Mustafa et al. Iraqi Journal of Science, 2021, Vol. 62, No. 6, pp: 1986-1993
DOI: 10.24996/ijs.2021.62.6.24

ISSN: 0067-2904

M_n – Polynomials of Some Special Graphs

Raghad A. Mustafa[1], Ahmed M. Ali[1], AbdulSattar M. Khidhir[2]

[1]University of Mosul, Colleges of Computer Science and Mathematics.

[2]Computer Center, Northern Technical University

Received: 2/9/2020 Accepted: 30/11/2020

Abstract

Let G be a connected graph with vertices set $V = V(G)$ and edges set $E = E(G)$. The ordinary distance between any two vertices of $V(G)$ is a mapping d from $V \times V$ into a nonnegative integer number such that $d(v, u)$ is the length of a shortest $(v - u) -$ path. The maximum distance between two subsets $\acute{S}$ and S of $V(G)$ is the maximum distance between any two vertices v and u such that v belong to $\acute{S}$ and u belong to S. In this paper, we take a special case of maximum distance when $\acute{S}$ consists of one vertex and S consists of $(n - 1)$ vertices, $n \geq 3$. This distance is defined by:

$d_{\max}(v, S) = \max\{d(v, u): u \in S\}, |S| = n - 1, 3 \leq n \leq p, v \in V(G), v \notin S$,

where p is the order of a graph G.

In this paper, we defined M_n – polynomials based on the maximum distance between a vertex v in $V(G)$ and a subset S that has $(n - 1) -$vertices of a vertex set of G and M_n – index. Also, we find $M_n -$polynomials for some special graphs, such as: complete, complete bipartite, star, wheel, and fan graphs, in addition to $M_n -$ polynomials of path, cycle, and Jahangir graphs. Then we determine the indices of these distances.

Keywords: max–n–distance, M_n –Polynomial , M_n –index .

متعددات حدود – M_n لبعض البيانات الخاصة

رغد عبدالعزيز مصطفى[1] , احمد محمد علي[1], عبدالستار محمد خضر[2]

[1]قسم الرياضيات, كلية علوم الحاسوب والرياضيات, جامعة الموصل.

[2]الجامعة التقنية الشمالية, المعهد التقني الموصل, قسم تقنيات انظمة الحاسوب

الخلاصة

ليكن G بيان متصل مجموعة رؤوسه $V = V(G)$ ومجموعة حافاته $E = E(G)$. تعرف المسافة الاعتيادية بين اي رأسين من رؤوس $V(G)$ على انها دالة d من $V \times V$ الى مجموعة الاعداد الصحيحة غير السالبة بحيث ان $d(u, v)$ هو الطول لأقصر درب– $(u - v)$.

*Email: raghad.math@uomosul.edu.iq

المسافة العظمى بين المجموعتين الجزئيتين $\hat{S}$ و S من $V(G)$ هي اعظم مسافة بين اي رأسين u و v بحيث ان الرأس v ينتمي الى المجموعة $\hat{S}$ والرأس u ينتمي الى المجموعة S . في هذا البحث تم اخذ حالة خاصة من المسافة العظمى عندما $\hat{S}$ تحوي على عنصر واحد والمجموعة S تتألف من $(n-1)$ من الرؤوس حيث $n \geq 3$ وتعرف كالاتي :

$d_{\max}(v,S) = \max\{d(v,u): u \in S\}, |S| = n-1, 3 \leq n \leq p, v \in V(G), v \notin S,$

حيث ان p تمثل رتبة البيان G .

في هذا البحث عرفنا متعددة الحدود M_n- والتي تعتمد على المسافة العظمى بين الرأس v في $V(G)$ والمجموعة الجزئية S التي لها $(n-1)$ من رؤوس البيان G والدليل M_n- . كذلك وجدنا متعددة الحدود $-$ M_n لبعض البيانات الخاصة مثل : البيان التام والبيان ثنائي التجزئة التام وبيان النجمة وبيان العجلة وبيان المروحة بالإضافة الى بيان الدرب والدارة وبيان جنكير مع ايجاد الادلة لهذه المسافة .

1. Introduction

In 1999, Dankelmann *et al*. defined the distance between two subsets of vertices in a connected graph G, as follows:

The minimum distance from U to V is:

$$d_{min}(U,V) = min\{ d(u,v): u \in U, v \in V \},$$

the average distance from U to V is:

$$d_{avg}(U,V) = \frac{1}{|U||V|}\sum_{u\in U, v\in V} d(u,v), \text{ and}$$

the maximum distance from U to V is:

$$d_{max}(U,V) = max\{ d(u,v): u \in U, v \in V \},$$

where U and V are not necessarily distinct subsets of vertices of G , and $d(u,v)$ is the number of edges in a shortest path between u and v [1] .

For metric axiom (2) for the maximum distance from U to V, we note that $d_{max}(U,V) = 0$ if $U = V = \{w\}$; thus for $U = V$ and $|U|$ or $|V| \geq 2$, this implies that $d_{max}(U,V) > 0$. Therefore, axiom (2) of the metric space does not in this case hold.

The literature is rich in papers on determining polynomials that depend on the minimum distance between a vertex v and a subset of vertices of G consisting of $(n-1)-$ vertices for many graphs and operations defined in graphs (see [2,3]). Also, there are many recent studies on other types of distances for some graphs (see ([4,5,6,7,8,9,10,11]). For additional information about the applications for some types of these distances, see [12,13,14], and about other applications see [15,16].

We assume that all graphs G in this paper are simple, undirected, and connected [17]. We define the max – n – distance in G as the maximum distance from a singleton v, $v \in V(G)$ to an $(n-1)-$subset S, $S \subseteq V(G)$, such that v does not belong to S, as follows:

$d_{max}(v,S) = max\{d(v,u) : u \in S\}$, $|S| = n-1$, $2 \leq n \leq p$, in which p is the order of a graph G.

It is clear that, for any vertex v and for any subset S of $V(G)$, we have: $d_{max}(v,S) \geq 1$. This means that the axiom (2) of metric space does not hold. If the vertex v dominates all vertices S, then $d_{max}(v,S) = 1$.

When $n = 2$, the ordinary distance between two vertices of $V(G)$ can be obtained [17]. Therefore, we let $n \geq 3$. The M_n-eccentricity of a vertex v is also the maximum distance between v and a set S of $(n-1)$vertices. This means that:

$$e_{max}(v,n) = max_{\substack{S \subseteq V(G) \\ |S|=n-1}}\{d_{max}(v,S)\},$$

and the M_n-radius and the M_n- diameter of G with respect to this distance (max – n –distance), respectively, are defined by:

$rad_{max}(G,n) = r_{max}(G,n) = min_{v\in V}\{e_{max}(v,n)\}$

and

$diam_{max}(G,n) = \delta_{max}(G,n) = max_{v\in V}\{e_{max}(v,n)\}$.

The M_n-index of a graph G of order p, where, $3 \leq n \leq p$, is the sum of max – n – distances of all pairs (v,S) in G :

$M_n(G) = \sum_{\substack{v\in V-S \\ S\subseteq V}} d_{max}(v,S).$

The M_n-polynomial of a graph G of order p is denoted by $M_n(G;x)$ and defined by :

$M_n(G;x) = \sum_{k=m}^{\delta_{max}(G,n)} C_n(G,k)x^k$,
where $m = min\{d_{max}(v,S), v \in V - S, S \subseteq V\}$, and $C_n(G,k)$ is the number of pairs $(v,S), S \subseteq V(G), |S| = n-1, 3 \le n \le p$, such that $d_{max}(v,S) = k$, for each $m \le k \le \delta_{max}(G,n)$. It is clear that the M_n- index of any connected graph G can be obtained from the M_n-polynomial, as follows :
$M_n(G) = \frac{d}{dx} M_n(G;x)|_{x=1} = \sum_{k=m}^{\delta_{max}(G,n)} kC_n(G,k)$.
If $\lambda_k(v)$ represents the number of pairs (v,S) such that $d_{max}(v,S) = k$, then
$C_n(G,k) = \sum_{v \in V(G)} \lambda_k(v)$.
Properties 1.1: For all $3 \le n \le p$, we have:
a. $C_n(G,1) = \sum_{\forall v \in V(G)} \binom{degv}{n-1}$.
b. $\sum_{k=1}^{\delta_{max}(G,n)} C_n(G,k) = p\binom{p-1}{n-1}$.
c. $\delta_{max}(G,n) = \delta$, where δ is the diameter of ordinary distance of G.
Proof
a. Since the number of vertices that are lying at a distance one from the vertex $v \in V(G)$ is equal to the degree of v, $\deg v$, and there is no vertex lying at a distance less than one to v, then
$\lambda_1(v) = \sum_{j=1}^{\deg v} \binom{\deg v}{j} \binom{0}{n-j-1} = \binom{\deg v}{n-1}$.
Therefore, $C_n(G,1) = \sum_{v \in V(G)} \lambda_1(v) = \sum_{\forall v \in V(G)} \binom{degv}{n-1}$.
b. Let $C_n(v,G,k)$ be the number of pairs (v,S) , $S \subseteq V(G)$,$|S| = n-1, 3 \le n \le p$, $v \notin S$, such that $d_{max}(v,S) = k$, for each $1 \le k \le \delta_{max}(G,n)$ then

$$C_n(G,k) = \sum_{v \in V(G)} C_n(v,G,k) \text{ , so}$$

$\sum_{k=1}^{\delta_{max}(G,n)} C_n(G,k) = \sum_{v \in V(G)} \sum_{k=1}^{\delta_{max}(G,n)} C_n(v,G,k) = \sum_{v \in V(G)} \binom{p-1}{n-1} = p\binom{p-1}{n-1}$.
c. By the definition of the diameter of a graph G with respect to the max- n distance, we have

$$\delta_{max}(G,n) = \max\{d_{max}(v,S): v \in V(G), v \notin S, S \subseteq V(G), |S| = n-1\}$$
$$= \max\{maxd(v,u): v \in V(G), u \in S\}$$
$$= \max\{d(v,u): v,u \in V(G)\} = \delta.$$

Example (1): Let K_p be a complete graph of order , $p \ge 3$, then
$M_n(K_p;x) = p\binom{p-1}{n-1}x$, where $|S| = n-1$, $\emptyset \ne S \subseteq V(K_p)$, $n \ge 3$.

2. Main Results

2.1 M_n –Polynomials of some special graphs which have a diameter equal to two

We find the M_n–Polynomials for some special graphs which have a diameter equal to two, such as complete bipartite, star, wheel, and fan graphs.
Theorem 2.1.1: For all $p \ge 4$, $p = p_1 + p_2$, we have:
$M_n(K_{p_1,p_2};x) = [p_1\binom{p_2}{n-1} + p_2\binom{p_1}{n-1}]x + [p\binom{p-1}{n-1} - p_1\binom{p_2}{n-1} - p_2\binom{p_1}{n-1}]x^2$
Proof : From property (1), we have:
$C_n(K_{p_1,p_2},1) = p_1\binom{p_2}{n-1} + p_2\binom{p_1}{n-1}$.
Since $diam(K_{p_1,p_2}) = 2$, then
$C_n(K_{p_1,p_2},2) = p\binom{p-1}{n-1} - C_n(K_{p_1,p_2},1)$ (by property 2) .
$= p\binom{p-1}{n-1} - p_1\binom{p_2}{n-1} - p_2\binom{p_1}{n-1}$, where $= p_1 + p_2$.
Since the star graph S_p is a special case of complete bipartite graph $K_{p_1,1}$, then
$M_n(S_p;x) = \binom{p-1}{n-1}x + (p-1)\binom{p-1}{n-1}x^2$, where $= p_1 + 1$.
By the same method, we obtain the M_n –Polynomials of wheel W_p and fan F_p graphs .
Theorem 2.1.2: For $p \ge 5$, we have
1. $M_n(F_p;x) = [\binom{p-1}{n-1} + (p-1)\binom{2}{n-1}]x + (p-1)[\binom{p-1}{n-1} - \binom{2}{n-1}]x^2$,
2. $M_n(W_p;x) = [\binom{p-1}{n-1} + (p-1)\binom{3}{n-1}]x + (p-1)[\binom{p-1}{n-1} - \binom{3}{n-1}]x^2$.
Proof : Obvious.
Proposition 2.1.3 : The M_n– index of G has a diameter equal to two, for all $n \ge 3$, that is
$M_n(G) = 2p\binom{p-1}{n-1} - \sum_{\forall v \in V(G)} \binom{degv}{n-1}$, where p is order of G.
Proof

Since $\delta = 2$, then from properties 1.1, (a) and (b), we obtain the proposition 2.1.3.

Corollary 2.1.4: For all $n \geq 3$, we have :

1. $M_n(K_{p_1,p_2}) = 2p\binom{p-1}{n-1} - p_1\binom{p_2}{n-1} - p_2\binom{p_1}{n-1}$, where $p = p_1 + p_2, p_i \geq 2$, $i = 1,2$.

2. $M_n(S_p) = (2p-1)\binom{p-1}{n-1}$, $p \geq 4$.

3. $M_n(W_p) = (2p-1)\binom{p-1}{n-1} - (p-1)\binom{3}{n-1}$, $p \geq 4$.

4. $M_n(F_p) = (2p-1)\binom{p-1}{n-1} - (p-1)\binom{2}{n-1}$, $p \geq 5$.

Proof:

1. Since $degv = p_1$, when $v \in V_2$ and $degv = p_2$, when $v \in V_1$, where $V(K_{p_1,p_2}) = V_1 \cup V_2$, then from Proposition 2.1.3 , we obtain what is required.

2. Since $degv = 1$, when $v \in V(S_p) - \{center\ vertex\}$ and $degv = p - 1$, when v is center vertex of S_p, then from Proposition 2.1.3 , we obtain

$M_n(S_p) = (2p-1)\binom{p-1}{n-1}$, $p \geq 4$.

3. Since $degv = 3$, when $v \in V(W_p) - \{center\ vertex\}$, and $degv = p - 1$, when v is center vertex of W_p, then from Proposition 2.1.3 , we have

$M_n(W_p) = (2p-1)\binom{p-1}{n-1} - (p-1)\binom{3}{n-1}$, $p \geq 4$.

4. Since $degv = 2$, when $v \in V(F_p) - \{center\ vertex\}$ and $degv = p - 1$, when v is center vertex of F_p, then from Proposition 2.1.3 , we get

$M_n(F_p) = (2p-1)\binom{p-1}{n-1} - (p-1)\binom{2}{n-1}$, $p \geq 5$.

2.2. M_n–Polynomials of some other Special Graphs:

In this section, we shall find the M_n −Polynomials of a cycle C_p and a path P_p graphs and find the indices of the maximum distance of them.

Theorem 2.2.1: Let C_p be a cycle graph of order p, $p \geq 6$. Then for all $3 \leq n \leq p$, we have

$$M_n(C_p\,;x) = p\binom{2}{n-1}x + p\sum_{k=2}^{\left\lfloor\frac{p}{2}\right\rfloor-1}\left[\binom{2k}{n-1} - \binom{2k-2}{n-1}\right]x^k + p\begin{cases}\binom{p-2}{n-2}x^{\frac{p}{2}} & ;if\ p\ is\ even\,, \\ \left[\binom{p-1}{n-1} - \binom{p-3}{n-1}\right]x^{\frac{p-1}{2}} & ;\text{if } p \text{ is odd.}\end{cases}$$

Proof: We have from property (1) that $C_n(C_p\,,1) = p\binom{2}{n-1}$.

To find $C_n(C_p\,,k)$ for all $2 \leq k \leq \left\lfloor\frac{p}{2}\right\rfloor$, let S be a subset of vertices of $V(C_p)$ that has length $(n-1)$ and let $U = \{u_{i+1}\,,u_{i+2}\,,\dots\,,u_{i+k-1}\,,u_{i-1}\,,u_{i-2}\,,\dots\,,u_{i-k+1}\}$, then the cardinality of U is $2k-2$.

Now, if p is an odd number, then we have three cases:

Case I :$S = \{u_{i+k}\} \cup S_1\,, \emptyset \neq S_1 \subseteq U.$

Case II :$S = \{u_{i-k}\} \cup S_1\,, \emptyset \neq S_1 \subseteq U.$

Case III :$S = \{u_{i+k}\,,u_{i-k}\} \cup S_1\,, S_1 \subseteq U.$

Then from these cases, we obtain:

$\lambda_k(v_i) = \sum_{j=1}^{2}\binom{2}{j}\binom{2k-2}{n-1-j} = \binom{2k}{n-1} - \binom{2k-2}{n-1},\ 1 \leq i \leq p.$

Since $C_n(C_p\,,k) = \sum_{v \in V(C_p)}\lambda_k(v)$, then we have:

$$M_n(C_p\,;x) = p\binom{2}{n-1}x + p\sum_{k=2}^{\left\lfloor\frac{p}{2}\right\rfloor}\left[\binom{2k}{n-1} - \binom{2k-2}{n-1}\right]x^k\ .$$

If p is an even number, then we get the same coefficient $C_n(C_p\,,k)$, and when p is an odd for all $2 \leq k \leq \left\lfloor\frac{p}{2}\right\rfloor - 1$ and $k = \frac{p}{2}$, then $u_{i+k} = u_{i-k}$. Thus, we have:

$$\lambda_{\frac{p}{2}}(v_i) = \binom{p-2}{n-2}\,,\ 1 \leq i \leq p\ .$$

This completes the proof

Corollary 2.2.2: For all $3 \leq n \leq p$, $p \geq 6$, we have **:**

$$M_n(C_p) = p\binom{2}{n-1} + p\sum_{k=2}^{\left\lfloor\frac{p}{2}\right\rfloor-1}k\left\{\binom{2k}{n-1} - \binom{2k-2}{n-1}\right\} + p\left\lfloor\frac{p}{2}\right\rfloor\begin{cases}\binom{p-2}{n-2} & ;if\ p\ is\ even\,, \\ \binom{p-1}{n-1} - \binom{p-3}{n-1}; & if\ p\ is\ odd.\end{cases}$$

Theorem 2.2.3 : Let P_p be a path graph of order p, $p \geq 5$. Then for all $3 \leq n \leq p$, we have:
$M_n(P_p\,;x) = (p-2)\binom{2}{n-1}x + \sum_{k=2}^{p-1} C_n(P_p\,,k)x^k$, where

$$C_n(P_p\,,k) = \binom{p-2k}{1}\left\{\binom{2k}{n-1} - \binom{2k-2}{n-1}\right\} + 2\begin{cases}\sum_{i=1}^{k}\binom{k+i-2}{n-2}; & 2 \leq k \leq \left\lfloor\frac{p}{2}\right\rfloor, \\ \sum_{i=1}^{p-k}\binom{p-i-1}{n-2}; & \left\lfloor\frac{p}{2}\right\rfloor + 1 \leq k \leq p-1.\end{cases}$$

Proof: It is clear that $C_n(P_p\,,1) = (p-2)\binom{2}{n-1} + 2\binom{1}{n-1} = (p-2)\binom{2}{n-1}$, for all $n \geq 3$, by property (1).

Now, to find $C_n(P_p\,,k)$ for all $2 \leq k \leq p-1$, let S be a subset of vertices of $V(P_p)$ that has length $n-1$. Then, there are two cases:

Case I : For all $2 \leq k \leq \left\lfloor\frac{p}{2}\right\rfloor$, then there are two subcases:

a. If $S = \{u_{i+k}\} \cup S_1$, where $\emptyset \neq S_1 \subseteq S_2 \cup \{u_{i+1}, \dots, u_{i+k-1}\}$, $S_2 = \{u_1, \dots, u_{i-1}\}$, such that $S_2 \neq \emptyset$ when $i = 1$, then
$\lambda_k(v_i) = \binom{k+i-2}{n-2}$, $\quad 1 \leq i \leq k$.
By symmetry, we have $\lambda_k(v_i) = \lambda_k(v_{p-i+1})$, $1 \leq i \leq k$.

b. If $S = \{u_{i+k}\,, u_{i-k}\} \cup S_3$, where S is required to contain at least one of $\{u_{i+k}\,, u_{i-k}\}$ and $S_3 = \{u_{i-k+1}, \dots, u_{i-1}\,, u_{i+1}\,, \dots, u_{i+k-1}\}$, $k+1 \leq i \leq p-k$,$|S_3| = 2k-2$, then
$\lambda_k(v_i) = \sum_{j=1}^{2}\binom{2}{j}\binom{2k-2}{n-1-j} = \binom{2k}{n-1} - \binom{2k-2}{n-1}$.

Case II : For all $\left\lfloor\frac{p}{2}\right\rfloor + 1 \leq k \leq p-1$, we have:
$\lambda_k(v_i) = \binom{p-i-1}{n-2}$, $1 \leq i \leq p-k$,
where $S = \{u_{i+k}\} \cup S_4$, $\emptyset \neq S_4 \subseteq \{u_1\,, \dots, u_{i-1}\,, u_{i+1}\,, \dots, u_{i+k-1}\}$.
From the two cases and $C_n(P_p\,,k) = \sum_{v \in V(P_p)} \lambda_k(v)$, we have:

$$C_n(P_p\,,k) = \binom{p-2k}{1}\left\{\binom{2k}{n-1} - \binom{2k-2}{n-1}\right\} + 2\begin{cases}\sum_{i=1}^{k}\binom{k+i-2}{n-2}; & 2 \leq k \leq \left\lfloor\frac{p}{2}\right\rfloor, \\ \sum_{i=1}^{p-k}\binom{p-i-1}{n-2}; & \left\lfloor\frac{p}{2}\right\rfloor + 1 \leq k \leq p-1.\end{cases}$$

Corollary 2.2.4 : For all $3 \leq n \leq p$, $p \geq 5$, is

$$M_n(P_p) = (p-2)\binom{2}{n-1} + \sum_{k=2}^{p-1} k\left[\binom{p-2k}{1}\left\{\binom{2k}{n-1} - \binom{2k-2}{n-1}\right\} + 2\begin{cases}\sum_{i=1}^{k}\binom{k+i-2}{n-2}; & 2 \leq k \leq \left\lfloor\frac{p}{2}\right\rfloor, \\ \sum_{i=1}^{p-k}\binom{p-i-1}{n-2}; & \left\lfloor\frac{p}{2}\right\rfloor + 1 \leq k \leq p-1.\end{cases}\right]$$

2.3. The M_n-Polynomial of Jahangir Graph $J_{2,m}$

Definition [18]: Jahangir graph $J_{n,m}$ is a graph of $nm + 1$ vertices, for $m \geq 3$, that is, a graph consisting of a cycle of order nm,C_{nm} with a new additional vertex which is adjacent to the m vertices of C_{nm} at distance n between each other on C_{nm}. We consider the graph $J_{n,m}$ when $n = 2$ (Figure 1).

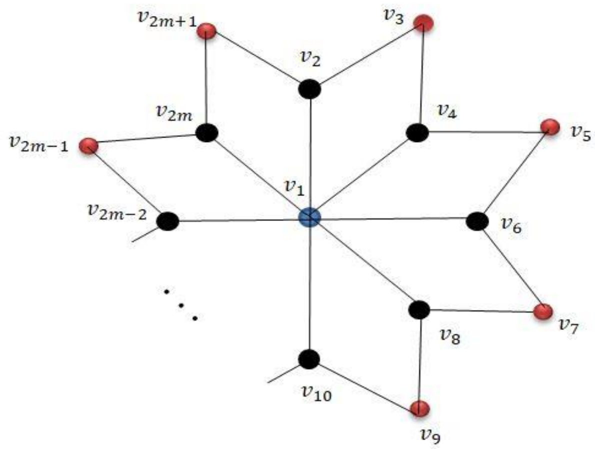

Figure 1-Jahangir graph $J_{2,m}$,has $2m + 1$ vertices, $3m$ edges of degree three and one vertex of degree m

Some Properties of $J_{2,m}$

- **Diameter property:** The graph $J_{2,m}$ has a diameter of 4, for all $m \geq 4$.
- **The distance property:** The vertices of the graph $J_{2,m}$ have the max- n - distance between the vertex v_i and any subset S of vertices of $V(J_{2,m})$, as follows:

$d_{max}(v_1, S) \leq 2$,
$d_{max}(v_i, S) \leq 3$,for all$i = 2,4, \dots, 2m$,
$d_{max}(v_i, S) \leq 4$, for all $i = 3,5, \dots 2m + 1$.

- **Symmetry property :**Vertices of the graph $J_{2,m}$ that have the same max – n –

distance are called the symmetric vertices, which are:
$v_2 \equiv v_i$, for all$i = 4, 6, \dots, 2m$, and $v_3 \equiv v_i$, for all $i = 5,7, \dots 2m + 1$.

Theorem 2.3.1: For all $m \geq 4$, we have
$M_n(J_{2,m}; x) = \sum_{k=1}^{4} C_n(J_{2,m}, k)x^k$, where
$C_n(J_{2,m}, 1) = \binom{m}{n-1} + m\binom{3}{n-1} + m\binom{2}{n-1}$,
$C_n(J_{2,m}, 2) = \binom{2m}{n-1} - \binom{m}{n-1} + m\left[\binom{m+2}{n-1} - \binom{3}{n-1}\right] + m\left[\binom{5}{n-1} - \binom{2}{n-1}\right]$,
$C_n(J_{2,m}, 3) = m\left[\binom{2m}{n-1} - \binom{m+2}{n-1}\right] + m\left[\binom{m+3}{n-1} - \binom{5}{n-1}\right]$,
$C_n(J_{2,m}, 4) = m\left[\binom{2m}{n-1} - \binom{m+3}{n-1}\right]$.

Proof: It is obvious that $C_n(J_{2,m}, 1) = \binom{m}{n-1} + m\binom{3}{n-1} + m\binom{2}{n-1}$.
To prove $C_n(J_{2,m}, k)$ for all $k = 2,3,4$, we will find the coefficients $C_n(v_i, J_{2,m}, k)$ for $i = 1,2,3$ and use the symmetry property to get the required result. Let $S \subseteq V(J_{2,m})$ be a subset of $(n-1)$ elements, $3 \leq n \leq 2m + 1$, and let $\lambda_k(v)$ be the number of subsets S for which the max-n-distance between v and S is equal to k.
Now, when $k = 2$, then there are three cases:
Case I : If $i = 1$, then there are m vertices $\{v_3, v_5, \dots, v_{2m+1}\}$ lying at a distance of 2 from v_1 and there are m vertices $\{v_2, v_4, \dots, v_{2m}\}$ lying at a distance of 1 to v_1 . Thus,
$C_n(v_1, J_{2,m}, 2) = \lambda_2(v_1) = \sum_{j=1}^{m} \binom{m}{j}\binom{m}{n-j-1} = \binom{2m}{n-1} - \binom{m}{n-1}$.
Case II : If $i = 2$, then there are $(m-1)$ vertices $\{v_4, v_6, \dots, v_{2m}\}$ lying at a distance of 2 from v_2 and there are 3 vertices $\{v_1, v_3, v_{2m+1}\}$ lying at a distance of 1 to v_1. Thus
$C_n(v_2, J_{2,m}, 2) = \lambda_2(v_2) = \sum_{j=1}^{m-1} \binom{m-1}{j}\binom{3}{n-j-1} = \binom{m+2}{n-1} - \binom{3}{n-1}$.
Case II : If $i = 3$, then there are 3 vertices $\{v_1, v_5, v_{2m+1}\}$ lying at a distance of 2 from v_3 and there are only 2 vertices $\{v_2, v_4\}$ lying at a distance of 1 to v_3. Then
$C_n(v_3, J_{2,m}, 2) = \lambda_2(v_3) = \sum_{j=1}^{3} \binom{3}{j}\binom{2}{n-j-1} = \binom{5}{n-1} - \binom{2}{n-1}$.
From the three cases and the symmetry property, we have
$C_n(J_{2,m}, 2) = \binom{2m}{n-1} - \binom{m}{n-1} + m\left[\binom{m+2}{n-1} - \binom{3}{n-1}\right] + m\left[\binom{5}{n-1} - \binom{2}{n-1}\right]$.
Now, when $= 3$, then
$\lambda_3(v_1) = 0$, from the distance property.
If $i = 2$, then there are $(m-2)$ vertices $\{v_5, v_7, \dots, v_{2m-1}\}$ lying at a distance of 3 from v_2 and there are $(m+2)$vertices $\{v_1; v_2, v_4, \dots, v_{2m}\}$ lying at a distance of less than 3 to v_2 . Then
$C_n(v_2, J_{2,m}, 3) = \lambda_3(v_2) = \sum_{j=1}^{m-2} \binom{m-2}{j}\binom{m+2}{n-j-1} = \binom{2m}{n-1} - \binom{m+2}{n-1}$.
If $i = 3$, then there are $(m-2)$ vertices $\{v_6, v_8, \dots, v_{2m}\}$ lying at a distance of 3 from v_3 and there are 5 vertices $\{v_1, v_2, v_4, v_5, v_{2m+1}\}$ lying at a distance of less than 3 to v_3 . Then
$C_n(v_3, J_{2,m}, 3) = \lambda_3(v_3) = \sum_{j=1}^{m-2} \binom{m-2}{j}\binom{5}{n-j-1} = \binom{m+3}{n-1} - \binom{5}{n-1}$.
From the above and the symmetry property, we have

$C_n(J_{2,m}, 3) = m\left[\binom{2m}{n-1} - \binom{m+2}{n-1}\right] + m\left[\binom{m+3}{n-1} - \binom{5}{n-1}\right]$.
Finally, when $= 4$, then we have
$\lambda_4(v_i) = 0$, for $i = 1,2,$ from the distance property.

If $i = 3$, then there are $(m-3)$vertices $\{v_7, v_9, \ldots, v_{2m-1}\}$ lying at a distance of 4 from v_3 and there are $(m+3)$ vertices $\{v_1, v_5, v_{2m+1}\} \cup \{v_2, v_4, \ldots, v_{2m}\}$ lying at a distance of less than 4 to v_3. Then

$C_n(v_3, J_{2,m}, 4) = \lambda_4(v_3) = \sum_{j=1}^{m-3} \binom{m-3}{j} \binom{m+3}{n-j-1} = \binom{2m}{n-1} - \binom{m+3}{n-1}$.

From the above and the symmetry property, we have

$C_n(J_{2,m}, 4) = m\left[\binom{2m}{n-1} - \binom{m+3}{n-1}\right]$.

Remark: When $m = 3$, then the Jahangir graph $J_{2,3}$ has 3 diameters and we have :

$$M_n(J_{2,3}; x) = \left[4\binom{3}{n-1} + 3\binom{2}{n-1}\right]x + \left\{\binom{6}{n-1} - \binom{3}{n-1} + 3\left[\binom{5}{n-1} - \binom{3}{n-1}\right] + 3\left[\binom{5}{n-1} - \binom{2}{n-1}\right]\right\}x^2 + 6\binom{5}{n-2}x^3.$$

3. The Relation Between the M_n - polynomial and the Hosoya polynomial

In this section, we expound the relation between M_n −polynomial and Hosoya polynomial when $n = 2$. Let G be any connected graph of order p, then Hosoya polynomial is defined as follows [19]:

$H(G; x) = \sum_{k=1}^{\delta} d(G, k)x^k$,

where $d(G, k)$ is the number of pairs unordered of distinct vertices that are at a distance k, and Wiener index of G is defined as:

$W(G) = \frac{d}{dx} H(G; x)|_{x=1} = \sum_{k=1}^{\delta} kd(G, k)$.

From properties 1.1, we get:

$C_2(G, 1) = \sum_{\forall v \in V(G)} \binom{degv}{1} = \sum_{\forall v \in V(G)} degv = 2q$.

Hence $C_2(G, 1) = 2d(G, 1)$.

Theorem 3.1: Let G be a connected graph of order p, then $H_2(G; x) = 2H(G; x)$.

Proof: Let $C_n(G, k)$ be the number of all order pairs (v, S) such that $d_{\max}(v, S) = k$, $k \geq 1$, $|S| = n - 1$, $S \subseteq V(G)$. If we take $n = 2$, then $C_2(G; k)$ is the number of all order pairs $(v, \{u\})$ such that $d_{\max}(v, \{u\}) = k$, $k \geq 1$, $v \neq u$.

Since $(v, \{u\}) \neq (u, \{v\})$ for $v \neq u$, then

$C_2(G, k) = \sum_{\{u,v\} \subseteq V(G)} d_{max}(v, \{u\})$ such that $d_{max}(v, \{u\}) = k$.

Since $d_{max}(v, \{u\}) = d_{max}(u, \{v\})$ and $d(G, k)$ are the numbers of all distinguishable unordered pairs (v, u) of vertices that are of distance k apart, then

$C_2(G, k) = 2d(G, k)$, for$k \geq 1$, which implies that

$\sum_{k \geq 1} C_2(G, k) = 2 \sum_{k \geq 1} d(G, k)$.

Hence, $M_2(G; x) = 2H(G; x)$.

Also, we can obtain Wiener index from $W_2(G)$ by :

Hence, $M_2(G) = \frac{d}{dx} M_2(G; x)|_{x=1} = \frac{d}{dx} 2H(G; x)|_{x=1} = 2W(G)$.

In the table below, we explain the difference between the M_n − polynomial and Hosoya polynomial when $n = 2$ for the special graphs. We obtained Hosoya Polynomial results of the special graphs K_p , K_{p_1,p_2} , S_p, W_p, F_p, C_p, P_p, and $J_{2,m}$from the new results which we obtained in this paper, after substituting the value of $n = 2$ and dividing $M_2(G; x)$ by 2.

Table 1-Comparison between M_n − polynomial and Hosoya polynomial for some special graphs

Special graphs	M_2 − Polynomial ($M_2(G; x)$)	Hosoya Polynomial ($H(G; x)$)
K_p	$p(p-1)x$	$\frac{p(p-1)}{2}x$
K_{p_1,p_2}	$2p_1p_2x + [p(p-1) - 2p_1p_2]x^2$	$p_1p_2x + \left[\frac{p(p-1)}{2} - p_1p_2\right]x^2$
S_p	$2(p-1)x + (p-1)(p-2)x^2$	$(p-1)x + \frac{(p-1)(p-2)}{2}x^2$
W_p	$4(p-1)x + (p-1)(p-5)x^2$	$2(p-1)x + \frac{(p-1)(p-5)}{2}x^2$
F_p	$3(p-1)x + (p-1)(p-3)x^2$	$\frac{3(p-1)}{2}x + \frac{(p-1)(p-3)}{2}x^2$

C_p	$2p\sum_{k=1}^{\lfloor\frac{p}{2}\rfloor-1} x^k+\begin{cases} px^{\lfloor\frac{p}{2}\rfloor} & ; if\ p\ is\ even\ , \\ 2px^{\lfloor\frac{p}{2}\rfloor} & ; if\ p\ is\ odd. \end{cases}$	$2p\sum_{k=1}^{\lfloor\frac{p}{2}\rfloor-1} x^k+\begin{cases} px^{\lfloor\frac{p}{2}\rfloor} & ; if\ p\ is\ even\ , \\ 2px^{\lfloor\frac{p}{2}\rfloor} & ; if\ p\ is\ odd. \end{cases}$
P_p	$2\sum_{k=1}^{p-1}(p-k)x^k$	$\sum_{k=1}^{p-1}(p-k)x^k$
$J_{2,m}$	$6mx+m(m+3)x^2+2m(m-2)x^3$ $+m(m-3)x^4$	$3mx+\frac{m(m+3)}{2}x^2+m(m-2)x^3$ $+\frac{m(m-3)}{2}x^4$

Also, we can obtain Wiener index for special graphs by deriving $M_2(G;x)$ with respect to x, and then $x=1$, followed by division by 2.

4. Conclusions

This paper investigated M_n −polynomials with special structures and properties based on the maximum distance between the subset S of vertices of $V(G)$ with $(n-1)-$ vertices, ($S\subseteq V(G)$,$|S|=n-1$, ≥ 3), and a singleton vertex v in $V(G)$ which does not belong to S. In this paper, we determined the Hosoya polynomial from M_n- polynomial by dividing the M_2- polynomial by 2 when $n=2$.

This type of distance can have an application in several areas, for example in networks, by sending a set of signals or messages from a specific site to the farthest set of sites, so that these signals or messages from that set are then sent to sets closer to them in order to ensure that some data is not lost.

References

1. P. Dankelmann, W. Goddard, M. A. Henning, H. C. Swart, **1999**. *Generalized Eccentricity, Radius and Diameter in Graphs; Foundation for Research Development*, John Wiely & Sons, Inc. Networks **34**: 312–319.
2. A. M. Ali , A. A. Ali , T. H. Ismail , **2011**. *Hosoya Polynomial and Wiener Indices of Distances in Graphs*. LAP LAMPART Academic Publishing GmbH &Co. KG and licensors.
3. G. A. Mohammed-Saleh, **2013**. On the Detour Distance and Detour polynomials of graphs", Ph.D. Thesis, University of Salahaddin \ Erbil, Erbil, Iraq.
4. A. A. Kinsley, P. S. Ananthi, **2017**. Dd – Distance in Graphs , *Imperial Journal of Interdisciplinary Research*, **3**: 1457 – 1459.
5. H. Guo, B. Zhow, **2017**. Properties of Degree Distance and Gutman Index of Uniform Hypergraphs, MATCH Commun. Math. Comput Chem., **78**(2017): 213-220.
6. A. Sadeghieh, S. Alikhani, N. Ghanbari, A. Khalaf, Hosoya polynomial of some cactus chains. Cogent Mathematics, 1-7.
7. A. M. Ali , H. N. Mohammed, **2019**. Schultz and Modified Schultz Polynomials of Some Cog-Special Graphs, *Open Access Library Journal*, **6**: 1-13.
8. R. V. Venkateswara, P. L. N. Varm, B. D. Reddy, **2019**. Average Eccentricity of Graphs W.R.T. D Distance, *International Journal of Recent Technology and Engineering*, **7**: 208-210.
9. A. M. Ali, A. S. Aziz, **2020**. A Relation between D-Index and Wiener Index for r-Regular Graphs. International Journal of Mathematics and Mathematical Sciences, **2020** (2020): 1-6.
10. H. O. Abdullah, I. D. Ali, The Restricted Detour Polynomial of the Theta Graph, *Raf. J. of Comp. & Math's*., **14**(1): 13-20.
11. J. Devillers, A. T. Balaban, **1999**. *Topological Indices and Related Descriptors in QSAR and QSPR*, Gordon & Breach, Amsterdam.
12. M. V. Diudea, **2001**. *QSPR/QSAR Studies by Molecular Descriptors*, Nova, Huntington, New York.
13. Diudea, M. V., Gutman, I. and J¨antschi, L. **2001**. *Molecular Topology*, Nova, Huntington, New York.
14. G. Chartrand, L. Lesniak, **2016**. Graphs and *Digraphs*, 6th ed. Wadsworth and Brooks / Cole, California.
15. T. A. Khaleel , A. A. Al-Shumam, **2020**. *A Study of Graph Theory Applications in IT Security*. *Iraqi Journal of Science*, **61**(10): 2705-2714.

16. M. Rezaei, M. R. Farahani, M. K. Jamil, H. M. Rehman, M. Imran, **2017**. On the Atom Bond Connectivity Index of Titania Nanotubes TiO2 (m, n) . *Iraqi Journal of Science*, **58**(4C): 2392-2400.

17. F. Buckley, F. Harary, **1990**. *Distance in Graphs*. Addison -Wesley, Redwood, California.

18. D.A. Mojdeh, A.N. Ghameshlou, **2007**. Domination in Jahangir graph $J_{2,m}$. *Int. J. Contemp. Math. Sciences*, **2**(24):193– 199.

19. H. Hosoya, **1988**. On some counting polynomials in chemistry. *Discrete Applied Math*., **19**: 239-257.