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Blow-up Properties of a Coupled System of Reaction-Diffusion Equations

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Abstract

This paper is concerned with a Coupled Reaction-diffusion system defined in a ball with homogeneous Dirichlet boundary conditions. Firstly, we studied the blow-up set showing that, under some conditions, the blow-up in this problem occurs only at a single point. Secondly, under some restricted assumptions on the reaction terms, we established the upper (lower) blow-up rate estimates. Finally, we considered the Ignition system in general dimensional space as an application to our results.

Keywords: Reaction-diffusion system; Ignition system; Dirichlet boundary conditions; Blow-up set; Blow-up rate estimate.

خصائص التفجير لنظام مزدوج من معادلات الانتشار ورد الفعل

معن عبد الكاظم رشيد

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الخلاصة

يهتم هذا البحث بنظام مزدوج من معادلات الانتشارورد الفعل المعرفة في كرة،مع شروط ديريتشليت الحدودية المتجانسة. أولاً، نقوم بدراسة مجموعة التفجير التي توضح أنه ، مع فرض بعض الشروط على النظام، يحدث الانفجار في هذه المسألة في نقطة واحدة فقط. ثانيًا، مع بعض الافتراضات المقيدة على حدود ردود الفعل، أنشأنا تقديرات نسبة الانفجار العلوي (السفلي). أخيرًا ، قمنا بدراسة نظام الإشتعال في الفضاء ذي البعد العام كتطبيق للنتائج.

1. Introduction

It is well known that many phenomena in the world can be described using partial differential equations. Therefore, since the last decades, the analytical and numerical solutions of partial differential equations have been studied by many authors, see for instance [1,2]. One of the remarkable phenomena in time-dependent problems is the blow-up, which has been considered by many authors (for a single equation and systems), see for instance [3-5]. This work is concerned with the blow-up properties of a Coupled Reaction-diffusion system defined in a ball with homogeneous Dirichlet boundary conditions:

$$\left.\begin{array}{lll} u_t = \Delta u + f(v), & v_t = \Delta v + g(u), & (x,t) \in B_R \times (0,T), \\ u(x,t) = 0, & v(x,t) = 0, & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in B_R, \end{array}\right\} \quad (1)$$

where $f, g \in C^1(R) \cap C^2(R\backslash\{0\})$ are positive and increasing superlinear functions on $(0,\infty)$, $1/f, 1/g$ being integrable at infinity. Moreover, $f'g', f'', g''$ are positive functions in $(0,\infty)$, u_0 and v_0 are nonnegative, smooth, redial non-increasing, and vanishing on ∂B_R. That is:

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$$\left.\begin{array}{lll} u_0(x) = u_0(|x|), & v_0(x) = v_0(|x|), & x \in B_R, \\ u_0(x) = 0, & v_0(x) = 0, & x \in \partial B_R, \\ u_{0r}(|x|) \le 0, & v_{0r}(|x|) \le 0, & x \in B_R. \end{array}\right\} \quad (2)$$

In addition, the following conditions are assumed to be satisfied:

$$\Delta u_0 + f(v_0) \ge 0, \quad \Delta v_0 + g(u_0) \ge 0, \quad \forall x \in B_R. \quad (3)$$

In fact, Problem (1) has been used to describe physical models arising in many fields of sciences [6]; for instance, the chemical concentration, the temperature, and in the chemical reaction process. The coupled reaction-diffusion systems defined in a ball with homogeneous Dirichlet boundary conditions have been studied in [6- 9].

In [7], problem (1) has been considered in one dimensional space:

$$u_t = u_{xx} + f(v), \qquad v_t = v_{xx} + g(u), \quad (x,t) \in (-R,R) \times (0,T),$$

Under some assumptions on f and g, it has been shown that the blow-up can only occur at a single point. As applications to that result, two special cases of f, g where considered: the power forms and the exponential forms.

Later, problem (1) has been studied in a general dimensional space [9], where f and g are of power type functions:

$$u_t = \Delta u + v^p, \quad v_t = \Delta v + u^q, \quad (x,t) \in B_R \times (0,T) \quad p, q > 1. (4)$$

It has been proved that the blow-up can only accour at a single point. In addition, the lower point-wise estimates are as follows:

$$u(x,T) \ge c_1 |x|^{-2\alpha}, \qquad v(x,T) \ge c_2 |x|^{-2\beta},$$

where

$$\alpha = \frac{p+1}{pq-1}, \quad \beta = \frac{q+1}{pq-1}.$$

In [6], it was shown that the upper and lower blow-up rate estimates of this problem are as follows:

$$c_1(T-t)^{-\alpha} \le u(0,t) \le c_2(T-t)^{-\alpha}, \quad t \in (0,T),$$
$$c_3(T-t)^{-\beta} \le u(0,t) \le c_4(T-t)^{-\beta}, \quad t \in (0,T).$$

For another special case of problem (1), where f, g are of exponential type, we have

$$u_t = \Delta u + e^{pv}, \qquad v_t = \Delta v + e^{qu}, \quad (x,t) \in B_R \times (0,T), \quad p, q > 0. \quad (5)$$

It has been proved that the only blow-up point is $x = 0$ and the upper (lower) blow-up rate estimates are as follows [6]:

$$\log c - \log[q(T-t)] \le qu(0,t) \le \log C - \log[q(T-t)], \quad t \in (0,T),$$
$$\log c - \log[p(T-t)] \le pv(0,t) \le \log C - \log[p(T-t)], \quad t \in (0,T).$$

In this paper, under some conditions on the reaction terms, f and g, we prove that blow-up, in problem (1), occurs only at a single point. Moreover, we established the upper (lower) blow-up rate estimates. In addition, the Ignition system [10] will be considered in a general dimensional space as an application to our result.

2. Preliminaries

By the standard parabolic theory, the local existence and uniqueness of classical solutions to problem (1) are guaranteed [11]. In addition, for many types of the functions, $f$and g, if the initial functions (u_0, v_0) are suitably large, then $T < \infty$ [12, 13]. Moreover, only simultaneous blow-up can occur and that is because the system in (1) is coupled.

In the next lemma, we present some properties to the solutions of problem (1)-(2). For simplicity, we denote $u(r,t) = u(x,t), v(r,t) = v(x,t)$.

Lemma 1,[14]: Let (u, v) be a classical solution to the problem (1), (2). Then

1. u and v are positive and radial.
2. $u_r \le 0, v_r \le 0$ in $[0,R) \times (0,T)$. Moreover, $u_r < 0, v_r < 0$ in $(0,R] \times (0,T)$.
3. $u_t > 0, v_t > 0, \quad (x,t) \in B_R \times (0,T)$.
4. If (u, v) blows-up, then $x = 0$ belongs to the blow-up set.

3. Blow-up Set

Under some assumptions, the next theorem shows that the only possible blow-up point to problem (1)-(2) is $x = 0$.

Theorem 1: Let (u, v) be a blow-up solution to problem (1)-(2). Assume that

$$u_{0r}(r) \le -\delta_1 r, v_{0r}(r) \le -\delta_2 r \quad for\ 0 < r \le R, \quad \text{where } \delta_1, \delta_2 > 0. \quad (6)$$

If there exist two functions $F,G \in C^2([0,\infty))$ such that $F,G > 0; F',G',F'',G'' \geq 0$ in $(0,\infty)$. In addition, the following conditions are assumed to be satisfied:

$$\int_s^\infty \frac{dv}{F(v)} < \infty, \quad \int_s^\infty \frac{du}{G(u)} < \infty, \quad for \quad s > 0,$$

$$\left.\begin{aligned} &f'(v)F(v) - f(v)G'(u) \geq 2\varepsilon G(u)G'(u), \quad in \quad (0,R)\times(0,T), \\ &g'(u)G(u) - g(u)F'(v) \geq 2\varepsilon F(v)F'(v), \quad in \quad (0,R)\times(0,T), \end{aligned}\right\} \tag{7}$$

for some $\varepsilon \in (0,1)$, then $x = 0$ is the only possible blow-up point .

Proof

Following the technique used in [15] for the scalar problem:

As in Lemma 1, and since (u,v) are radial, for simplicity we denote

$$u(r,t) = u(x,t), \quad v(r,t) = v(x,t).$$

By the new variables, system (1) can be rewritten as follows:

$$\left.\begin{aligned} &u_t = u_{rr} + \tfrac{n-1}{r}u_r + f(v), \qquad (r,t) \in (0,R)\times(0,T), \\ &v_t = v_{rr} + \tfrac{n-1}{r}v_r + g(u), \qquad (r,t) \in (0,R)\times(0,T). \end{aligned}\right\} \tag{8}$$

We set that $J_1 = r^{n-1}u_r + \varepsilon r^n G(u), \quad J_2 = r^{n-1}v_r + \varepsilon r^n F(v).$

By the parabolic regularity results, we obtain

$$u_r, v_r \in C^{2,1}((0,R)\times(0,T)) \cap C([0,R]\times[0,T)).$$

Since $F,G \in C^2([0,\infty))$, it follows that

$$J_1, J_2 \in C^{2,1}\big((0,R)\times(0,T)\big) \cap C\big([0,R]\times[0,T)\big),$$

For convenience, we denote

$$w_1 = r^{n-1}u_r, \quad w_2 = r^{n-1}v_r, \quad , c(r) = \varepsilon r^n,$$

thus

$$J_1 = w_1 + c(r)G(u), \quad J_2 = w_2 + c(r)F(v).$$

For simplicity, we write $w_1 = w$.

A direct calculation shows

$$\begin{aligned} &w_t = r^{n-1}u_{rt}, \\ &w_r = r^{n-1}u_{rr} + (n-1)r^{n-2}u_r, \\ &w_{rr} = r^{n-1}u_{rrr} + (n-1)r^{n-2}u_{rr} + (n-1)(n-2)r^{n-3}u_r \\ &+(n-1)r^{n-2}u_{rr}. \end{aligned}$$

This leads to

$$\begin{aligned} w_t + \tfrac{n-1}{r}w_r - w_{rr} &= r^{n-1}u_{rt} + (n-1)r^{n-2}u_{rr} + (n-1)^2 r^{n-3}u_r \\ &-r^{n-1}u_{rrr} - (n-1)r^{n-2}u_{rr} - (n-1)(n-2)r^{n-3}u_r - (n-1)r^{n-2}u_{rr}. \end{aligned}$$

From (8), it follows that

$$u_{rrr} = u_{tr} - \tfrac{n-1}{r}u_{rr} + \tfrac{n-1}{r^2}u_r - f'v_r.$$

Thus

$$w_{1t} + \tfrac{n-1}{r}w_{1r} - w_{1rr} = w_2 f'(v).$$

In the same, way we can show that

$$w_{2t} + \tfrac{n-1}{r}w_{2r} - w_{2rr} = w_1 g'(u).$$

Also, it is clear that

$$\begin{aligned} &[c(r)G(u)]_t = c(r)G'(u)u_t = \varepsilon r^n G'(u)(u_{rr} + \tfrac{n-1}{r}u_r + f(v)), \\ &[c(r)G(u)]_r = \varepsilon r^n G'(u)u_r + \varepsilon n G(u)r^{n-1}, \\ &\tfrac{(n-1)}{r}[c(r)G(u)]_r = \varepsilon(n-1)r^{n-1}G'(u)u_r + \varepsilon n(n-1)G(u)r^{n-2}, \\ &[c(r)G(u)]_{rr} = \varepsilon r^n(G'(u)u_{rr} + u_r^2 G''(u)) + \varepsilon G'(u)u_r n r^{n-1} \\ &\quad +\varepsilon n G(u)(n-1)r^{n-2} + \varepsilon n r^{n-1}G'(u)u_r. \end{aligned}$$

From above, it follows that

$$\begin{aligned} J_{1t} + \tfrac{n-1}{r}J_{1r} - J_{1rr} &= f'(v)[J_2 - \varepsilon r^n F(v)] + \varepsilon r^n G'(u)f(v) \\ &-2\varepsilon G'(u)[r^{n-1}u_r] - \varepsilon r^n G''(u)u_r^2. \end{aligned}$$

Using the relation $r^{n-1}u_r = w_1 = J_1 - \varepsilon r^n G(u)$, we obtain

$$J_{1t}+\frac{n-1}{r}J_{1r}-J_{1rr}\le f'(v)[J_2-\varepsilon r^n F(v)]$$
$$+\varepsilon r^n G'(u)f(v)-2\varepsilon G'(u)[J_1-\varepsilon r^n G(u)]$$

Thus

$$J_{1t}+\frac{n-1}{r}J_{1r}-J_{1rr}-bJ_1-cJ_2\le -\varepsilon r^n H, \qquad (9)$$

where

$$H=F(v)f'(v)-f(v)G'(u)-2\varepsilon G(u)G'(u);\ b=-2\varepsilon G'(u), c=f'(v).$$

From our assumption (7), it follows that $H\ge 0$ in $(0,R)\times(0,T)$.
Thus

$$J_{1t}+\frac{n-1}{r}J_{1r}-J_{1rr}-bJ_1-cJ_2\le 0,\quad (x,t)\in(0,R)\times(0,T).$$

In the same way, we can show that

$$J_{2t}+\frac{n-1}{r}J_{2r}-J_{2rr}-dJ_2-hJ_1\le 0,\quad (x,t)\in(0,R)\times(0,T),$$

where $d=-2\varepsilon F'(v), h=g'(u)$.
Clearly, c,h,d and b are bounded functions on $(0,R)\times[0,t]$ for any fixed $t\in(0,T)$, moreover, $c,h\ge 0$.
Also,

$$J_1(0,t)=J_2(0,t)=0,\quad t\in(0,T).$$

By (6), we obtain

$$J_1(r,0)=r^{n-1}[u_{0r}(r)+\varepsilon rG(u_0(r))]\le r^n[-\delta_1+\varepsilon G(u_0(r))],$$
$$J_2(r,0)=r^{n-1}[v_{0r}(r)+\varepsilon rF(v_0(r))]\le r^n[-\delta_2+\varepsilon F(v_0(r))].$$

Since u,v are increasing in time, then we obtain

$$u>u_0, v>v_0,\quad (x,t)\in B_R\times(0,T),$$

Moreover, we can easily show that

$$u_r(R,t)<u_{0r}(R)<0,\quad v_r(R,t)<v_{0r}(R)<0,\quad t\in(0,T).$$

Thus

$$J_1(R,t)\le r^{n-1}[u_{0r}(R)+\varepsilon rG(0)]\le R^n[-\delta_1+\varepsilon G(0)],\quad t\in(0,T),$$
$$J_2(R,t)\le r^{n-1}[v_{0r}(R)+\varepsilon rF(0)]\le R^n[-\delta_2+\varepsilon F(0)],\quad t\in(0,T).$$

Therefore, each of the functions $J_1(r,0), J_2(r,0), J_1(R,t)$, and $J_2(R,t)$ are nonpositive, for $r\in(0,R), t\in(0,T)$, provided that

$$\varepsilon\le\min\{\frac{\delta_1}{\max_{[0,R]}G(u_0)},\frac{\delta_2}{\max_{[0,R]}F(v_0)}\}.$$

From above and the maximum principle [16], it follows that

$$J_1, J_2\le 0,\quad (x,t)\in B_R\times(0,T). \qquad (10)$$

We define

$$G^*(s)=\int_s^\infty\frac{du}{G(u)},\quad F^*(s)=\int_s^\infty\frac{dv}{F(v)}.$$

From (10), it follows that

$$\frac{-u_r}{G(u)}\ge\varepsilon r$$

Clearly,

$$\frac{d}{dr}G^*(u(r,t))=\frac{d}{dr}\int_u^\infty\frac{du}{G(u)}=-\frac{d}{dr}\int_\infty^u\frac{du}{G(u)}=-\frac{d}{du}\int_\infty^u\frac{u_r}{G(u)}du=-\frac{u_r}{G(u)}.$$

Thus

$$G^*(u(r,t))_r\ge\varepsilon r.$$

Now, we integrate the last equation from 0 to r, as follows

$$G^*(u(r,t))-G^*(u(0,t))\ge\frac{1}{2}\varepsilon r^2.$$

It follows that

$$G^*(u(r,t))\ge\frac{1}{2}\varepsilon r^2. \qquad (11)$$

In the same way, we can show that

$$F^*(v(r,t))\ge\frac{1}{2}\varepsilon r^2. \qquad (12)$$

If for some $r>0$; $u(r,t)\to\infty$ or $v(r,t)\to\infty$ as $t\to T$, then $G^*(u(r,t))\to 0$ or $F^*(v(r,t))\to 0$ as $t\to T$, which is a contradiction to (11), (12).

It follows that if $x \neq 0$, then it cannot be a blow-up point. Therefore, under the assumption of this theorem, the blow-up in problem (1)-(2) can only occur at a single point, which is $x = 0$.

4. Blow-up Rate Estimates

In this subsection, we consider the lower (upper) blow-up rate estimates for problem (1)-(2) with some restricted assumptions on f, g.

Theorem 2: Let (u, v) be a solution to (1), (2), which blows up at only one point (x = 0). Assume that there exsits $\gamma > 1$, such that

$$g(u) \leq \gamma f(v), \quad f(v) \leq \gamma g(u), \quad (x,t) \in B_R \times (0,T). \tag{13}$$

Then, there are four positive constants c_1, c_2, c_3 and c_4, such that

$$G_1^{-1}(c_1(T-t)) \leq u(0,t) \leq G_1^{-1}(c_2(T-t)), \quad t \in (0,T), \tag{14}$$

$$F_1^{-1}(c_3(T-t)) \leq v(0,t) \leq F_1^{-1}(c_4(T-t)), \quad t \in (0,T), \tag{15}$$

where

$$G_1(s) = \int_s^\infty \frac{du}{g(u)}, \quad F_1(s) = \int_s^\infty \frac{dv}{f(v)}. \tag{16}$$

Proof

Firstly, we derive the lower blow-up rate estimates.

We set $U(t) = u(0,t), \quad V(t) = v(0,t), \quad t \in [0,T).$

Since (u, v) attains its maximum at $x = 0$, we obtain

$$\Delta U(t) \leq 0, \quad \Delta U(t) \leq 0, \quad 0 \leq t < T.$$

From (1), it follows that

$$U_t(t) \leq f(V(t)), \quad V_t(t) \leq g(U(t)), \quad 0 < t < T. \tag{17}$$

From (13) and (17), it follows that

$$U_t(t) \leq \gamma g(U(t)), \quad V_t(t) \leq \gamma f(V(t)), \quad 0 < t < T.$$

Thus

$$\frac{U_t(t)}{g(U(t))} \leq \gamma, \quad \frac{V_t(t)}{f(V(t))} \leq \gamma, \quad 0 < t < T. \tag{18}$$

Clearly,

$$-\frac{dG_1(u(0,t))}{dt} = -\frac{d}{dt}\int_{u(0,t)}^\infty \frac{du}{g(u(0,t))} = -\frac{d}{dt}\int_t^T \frac{(du/dt)}{g(u(0,t))}dt = \frac{d}{dt}\int_T^t \frac{u_t}{g(u(0,t))}dt,$$

this leads to

$$-\frac{dG_1(u(0,t))}{dt} = \frac{u_t(0,t)}{g(u(0,t))},$$

where G_1 is defined as in (16).

From the above and equation (18), we obtain

$$-\frac{dG_1(u)}{dt} \leq \gamma, \quad 0 < t < T. \tag{19}$$

By integrating (19) from t to T, we obtain

$$G_1(u(0,t)) - G_1(u(0,T)) \leq \gamma(T-t).$$

Clearly, $G_1(u(0,T)) = 0$.

Thus

$$G_1(u(0,t)) \leq \gamma(T-t), \quad 0 < t < T.$$

Since G_1 is decreasing, then by the last equation, we have

$$u(0,t) \geq G_1^{-1}(\gamma(T-t)), \quad 0 < t < T.$$

For v, in the same way, we can show that

$$v(0,t) \geq F_1^{-1}(\gamma(T-t)), \quad 0 < t < T.$$

Next, we consider the upper bounds.

We define the functions Q, H as follows

$$Q(x,t) = u_t - \theta f(v), \quad H(x,t) = v_t - \theta g(u), \quad (x,t) \in B_R \times (0,T),$$

where $\theta > 0$. By the parabolic regularity, we have

$$u_t, v_t \in C^{2,1}(B_R \times (0,T)) \cap C(\overline{B}_R \times [0,T)),$$

and since $f, g \in C^2(0,\infty) \cap C([0,\infty))$, it follows that

$$F, G \in C^{2,1}(B_R \times (0,T)) \cap C(\overline{B}_R \times [0,T)).$$

A direct calculation shows

$$Q_t - \Delta Q = u_{tt} - \theta f' v_t - \Delta u_t + \theta \Delta f(v),$$

$$= u_{tt} - \Delta u_t - \theta f'[v_t - \Delta v] + \theta |\nabla v|^2 f'',$$

$= f'v_t - \theta f' g(u) + \theta |\nabla v|^2 f''.$

Thus

$Q_t - \Delta Q - f'(v)H = \theta |\nabla v|^2 f'' \geq 0, \quad in \quad B_R \times (0,T),$ due to the positivity of the functions f'' in $(0,\infty)$.

In the same we can show that

$$H_t - \Delta H - g'(u)Q = \theta |\nabla u|^2 g'' \geq 0, \quad in \quad B_R \times (0,T).$$

Since f', g' are continuous functions, so, $f'(v), g'(u)$ are bounded in $\overline{B}_R \times [0,t]$ for $t < T$.

By Lemma1, $u_t, v_t > 0, in\ B_R \times (0,T)$, and since u, v blow up at $x = 0$, therefore, there exist $k_1 > 0, k_2 > 0, \varepsilon \in (0,R), \tau \in (0,T)$ such that

$$u_t(x,t) \geq k_1, \quad v_t(x,t) \geq k_2, \quad (x,t) \in \overline{B}_\varepsilon \times [\tau, T).$$

Also, we can find $\theta > 0$ such that

$$u_t(x,\tau) \geq \theta f(v(x,\tau)), v_t(x,\tau) \geq \theta g(u(x,\tau)), \quad for\ x \in B_\varepsilon.$$

Thus

$$F(x,\tau) \geq 0, G(x,\tau) \geq 0 \quad for\ x \in B_\varepsilon.$$

Since u, v blow up only at $x = 0$, then there exists $C_1, C_2 > 0$ such that

$$f(v(x,t)) \leq C_1 < \infty, g(u(x,t)) \leq C_2 < \infty, \quad in \quad \partial B_\varepsilon \times (0,T),$$

If we choose that θ is small enough such that

$$\theta \leq \min\{\frac{k_1}{C_1}, \frac{k_2}{C_2}\},$$

then, we can get

$$F(x,t) \geq 0, G(x,t) \geq 0 \quad (x,t) \in \partial B_\varepsilon \times [\tau, T),$$

From the above and by the maximum principle [16], starting from τ instead of zero, it follows that

$$F(x,t) \geq 0, G(x,t) \geq 0 \quad (x,t) \in \overline{B}_\varepsilon \times (\tau, T).$$

This leads to

$$u_t(0,t) \geq \theta f(v(0,t)), \quad v_t \geq \theta g(u(0,t)), \quad for \quad \tau \leq t < T. \tag{20}$$

By (13), we obtain

$$u_t(0,t) \geq \frac{\theta}{\gamma} g(u(0,t)), \quad v_t \geq \frac{\theta}{\gamma} f(v(0,t)), \quad \tau \leq t < T. \tag{21}$$

Since $-\frac{dG_1(u(0,t))}{dt} = \frac{u_t(0,t)}{g(u(0,t))}$, from (21) and the last equation, it follows that

$$-\frac{dG_1(u(0,t))}{dt} \geq \frac{\theta}{\gamma}, \quad \tau \leq t < T.$$

By integrating the last inequality from t to T, we obtain

$$\int_t^T - dG_1(u(0,t)) = G_1(u(0,t)) - G_1(u(0,T)) \geq \frac{\theta}{\gamma}(T-t).$$

Thus

$$G_1(u(0,t)) \geq \frac{\theta}{\gamma}(T-t), \quad \tau \leq t < T. \tag{22}$$

Since G_1 is decreasing, by (22), it follows that

$$u(0,t) \leq G_1^{-1}(\frac{\theta}{\gamma}(T-t)), \quad \tau \leq t < T.$$

So, there is $c_2 > 0$ such that

$$u(0,t) \leq G_1^{-1}(c_2(T-t)), \quad 0 < t < T.$$

Similarly, we can find $c_4 > 0$ such that

$$v(0,t) \leq F_1^{-1}(c_4(T-t)), \quad 0 < t < T.$$

5. The Ignition System

In this section, we apply Theorem 1 and Theorem 2 to the so called Ignition system [7] which takes the following form:

$$\left.\begin{array}{lll} u_t = \Delta u + Ae^v, & v_t = \Delta v + Be^u, & (x,t) \in B_R \times (0,T), \\ u(x,t) = 0, & v(x,t) = 0, & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in B_R, \end{array}\right\} \tag{23}$$

where $A, B > 0$

In order to show that the condition (13) is satisfied for system (23), we need to prove the following lemma.

Lemma 2: Let (u,v) be a solution to problem (23) with (2).Then there exists $M > 1$ such that

$$e^v \le Me^u, \quad e^u \le Me^v, \quad (x,t) \in B_R \times (0,T). \tag{24}$$

Proof

We define $J = Me^u - e^v, \quad (x,t) \in B_R \times (0,T)$.

Clearly, $J \in C^{2,1}(\overline{\Omega} \times [0,T))$. A direct calculation shows that

$$J_t = Me^u u_t - e^v v_t,$$
$$\nabla J = Me^u \nabla u - \nabla v e^v, \tag{25}$$
$$\Delta J = Me^u \Delta u + Me^u |\nabla u|^2 - e^v \Delta v - e^v |\nabla v|^2.$$

Thus

$$J_t - \Delta J = Me^u[u_t - \Delta u] - e^v[v_t - \Delta v] + e^v|\nabla v|^2 - Me^u|\nabla u|^2$$
$$= (MA - B)e^{u+v} + e^v|\nabla v|^2 - Me^u|\nabla u|^2. \tag{26}$$

From (25), we obtain

$$\nabla u = \frac{1}{Me^u}[\nabla v e^v + \nabla J].$$

This leads to

$$|\nabla u|^2 = \frac{1}{M^2 e^{2u}}[e^{2v}|\nabla v|^2 + 2e^v \nabla v \cdot \nabla J + |\nabla J|^2].$$

Therefore, (26) becomes

$$J_t - \Delta J = (MA - B)e^{u+v} + [e^v - \frac{e^{2v}}{Me^u}]|\nabla v|^2 - [\frac{2e^v}{Me^u}\nabla v + \frac{1}{Me^u}\nabla J] \cdot \nabla J\,,$$

Since $e^v - \frac{e^{2v}}{Me^u} = e^v \frac{J}{Me^u}$, we can rewrite the last equation as follows:

$$J_t - \Delta J - b \cdot \nabla J - cJ = (MA - B)e^{u+v} \ge 0, \quad (x,t) \in B_R \times (0,T),$$

Provided that $M \ge B/A$, where

$$b = -[\frac{2e^v}{Me^u}\nabla v + \frac{1}{Me^u}\nabla J], c = \frac{e^v}{Me^u}|\nabla v|^2.$$

It is clear that, c is bounded in $B_R \times (0,t]$, for $t < T$.

Moreover, $J(R,\cdot) = M - 1 > 0$ and $J(\cdot,0) = Me^{u_0} - e^{v_0} \ge 0$, provided that M is large enough.

From the above and the maximum principle [16], we deduce that

$$J \ge 0, \quad (x,t) \in B_R \times (0,T).$$

In a similar way, one can show that the function $H = Me^v - e^u \ge 0$ in $B_R \times (0,T)$.

Next, we apply Theorem 1 to the ignition system (23) with (2) by choosing appropriate forms for F, G.

Theorem 3: Let (u,v) be a blow-up solution to problem (23) with (2), where (u_0, v_0) satisfies (6). Then, the blow-up can only accour at a single point. In addition, the point-wise estimates are as follows

$$u \le \log C - \frac{2}{\alpha}\log(r), \quad v \le \log C - \frac{2}{\alpha}\log(r), \quad (r,t) \in (0,R] \times (0,T).$$

Proof

$$\text{Let} \quad F(v) = e^{\alpha v}, G(u) = e^{\alpha u}, \quad \alpha \in (0,1). \tag{27}$$

Firstly, we aim to show that F, G satisfy the condition (7).

By some direct calculations, we obtain

$$f'F - fG' = Ae^v[e^{\alpha v} - \alpha e^{\alpha u}]. \tag{28}$$

By (24) , it follows that

$$e^v \ge \frac{1}{M}e^u, \quad (x,t) \in B_R \times (0,T).$$

Thus, (28) becomes

$$f'F - fG' \ge \frac{A}{M}e^u[\frac{1}{M^\alpha}e^{\alpha u} - \alpha e^{\alpha u}]$$
$$\ge \frac{A}{M}\left[\frac{1}{M^\alpha} - \alpha\right]e^{2\alpha u} \ge 2\varepsilon\alpha e^{2\alpha u} = 2\varepsilon GG',$$

Provided that $\alpha < \frac{1}{M}$, ε is small enough, such that

$$\varepsilon \le \frac{A}{2M}[\frac{1}{\alpha M^\alpha} - 1].$$

In the same way, we can show that

$$g'G - gF' \ge \frac{B}{M}[\frac{1}{M^\alpha} - \alpha]e^{2\alpha v} \ge 2\varepsilon\alpha e^{2\alpha v} = 2\varepsilon FF', \tag{29}$$

Provided that

$$\varepsilon \le \frac{B}{2M}[\frac{1}{\alpha M^\alpha} - 1].$$

So, the condition (7) is satisfied.

Thus, by Theorem 1, we obtain that $x = 0$ is the only possible blow-up point.

Next, we derive the point-wise estimates.
We define the functions G^*, F^* as follows

$$G^*(s) = \int_s^\infty \frac{du}{G(u)}, \quad F^*(s) = \int_s^\infty \frac{dv}{F(v)}, \quad s \geq 0.$$

By (27), we obtain

$$G^*(s) = F^*(s) = \frac{1}{\alpha e^{\alpha s}}, \quad s > 0.$$

Therefore, (11) and (12) become

$$\frac{1}{\alpha e^{\alpha u}} \geq \varepsilon \frac{r^2}{2}, \quad \frac{1}{\alpha e^{\alpha v}} \geq \varepsilon \frac{r^2}{2}.$$

Thus

$$e^{\alpha u} \leq \frac{2}{\alpha \varepsilon r^2}, \quad e^{\alpha v} \leq \frac{2}{\alpha \varepsilon r^2}, \quad (r,t) \in (0,R] \times (0,T),$$

or $\quad u \leq \log C - \frac{2}{\alpha}\log(r), \quad v \leq \log C - \frac{2}{\alpha}\log(r), \quad (r,t) \in (0,R] \times (0,T).$

Next, we apply theorem (2) to derive the upper (lower) blow-up rate estimates for problem (23) with (2).

Theorem 4: Let (u, v) be a blow-up solution to problem (23) with (2). Assume that (u_0, v_0) satisfies (6). Then, there exist positive constants C_i , $i = 1,2,3,4$, such that

$$\log C_1 - \log(T-t) \leq u(0,t) \leq \log C_2 - \log(T-t), \quad t \in (0,T),$$
$$\log C_3 - \log(T-t) \leq v(0,t) \leq \log C_4 - \log(T-t), \quad t \in (0,T),$$

Proof

We define the functions G_1, F_1 as follows

$$G_1(s) = \int_s^\infty \frac{du}{Be^u}, \quad F_1(s) = \int_s^\infty \frac{dv}{Ae^v} ds.$$

It is obviously that

$$G_1(s) = \frac{1}{Be^s}, \quad F_1(s) = \frac{1}{Ae^s}, \quad s \geq 0.$$

Moreover,

$$G_1^{-1}(s) = -\log(Bs), \quad F_1^{-1}(s) = -\log(As), \quad s > 0.$$

Therefore, from (14) it follows that

$$-\log(Bc_1(T-t)) \leq u(0,t) \leq -\log(Bc_2(T-t)), \quad t \in (0,T).$$

Thus, there exist $C_1, C_1 > 0$ such that

$$\log C_1 - \log(T-t) \leq u(0,t) \leq \log C_2 - \log(T-t), \quad t \in (0,T).$$

By using the same way, we can find $C_3, C_4 > 0$ such that

$$\log C_3 - \log(T-t) \leq v(0,t) \leq \log C_4 - \log(T-t), \quad t \in (0,T).$$

6. Conclusions

In this paper, we studied the blow-up set and the upper (lower) blow-up rate estimates for a Coupled Reaction-diffusion system defined in a ball with homogeneous Dirichlet boundary conditions, under some assumptions. We note that the results of the present work can be applied to many types of systems, including the ignition system. Therefore, by these results, we can easily understand the blow-up properties and profiles of such types of systems.

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