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Application of q-Mittag-Leffer Function on Certain Subclasses of Analytic Functions

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Abstract
The main objective of this paper is to introduce and study the generality differential operator involving the q-Mittag-Leffler function on certain subclasses of analytic functions. Also, we investigate the inclusion properties of these classes, by using the concept of subordination between analytic functions.
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تطبيق دالة كيو متاك لوفر على عدد محدد من الاصناف الجزئية للدوال التحليلية

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الخلاصه
الهدف الرئيسي من هذا البحث هو تقديم ودراسة المؤثر التفاضلي المعمم والمرتبط بدالة كيو متاك لوفر في اصناف فرعية معينة من الدوال التحليلية ، وكذلك البحث في خصائص الاحتواء لهذه الاصناف، باستخدام مفهوم التابعيه بين الدوال التحليلية.

1. Introduction

Let A denotes the class of univalent functions $f\ (w)$ normalized by

$$f(w) = w + \sum_{n=2}^{\infty} a_n\, w^n, \qquad (1)$$

which are analytic in the open unit disk

$$U = \{w\colon w \in \mathbb{C}\ , |w| < 1\}.$$

Let f and g be analytic functions such that both are in U. The subordination between f and g is written as $f \prec g$ or $f(w) \prec g(w)$. In addition, we say that $f(w)$ is subordinate to $g(w)$ if there is a Schwarz function w with $\omega(w) = 0, |\omega(w)| < 1$,
$w \in U$, such that $f(w) = g(\omega(w))$ for all $w \in U$. Furthermore, if $g(w)$ is univalent in U, then we have the following equivalence [1, 2, 3]:

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$f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.
A function$f$ is said to be convex, respectively starlike, of order ρ if and only if

$$Re\left\{1+\frac{wf''(w)}{f'(w)}\right\} > \rho, 0 \leq \rho < 1\,, |w| < 1$$

and

$$Re\left\{\frac{wf'(w)}{f(w)}\right\} > \rho, 0 \leq \rho < 1\,, |w| < 1.$$

Remark 1.1. It is clear from the above that f is convex if and only if $wf'(w)$ is starlike.
Now , let a function f in A and g in $S^*(\rho)$ where g is starlike of order ρ, then f belongs to $K(\beta, \rho)$ if and only if

$$Re\left\{\frac{wf'(w)}{g(w)}\right\} > \rho, w \in U.$$

These functions are called close-to-convex functions of order β type ρ.

For $f \in A$, we introduce the subclasses of starlike, convex, and close-to-convex functions $S^*(\mu, \Psi)$, $C(\mu, \Psi)$ and $K(\mu, \xi, \Psi, \Phi)$ of order μ, which were studied by several authors [2,4,5] and are respectively defined by:

$$S^*(\mu, \Psi) = \left\{f \text{ } in \text{ } A : \frac{1}{1-\mu}\left(\frac{wf'(w)}{f(w)} - \mu\right) \prec \Psi(w), w \in U\right\}$$

$$C(\mu, \Psi) = \left\{f \text{ } in \text{ } A : \frac{1}{1-\mu}\left(1+\frac{wf''(w)}{f'(w)} - \mu\right) \prec \Psi(w), w \in U\right\}$$

$$K(\mu, \xi, \Psi, \Phi) = \left\{f \text{ } in \text{ } A : \frac{1}{1-\xi}\left(\frac{wf'(w)}{g(w)} - \xi\right) \prec \Phi(w), w \in U, g(w) \in S^*(\mu, \Psi)\right\}.$$

The study begins with definitions of the main terms and in-depth designs used for q-calculus applications. It is assumed, in this report, that $0 < q < 1$. Definitions are first given for the fractional q-calculus operator in a complex-assessed function $f(w)$, as follows:

Definition 1.1. Let $0 < q < 1$ and define the q-number $[n]_q$ by

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q} & (n \in \mathbb{C}) \\ \displaystyle\sum_{k=0}^{m-1} q^k = 1 + q + q^2 + \cdots + q^{m-1} & (n = m \in \mathbb{N}). \end{cases}$$

Definition 1.2. [6, 7]. The q-derivative (or the q-difference) operator $D_q$of a function f is defined by

$$D_q f(w) = \begin{cases} \dfrac{f(qw) - f(w)}{(q-1)w} & (w \neq 0) \\ f'(w) & (w = 0) \end{cases} \qquad (2)$$

In case $f(w) = w^n$ for $n \in N_0 = \{1, 2, 3, \dots\}$, the q-derivative of $f(w)$ is given by

$$D_q w^n = \frac{w^n - (wq)^n}{w(1-q)} = [n]_q w^{n-1},$$

where $[n]_q$ is defined in Definition 1.1.
From Definition 1.2, we note that

$$\lim_{q \to 1-} (D_q f)(w) = \lim_{q \to 1-} \frac{f(qw) - f(w)}{(q-1)w} = f'(w).$$

Next, we define the familiar Mittag-Leffer function $E_{\varrho}(w)$ introduced by Mittag-Leffer [8,9] and its generalization $E_{\varrho,\Upsilon}(w)$ introduced by Wiman [10], respectively, as follows

$$E_{\varrho}(w) = \sum_{n=0}^{\infty} \frac{w^n}{\Gamma(\varrho\, n+1)},$$

and

$$E_{\varrho,\Upsilon}(w) = \sum_{n=0}^{\infty} \frac{w^n}{\Gamma(\varrho\, n+\Upsilon)},$$

where $\Upsilon, \varrho \in \mathbb{C}, \mathrm{Re}(\Upsilon) > 0$ and $Re(\varrho) > 0$.

Sharma and Jain, in 2014 [11], introduced the q-analogue of generalized Mittag-Leffler function $E\,^{\delta}_{\varrho,\Upsilon}(w;q)(\delta,\varrho,\Upsilon \in \mathbb{C}, Re(\varrho) > 0,\ Re(\Upsilon) > 0, Re(\delta) > 0)$, which is defined by

$$E^{\delta}_{\varrho,\Upsilon}(w;q) = \sum_{n=0}^{\infty} \frac{(q^{\delta};q)_n}{(q;q)_n} \cdot \frac{w^n}{\Gamma_q(\varrho n+\Upsilon)}, \qquad (|q| < 1])$$

where $\Gamma_q(w)$ is the q-gamma function and $\lim_{q\to 1}\Gamma_q(w) = \Gamma(w)$.

The q-analogue of the Pochhammer symbol (q-shifted factorial) is defined by [12]

$$(\alpha,q)_n = \begin{cases} (1-\alpha)(1-\alpha q)\dots(1-\alpha q^{n-1}), & n = 1,2,3,\dots \\ 1, & n = 0. \end{cases}$$

Further, the q-gamma function $\Gamma_q(w)$ satisfies the following functional equation [12, 13]

$$\Gamma_q(w+1) = \frac{1-q^w}{1-q}\Gamma_q(w) = [w]_q\Gamma_q(w)$$

Also,

$$(q^{\alpha},q)_n = \frac{(1-q)^n\Gamma_q(\alpha+n)}{\Gamma_q(\alpha)}. \quad (n > 0).$$

Now, we define the function $Q^{\delta}_{\varrho,\Upsilon}(w)$ by

$$Q^{\delta}_{\varrho,\Upsilon}(w) = w\Gamma_q(\Upsilon)E^{\delta}_{\varrho,\Upsilon}(w;q)$$

$$= w + \sum_{n=2}^{\infty} \frac{\Gamma(\Upsilon)(q^{\alpha};q)_{n-1}}{\Gamma_q(\varrho\,(n-1)+\Upsilon)(q;q)_{n-1}}\, w^n.$$

Then, for $f \in A$, we define the following differential operator $D^{\delta,m}_{\alpha,q}(\varrho,\Upsilon)$ as follows

$$D^{\delta,m}_{\alpha,q}(\varrho,\Upsilon): A \longrightarrow A$$

$$D^{\delta,0}_{\alpha,q}(\varrho,\Upsilon)f(w) = f(w) * Q^{\delta}_{\varrho,\Upsilon}(w), \tag{3}$$

$$D^{\delta,1}_{\alpha,q}(\varrho,\Upsilon)f(w) = (1-\alpha)\left(Q^{\delta}_{\varrho,\Upsilon}(w) * f(w)\right) + \alpha w D_q\left(Q^{\delta}_{\varrho,\Upsilon}(w) * f(w)\right) \tag{4}$$

$$\vdots$$

$$D^{\delta,m}_{\alpha,q}(\varrho,\Upsilon)f(w) = D^{\delta,1}_{\alpha,q}\left(D^{\delta,m-1}_{\alpha,q}(\varrho,\Upsilon)f(w)\right). \tag{5}$$

Now, form (4) and (5), we get

$$D^{\delta,m}_{\alpha,q}(\varrho,\Upsilon)f(w) = w + \sum_{n=2}^{\infty}\left[1+([n]_q-1)\alpha\right]^m \frac{\Gamma_q(\Upsilon)(q^{\alpha};q)_{n-1}}{\Gamma_q(\varrho\,(n-1)+\Upsilon)(q;q)_{n-1}}\, a_n w^n, \tag{6}$$

where $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \alpha \geq 0$.

We can simply verify from (2) that

$$D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)f(w)=(1-\alpha)D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)f(w)+\alpha\, wD_q\left(D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)\right). \qquad (7)$$

Note the for $q\to 1$ and $\delta=1$, we obtain the operator in [14], for $q\to 1$, $\varrho=0$, $\gamma=1$ and $\Upsilon=1$, we obtain Al-Oboudi operator [4], for $q\to 1$, $\varrho=0,\gamma=1$, $\Upsilon=1$ and $\lambda=1$, we obtain Sălăgean operator [15], and for $q\to 1, m=0$ and $\delta=1$, we have $\mathbb{E}_{\Upsilon,\varrho}(w)$[16] .

From equation (7), we have

$$\alpha wD_q\left(D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)f(w)\right)=D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)f(w)-(1-\alpha)D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)f(w) \quad (8)$$

Let $\mathcal{T}$ be the class of all functions Ψ which are univalent and analytic in U for which $\Psi(U)$ is convex, such that $\Psi(0)=1$ and $Re\big((w)\big)>0; w\in U$.

Next, we provide a differential operator on the class A. We make use of the principle of subordination between analytic functions to investigate the classes of starlike, convex, and close-to-convex functions of $S^*(\mu,\Psi), C(\mu,\Psi),$ and $K(\mu,\xi,\Psi,\Phi)$ of order μ, respectively, for the function $\Psi,\Phi\in\mathcal{T}$, which are defined by

$$S_\alpha^m(\mu,\Psi)=\left\{f\in A:\ D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)f(w)\in S^*(\mu,\Psi)\right\}$$

$$C_\alpha^m(\mu,\Psi)=\left\{f\in A:\ D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)f(w)\in C(\mu,\Psi)\right\}$$

$$K_\alpha^m(\mu,\xi,\Psi,\Phi)=\left\{f\in A:\ D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)f(w)\in K(\mu,\xi,\Psi,\Phi)\right\}.$$

We also note that

$$f(w)\in C_\alpha^m(\mu,\Psi)\Leftrightarrow wf'(w)\in S_\alpha^m(\mu,\Psi).$$

2. Preliminary Results

The following lemmas will be required in our investigation.

Lemma 2.1. [7,17]. Let ζ be convex, univalent in U with $\zeta(0)=1$ and $Re(k\zeta(w)+\upsilon)\geq 0$, $k,\upsilon\in C$. If p is analytic in U with $p(0)=1$, then $p(w)+\frac{wp'(w)}{kp(w)+\upsilon}\prec\zeta(w), w\in U$, implies $p(w)\prec\zeta(w), w\in U$.

Lemma 2.2. [6] Let ζ be convex, univalent in U and w be analytic in U with $Re(w(w))\geq 0$. If p is analytic in U with $p(0)=\zeta(0)$ then $p(w)+w(w)wp'(w)\prec\zeta(w), w\in U$ implies $p(w)\prec\zeta(w), w\in U$.

In what follows, we give some inclusion properties of the operator $D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)f(w)$ using the principle of subordination.

3. Main Results

Theorem 3.1. Let $f\in A$ and let $\Psi\in\mathcal{T}$ with $Re\left((1-\mu)\Psi(\mathrm{w})+\frac{1-\alpha}{\alpha}+\mu>0\right)$. Then,

$$S_\alpha^{m+1}(\mu,\Psi)\subset S_\alpha^m(\mu,\Psi).$$

Proof. Let $f(w)$ belongs to the class $S_\alpha^{m+1}(\mu,\Psi)$ and let

$$p(w)=\frac{1}{1-\mu}\left(\frac{wD_q\left(D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)f(w)\right)}{D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)f(w)}\right)-\frac{1}{1-\mu}(\mu). \qquad (9)$$

By applying (8) in (9), we get

$$\frac{D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)f(w)-D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)f(w)+\alpha D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)f(w)}{\alpha\, D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)f(w)}=(1-\mu)p(w)+\mu,$$

and we get

$$\frac{1}{\alpha}\,\frac{D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)f(w)}{D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)f(w)}=(1-\mu)p(w)+\frac{1-\alpha}{\alpha}+\mu \qquad (10)$$

Now from (10), we get

$$\frac{D_q\left(D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)f(w)\right)}{D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)f(w)}=\frac{D_q\left(D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)f(w)\right)}{D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)f(w)}+\frac{(1-\mu)p'(w)}{(1-\mu)p(w)+\frac{1-\alpha}{\alpha}+\mu}. \quad (11)$$

Otherwise

$$\frac{D_q\left(D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)f(w)\right)}{D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)f(w)}=\frac{(1-\mu)p(w)+\mu}{w}. \quad (12)$$

From (11) and (12), we get

$$\frac{1}{1-\mu}\left(\frac{wD_q\left(D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)f(w)\right)}{D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)f(w)}-\mu\right)=p(w)+\frac{wp'(w)}{(1-\mu)p(w)+\mu+\frac{1-\alpha}{\alpha}}. \quad (13)$$

Applying Lemma 2.1 to (13) shows that

$p(w)\prec\Psi(w)$, i.e. $f\in D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)f(w)$

Thus,

$$S_\alpha^{m+1}(\mu,\Psi)\subset S_\alpha^m(\mu,\Psi),$$

which proves the theorem.

Theorem 3.2. Let f belongs to the analytic function of the form (1) and let $\Psi\in\mathcal{T}$. Then, $Re\left\{\left((1-\mu)\Psi(w)+\mu+\frac{1-\alpha}{\alpha}\right)\right\}>0.$

$$C_\alpha^{m+1}(\mu,\Psi)\subset C_\alpha^m(\mu,\Psi).$$

Proof. From Remark 1.1, we get

$$f\in C_\alpha^{m+1}(\mu,\Psi)\Leftrightarrow wf'\in S_\alpha^{m+1}(\mu,\Psi),$$

Now, by Theorem 3.1, we obtain

$$f\in C_\alpha^{m+1}(\mu,\Psi)\Leftrightarrow wf'\in S_\alpha^{m+1}(\mu,\Psi)\subset S_\alpha^m(\mu,\Psi)$$
$$\Longrightarrow wf'\in S_\alpha^m(\mu,\Psi)$$
$$\Rightarrow w\in C_\alpha^m(\mu,\Psi).$$

Thus,

$$C_\alpha^{m+1}(\mu,\Psi)\subset C_\alpha^m(\mu,\Psi).$$

The function $\Psi(w)=\frac{1-Aw}{1+Bw}$ is analytic and satisfies $\Psi(0)=1$. Thus, we obtain the following corollaries.

Corollary 3.1. Let $f\in A$ and $\Psi(w)=\frac{1-Aw}{1+Bw}$, $-1\le B\le A\le 1$ in Theorem 3.1. Then

$$S_\alpha^{m+1}(\mu,A,B)\subset S_\alpha^m(\mu,A,B).$$

Corollary 3.2. Let $f\in A$ and $\Psi(w)=\frac{1-Aw}{1+Bw}$, $-1\le B\le A\le 1$ in Theorem 3.1. Then

$$K_\alpha^{m+1}(\mu,A,B)\subset K_\alpha^m(\mu,A,B).$$

Theorem 3.3. Let $f\in A$ and $\Psi,\Phi\in\mathcal{T}$ with $Re\left\{\left((1-\mu)\Psi(w)+\mu+\frac{1-\alpha}{\alpha}\right)\right\}>0$. Then

$$K_\alpha^{m+1}(\mu,\xi,\Psi,\Phi)\subset K_\alpha^m(\mu,\xi,\Psi,\Phi).$$

Proof. Let f in $K_\alpha^{m+1}(\mu,\xi,\Psi,\Phi)$, then there exists a function g in $S_\alpha^{m+1}(\mu,\Psi)$ such that

$$Re\left\{\frac{wD_q\left(D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)f(w)\right)}{D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)f(w)}\right\}>\xi,\ w\in U.$$

That is, we get

$$\frac{1}{1-\xi}\left(\frac{wD_q\left(D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)f(w)\right)}{D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)f(w)}-\xi\right)\prec\Phi(\mathrm{w}),w\in U.$$

Let

$$p(w)=\frac{1}{1-\xi}\left(\frac{wD_q\left(D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)f(w)\right)}{D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)f(w)}-\xi\right). \quad (14)$$

From (7), we have
$wD_q\left(D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)f(w)\right)=\frac{D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)f(w)-(1-\alpha)D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)f(w)}{\alpha}$.
From (14), we get

$$\frac{1}{\alpha}D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)f(w)f(w) = \left(\frac{1-\alpha}{\alpha}\right)\left(D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)f(w)\right)+\left((1-\xi)p(w)+\xi\right)D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)g(w).$$

This implies that

$$\frac{1}{\alpha}wD_q\left(D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)f(w)\right)=\left(\frac{1-\alpha}{\alpha}\right)wD_q\left(D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)f(w)\right)$$
$$+\left((1-\xi)wp'(w)\right)\left(D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)g(w)\right)+((1-\xi)p(w)) + \xi)w\left[D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)g(w)\right]'. \quad (15)$$

Now, from Theorem 3.1, we have $g\in S_\alpha^{m+1}(\mu,\Psi)\Rightarrow g\in S_\alpha^{m}(\mu,\Psi)$. Now let

$$q(w)=\frac{1}{1-\mu}\left(\frac{wD_q\left(D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)g(w)\right)}{D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)g(w)}-\mu\right). \quad (16)$$

By using (7) in (16), we get

$$\frac{1}{\alpha}\left(\frac{D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)g(w)}{D_{\alpha,q}^{\delta,m}(\varrho,\Upsilon)g(w)}\right)=(1-\mu)q(w)+\mu+\frac{1-\alpha}{\alpha}, \quad (17)$$

Further, from (10) and (12), we get

$$\frac{wD_q\left(D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)f(w)\right)}{D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)g(w)}=(1-\xi)p(w)+\xi+\frac{(1-\xi)p'(w)}{(1-\mu)q(w)+\mu+\left(\frac{1-\alpha}{\alpha}\right)}. \quad (18)$$

Algebraic manipulation in (18) gives

$$\frac{1}{1-\xi}\left(\frac{D_q\left(D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)f(w)\right)}{D_{\alpha,q}^{\delta,m+1}(\varrho,\Upsilon)g(w)}-\xi\right)=p(w)+\frac{wp'(w)}{(1-\mu)q(w)+\mu+\left(\frac{1-\alpha}{\alpha}\right)}.$$

Thus, we obtain

$$\frac{1}{(1-\mu)q(w)+\mu+\left(\frac{1-\alpha}{\alpha}\right)}=w(w),$$

And by applying Lemma 2.2, we get that $p(w)\prec\Phi(w)$, which implies that
$f\in K_\alpha^m(\mu,\xi,\Psi,\Phi)$, which proves the theorem.

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