases. The temperature inside the boundary layer increases when the Reynolds number increases. The temperature inside the boundary layer decreases when the Boltzmann number increases.

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