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S-K-nonsingular Modules

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Abstract
In this paper, we introduce a type of modules, namely S-K-nonsingular modules, which is a generalization of K-nonsingular modules. A comprehensive study of these classes of modules is given.

Keywords: Nonsingular modules, S- K-nonsingular modules**.**
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المقاسات غير المنفردة من النمط S−K

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الخلاصة
في هذا البحث نقدم نوع من المقاسات الغير المنفردة من النمط S−K والتي هي تعميم للمقاسات الغير منفرده من النمط K. دراسة مركزة اعطيت لهذا المقاس.

Introduction

Let M be a right R-module, where R is a ring with unity. A submodule N of M is called essential in M (denoted by $N \leq_{ess} M$). If $N \cap W = (0)$ and $W \leq M$, then $W = (0)$ [1]. Rizievi [2] introduced the concept of K-nonsingular modules, where an R-module M is called K-nonsingular if for each $f \in End(M)$, $Ker(f) \leq_{ess} M$, implies $f = 0$. Ali and Younis [3] called an R-module M as an essentially quasi-Dedekind if Hom$(\frac{M}{N}, M) = 0$ for each $N \leq_{ess} M$. Also, they proved that K-nonsingular modules and essentially quasi-Dedekind modules are coinciding concepts. In this paper, we introduce a generalization of K-nonsingular module which we call S-K-nonsingular, where an R-module M is called S-K-nonsingular if for each $f \in End(M), Ker(f) \leq_{ess} M$, implies $f(M) \ll M$. A submodule N of M is small and denoted by ($N \ll M$) if whenever $N + W = M, W \leq M$, then $W = M$ [4]. It is clear that the zero submodule is small, hence every K-nonsingular is S-K-nonsingular. However, the converse may be not true (see Remarks and Examples 1.2 [1].

This paper consists of three sections. In section one, we study the basic properties of S-K-nonsingular modules. In section two, we show that the direct summand of S-K-nonsingular is S-K-nonsingular. The direct sum of S-K-nonsingular might not be true (Examples 3.4 [2]).Also, we show that, under certain conditions, the direct sum of S-K-nonsingular modules is S-K-nonsingular (Theorem 2.4, Proposition 2.5, Proposition 2.6, Theorem 2.7).

In section three, we show that if $E(M)$(injective hull of module M) is an S-K-nonsingular, then M is

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not necessarily an S- K-nonsingular. Also, we show that if M is a faithful finitely generated multiplication R-module, then M is S- K-nonsingular if R is S- K-nonsingular.
Note that $N \leq^{\oplus} M$ (implies that N is a direct summand of M) and, for any $m \in M$, $r\text{-}ann_R(m)$ implies that $\{r \in R: m(t) = 0\}$ and Im f(implies an image of f).
Definition 1.1 An R-module M is called S-K-nonsingular if for each $f \in End(M), Ker(f) \leq_{ess} M$ implies $f(M) \ll M$.
Remarks and Examples 1.2
1- It is clear that every K-nonsingular module is S-K-nonsingular, but the converse is not true in general; for examples, each of the Z-modules Z_4, Z_{12} and Z_{P^∞}, where P is a prime number, is S-K-nonsingular, but they are not K-nonsingularmodules.
2- Every hollow module M (that is, every submodule of M is small see) X is S-K-nonsingular.
3- Every Rickart module M is K-nonsingular and hence S-K-nonsingular, where M is called Rickart module if for each $f \in End(M)$,
$Ker(f) \leq^{\oplus} M$ [4,P.21].
Proof: Let $f \in End(M)$ with $Ker(f) \leq_{ess} M$. Since M is Rickart, then $Ker(f) \leq^{\oplus} M$. It follow that $Ker(f) = M$, hence $f = 0$ and M is K-nonsingular.
4- S-K-nonsingular modules need not to be Rickart modules; for example, Z_4 as Z-module is S-K-nonsingular and it is not Rickart.
5- If M *is* S-K-nonsingular and dual Rickart module, then M is K-nonsingular, where M is called dual Rickart if for each $f \in End(M), Im(f) \leq^{\oplus} M$ [4,P.21].
Proof: Let $f \in End(M)$ and $Ker(f) \leq_{ess} M$. To prove that $f = 0$, we state that since M is S- K-nonsingular, then $f(M) \ll M$. But M is a dual Rickart, then $f(M) \leq^{\oplus} M$, so that $f(M) = 0$, that is $f = 0$.
6- For any R-module $M, \frac{M}{Z_2(M)}$ is nonsingular, so it is K-nonsingular, which implies that S-K-nonsingular, where $Z_2(M)$ is the second Z_2-torsion submodule of M.
7- Let M be an R-module with $RadM = 0$. Then M is S-K-nonsingular if and only if M is K-nonsingular.
Proof: $\Leftarrow$ It is clear by (1).
$\Rightarrow$ Let $f \in End(M)$ and $Ker(f) \leq_{ess} M$. Since M is S-K-nonsingular, then $f(M) \ll M$. Hence $f(M) \leq RadM = 0$. Thus $f = 0$.
8- Every nonsingular module is K-nonsingular [5], hence it is S- K-nonsingular.
9- Every polyform module is K-nonsingular, hence it is S- K-nonsingular, where a module M is said to be a polyform if for each $L \leq M$ and for any $0 \neq \varphi: L \mapsto M, Ker\varphi \nleq_{ess} L$ [6, P.44].
The following Proposition is a characterization of S- K-nonsingular module.
Proposition 1.3: Let M be an R-module. Then M is S- K-nonsingular if and only if for each $f \in Hom\left(\frac{M}{N}, M\right), N \leq_{ess} M$, implies $f(M) \ll M$.
Proof: $\Rightarrow$ Let $N \leq_{ess} M$ and $f \in Hom\left(\frac{M}{N}, M\right)$. Then $g = f \circ \pi \in End(M)$, where π is the natural projection from M to $M/_N$. $N \leq Kerg$, so $Kerg \leq_{ess} M$. But M is S- K-nonsingular, implies $g(M) \ll M$, that is $f\left(\frac{M}{N}\right) \ll M$.
$\Leftarrow$ Let $f \in End(M)$ such that $Ker(f) \leq_{ess} M$. f induces $\hat{f}$: $\frac{M}{Ker(f)} \mapsto M$ by $\hat{f}(m + Ker(f)) = f(m)$ for each $m \in M$. By hypothesis, $\hat{f}\left(\frac{M}{Ker(f)}\right) \ll M$. It follows that $f(M) \ll M$ and M is S- K-nonsingular.
Corollary 1.4: Let M be a S- K-nonsingular. If $N \leq_{ess} M$, then $Mr \ll M$ for each $r \in ann(N)$.
Recall that for an R-module M, $Z^k(M) = \sum_{\varphi \in S} Im\varphi$ and $Ker\varphi \leq_{ess} M$. M is K-nonsingular if and only if $Z^k(M) = 0$ [6, 2964].
We have the following.
Proposition 1.5: For an R-module M, M is S- K-nonsingular if $Z^k(M) \ll M$ and the converse holds if M satisfies the ascending chain condition on small submodules.
Proof: Let $\varphi \in S$ and $Ker\varphi \leq_{ess} M$. By the definition of $Z^k(M)$, $Im\varphi \leq Z^k(M) \ll M$. Hence $Im\varphi \ll M$. Thus M is S - K-nonsingular.

Conversely, since M satisfies the ascending chain condition on small submodules, then $RadM \ll M$ [7, Theorem 3.1].
Hence. for $\varphi \in S$, with $Ker\varphi \leq_{ess} M$, then $Im\varphi \ll M$. It follows that $Z^k(M) = \sum_{\varphi\in S, Ker\varphi \leq_{ess} M} Im\varphi \leq RadM \ll M$. Thus $Z^k(M)$ is a small submodule.
Remark 1.6: If $Z(M) \ll M$, then M is S-K-nonsingular.
Proof: Since $Z^k(M) \leq Z(M)$ [5, Proposition 2.11] and $Z(M) \ll M$, then $Z^k(M) \ll M$ and hence M is S-K-nonsingular, by Proposition 1.5.
Example 1.7: Let $M = Q \oplus Z_4$ as Z-module and $Z(M) = (0) \oplus (\bar{2}) \ll M$. By Remark 1.6, M is S-K-nonsingular. Also, M is not K-nonsingular since, if so, then Z_4(direct summand of M) is K-nonsingular, which is a contradiction.
Recall that an R-module is essentially prime if $ann_R M = ann_R N$ for each $N \leq_{ess} M$ [3].
Proposition 1.8: Let M be a divisible R-module (where R is an integral domain). If M is S-K-nonsingular, then M is essentially prime.
Proof: Assume that $N \leq_{ess} M$ and $annM \subsetneq annN$, that is, there exists $a \in annN$ and $a \notin annM$. Thus $aN = 0$ and $aM \neq 0$. But M is divisible, so $aM = M$. Define $f: M \mapsto M$ by $f(m) = am$ for each $m \in M$. It is clear that f is a well-defined R-homomorphism. Since $Ker(f) \supseteq N \leq_{ess} M$ and M is S-K-nonsingular, then $f(M) \ll M$, which is a contradication, since $f(M) = aM = M$. Thus $annM = annN$ for each $N \leq_{ess} M$.
Remark 1.9: Essentially prime modules need not to be S-K-nonsingular; for example, $M = Z \oplus Z_2$ as Z-module is an essentially prime [3], but M is not S-K-nonsingular.
Recall that an R-module M is called a SQD-module if every nonzero submodule N of M is a SQD-submodule of M, that is, for each $f \in Hom\left(\frac{M}{N}, M\right)$, $f\left(\frac{M}{N}\right) \ll M$"[9]. By applying Proposition 1.3, we have immediately the following.
Remark 1.10: Every SQD-module is S-K-nonsingular. However, the convers is not true; for example, the Z-module $Z \oplus Z$ is K-nonsingular (hence S-K-nonsingular) but it is not SQD-module [8].
The following Theorem is a characterization for S-K-nonsingular rings.
Theorem 1.11: For a ring R, R is S- K-nonsingular if and only if, for each ideal I in R, $I \leq_{ess} R$, implies $ann_R I \ll R$.
Proof:$\Rightarrow$ Suppose that $ann_R I + J = R$ for some ideal J of R. Then $1 = a + b$ for some $a \in ann_R I, b \in J$. Define $f: R \mapsto R$ by $f(a) = ra$ for each $r \in R$. f is a well-defined homomorphism. $f(I) = Ia = 0$, so that $I \leq Ker(f)$. Hence $Ker(f) \leq_{ess} R$, since $I \leq_{ess} R$. Now, by S- K-nonsingular of R, $f(R) \ll R$; that is $Ra \ll R$. But $R = Ra + Rb$, so $R = Rb$ and this implies that $1 = tb$ for some $t \in R$. Thus $1 \in J$ and hence $J = R$ and $ann(I) \ll R$.
$\Leftarrow$ Let $f \in End(R)$ with $Ker(f) \leq_{ess} R$. To prove that $f(R) \ll R$, since $f \in End(R)$, then there exists $a \in R, a \neq 0$ such that $f(r) = ra$, for all $r \in R$. Hence $f(R) = Ra$ and $Ker(f) = ann_R(a) \leq_{ess} R$. By the condition, $ann_R(Ker(f)) = ann_R(ann(a)) \ll R$. Thus $f(R) \ll R$.
Corollary 1.12: For a ring R, R is S- K-nonsingular if and only if, for each $f \in End(R)$, there exists $a \in R$, $ann_R(a) \leq_{ess} R$, implies $(a) \ll R$.

2. Direct summand of S-K-nonsingular modules and direct sum of S-K-nonsingular modules

First we have the following.
Proposition 2.1: A direct summand of S-K-nonsingular R-module is a S- K-nonsingular module.
Proof: Let M be a S-K-nonsingular module , $W \leq^{\oplus} M$. Then $W \oplus U = M$ for some $U \leq M$. To prove that W is a S-K-nonsingular, suppose that $f \in End(W)$ and $Ker(f) \leq_{ess} W$. Since $End(M) = \begin{pmatrix} End(W) & Hom(U,W) \\ Hom(W,U) & End(U) \end{pmatrix}$, take $g = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}$, then $Kerg = Ker(f) \oplus U \leq_{ess} W \oplus U = M$. But M is S-K-nonsingular, hence $g(M) \ll M$ and so $f(W) \oplus (0) \ll W \oplus U$. Thus $f(W) \ll W$ and W is S-K-nonsingular.
Remark 2.2: The direct sum of two S-K-nonsingular modules needs not to be S-K-nonsingular (see Example 3.4(2)).
Proposition 2.3: Let M be indecomposable S-K-nonsingular which has a maximal essential submodule N. Then $M \oplus \frac{M}{N}$ is not S-K-nonsingular, but M and $\frac{M}{N}$ are S-K-nonsingular.
Proof: Suppose that $M \oplus \frac{M}{N}$ is S-K-nonsingular, and let $\varphi \in End\left(M \oplus \frac{M}{N}\right)$ defined by $\varphi(m, \bar{n}) =$

$(0,\overline{m})$. Thus $Ker\varphi = N\oplus\frac{M}{N} \leq_{ess} M\oplus\frac{M}{N}$, so that $\varphi(M\oplus\frac{M}{N}) = (0)\oplus\frac{M}{N} \ll M\oplus\frac{M}{N}$, which is a contradiction. Thus $M\oplus\frac{M}{N}$ is not S-K-nonsingular, but it is clear that M and $\frac{M}{N}$ are S-K-nonsingular.

Recall that a submodule N of an R-module M is fully invariant if for each $f \in End(M), f(N) \subseteq N$. M is called Duo if every submodule is fully invariant [8].

Theorem 2.4: Let a module $M = M_1\oplus M_2$ where M_1 and M_2 are fully invariant submodules of M. Then M is S-K-nonsingular if and only if

1- M_i is S-K-nonsingular modules for each $i \in \{1,2\}$.

2- $Hom(M_i, M_j) = 0$ for each $i \neq j$.

Proof: $\Rightarrow$ The condition (1) holds by Proposition 2.1 and condition (2) holds by [10, Lemma 1.9].

$\Leftarrow$ $End(M) = \begin{pmatrix} End(M_1) & Hom(M_2,M_1) \\ Hom(M_1,M_2) & End(M_2) \end{pmatrix}$. Hence $End(M) = \begin{pmatrix} End(M_1) & 0 \\ 0 & End(M_2) \end{pmatrix}$ by condition (2). Let $f \in End(M)$, then $f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$ for some $f_1 \in End(M_1), f_2 \in End(M_2)$, and let $Ker(f) \leq_{ess} M = M_1\oplus M_2$. Since $Ker(f) = Kerf_1\oplus Ker(f)_2$, then $Kerf_1 \leq_{ess} M_1$, $Kerf_2 \leq_{ess} M_2$. Then by condition (1), $f_1(M_1) \ll M_1, f_2(M_2) \ll M_2$, so that $f(M) = f_1(M_1)\oplus f_2(M_2) \ll M_1\oplus M_2 = M$.

Proposition 2.5: Let M be a direct sum of R-modules M_1 and M_2, and let $ann_R M_1\oplus ann_R M_2 = R$. Then M is S-K-nonsingular module if and only if M_1 and M_2 are S-K-nonsingular modules.

Proof:$\Rightarrow$ It follows by Proposition 2.1.

$\Leftarrow$ $End(M) = \begin{pmatrix} End(M_1) & Hom(M_2,M_1) \\ Hom(M_1,M_2) & End(M_2) \end{pmatrix}$. Since $ann_R M_1\oplus ann_R M_2 = R$, then $Hom(M_2,M_1) = 0$ and $Hom(M_1,M_2) = 0$ by [10, Lemma 2.7]. Thus $End(M) = \begin{pmatrix} End(M_1) & 0 \\ 0 & End(M_2) \end{pmatrix}$. Let $f \in End(M)$, with $Ker(f) \leq_{ess} M$. Then we get $f(M) \ll M$(by the same procedure of Theorem 2.4.

Recall that M is an abelian module if all idempotent endomorphism commutes with any endomorphism [2, Definition 4.2.1]. Equivalently, every direct summand of M is fully invariant [5, Theorem 4.6].

Proposition 2.6: Let M be abelian module and $M = M_1\oplus M_2$ where $M_1, M_2 \leq M$. Then M is S-K-nonsingular if and only if M_1 and M_2 are S-K-nonsingular.

Proof: Since M is abelian, then M_1 and M_2 are fully invariant submodules and so $(M_1 , M_2) = 0$, $Hom(M_2 , M_1) = 0$, by [9, Lemma 1.9]. Thus the result follows by Theorem 2.4.

Theorem 2.7: Let $M = \oplus M_i$ (I is an index set) be a direct sum of modules M_i ($i \in I$) such that M is duo. Then M is S-K-nonsingular if and only if M_i is S-K-nonsingular, for each $i \in I$.

Proof: $\Leftarrow$ Let $f \in End(M)$ and $Ker(f) \leq_{ess} M$. Then $Ker(f)$ is fully invariant in M(since M is duo). It follows that $Ker(f) = \oplus_{i\in I}(Ker(f) \cap M_i$. Define $f_i: M_i \longmapsto M$ by $f_i = f \|_{M_i}$ for each i. It is easy to see that $Ker(f)_i = Ker(f)\cap M_i \leq_{ess} M\cap M_i = M_i$. But M_i is S-K-nonsingular for each $i \in I$, hence $f_i(M_i) \ll M_i$ for each $i \in I$.

Since $f(M)$ is a submodule of M and M is a duo module, then $f(M) = \oplus_{i\in I}(f(M) \cap M_i)$. It is easy to check that $f_i(M_i) = f(M)\cap M_i$ for each $i \in I$. Thus $f(M) = \oplus_{i\in I} f_i(M_i)$. Moreover, since M_i is S-K-nonsingular, then $f_i(M_i) \ll M_i$ for each $i \in I$. It follows that $f(M) = \oplus_{i\in I} f_i(M_i) \ll \oplus M_i = M$ and, hence, M is S-K-nonsingular.

$\Rightarrow$ It follows by Proposition 2.1.

Definition 2.8: An R-module M is called S-K-nonsingular relative to an R-module W if for each $f \in Hom(M,W)$, $Ker(f) \leq_{ess} M$, implies $Imf \ll W$.

Remarks and Examples 2.9

1- Every S-K-nonsingular module M is a S-K-nonsingular relative to M.

2- The Z-module Q is S-K-nonsingular relative to Z-module Z. Z-module Z is not S-K-nonsingular relative to Z-module Z_2.

3- Let M_1 and M_2 be two R-modules such that M_1 is S-K-nonsingular relative to $M_1\oplus M_2$. Then M_1 is S-K-nonsingular.

Proof: Let $f \in End(M_1)$ and $Ker(f) \leq_{ess} M_1$. Then $i \circ f \in Hom(M_1, M_1\oplus M_2)$ where i is the inclusion mapping from M_1 to $M_1\oplus M_2$. Then $Ker(i \circ f) \leq_{ess} M_1$ and so $(i \circ f)$ $(M_1) \leq_{ess} M_1\oplus M_2$, since M_1 is S-K-nonsingular relative to $M_1 \oplus M_2$. Hence $f(M_1) \ll M_1 \oplus M_2$. But $f(M_1) \leq M_1 \leq^{\oplus} M_1 \oplus M_2$. Thus $f(M_1) \ll M_1$ and M_1 is S-K-nonsingular.

4- Z_{12} is not S-K-nonsingular relative to Z_6, since there exists $f: Z_{12} \mapsto Z_6$ defined by $f(\bar{x}) = \begin{cases} \bar{0} & if\ x \in (\bar{2}) \\ \bar{3} & otherwise \end{cases}$. Hence $Ker(f) = (\bar{2}) \leq_{ess} Z_{12}$, but $Im\, f = \{\bar{0}, \bar{3}\}$ is not small on Z_6.

Theorem 2.10: Let $M = M_1 \oplus M_2$. Then M is an S-K-nonsingular module if and only if M_i S-K-nonsingular relative to M_j, for each $j \in \{1,2\}$.

Proof: $\Rightarrow M_1$ and M_2 are S-K-nonsingular modules by Proposition 2.1, that is, M_1 is S-K-nonsingular relative to M_1 and M_2 is S-K-nonsingular relative to M_2 . To prove that M_1 is S-K-nonsingular relative to M_2, suppose that $f \in Hom(M_1, M_2)$ and $f(M_1) \not\ll M_2$. Then $h = i \circ f \circ \rho$, where i is the inclusion mapping from M_2 to M and ρ is the natural projection from M to M_1. $h(M) = (i \circ f \circ \rho)(M) = f(M_1)$, hence $h(M) \not\ll M$ (because if $h(M) \ll M$), then $f(M_1) \ll M$, but $f(M_1) \leq M_2 \leq^{\oplus} M$, so $f(M_1) \ll M_2$, which is a contradiction. Since M S-K-nonsingular, then $Kerh \not\leq_{ess} M$. On the other hand, $Ker(f) = Kerh$ implies $Ker(f) \not\leq_{ess} M$. It follows that $Ker(f) \not\leq_{ess} M_1$, since if $Ker(f) \leq_{ess} M_1$, then $Kerh = Ker(f) \oplus M \leq_{ess} M_1 \oplus M_2 = M$, which is a contradiction. So, $Ker(f) \not\leq_{ess} M_1$ and M_1 is S-K-nonsingular module relative to M_2.

Similarly, M_2 is S-K-nonsingular module relative to M_1.

$\Leftarrow$ Let $\psi \in End(M)$ and $Ker(\psi) \leq_{ess} M$. To prove that $\psi(M) \ll M$, let $\psi_1 = \psi\big|_{M_1}: M_1 \mapsto M$ defined by $\psi_1(x) = \psi(x, 0)$ for each $x \in M_1$. $Ker\, \psi_1 = Ker\psi \cap M_1 \leq_{ess} M \cap M_1 = M_1$. $\rho_1 \circ \psi_1 \in End(M_1)$ and $\rho_2 \circ \psi_1 \in Hom(M_1, M_2)$, where ρ_1 is the natural projection from M onto M_1 and ρ_2 is the natural projection from M onto M_2. Then

$Ker\, \psi_1 \leq Ker(\rho_1 \circ \psi_1) \cap Ker(\rho_2 \circ \psi_1)$. But $Ker\psi_1 \leq_{ess} M$ and $Ker\psi_1 \leq M_1$, so $Ker\psi_1 \leq_{ess} M_1$. It follows that $Ker(\rho_1 \circ \psi_1) \leq_{ess} M_1$ and $Ker(\rho_2 \circ \psi_1) \leq_{ess} M_2$. But M_1 is S-K-nonsingular and M_1 is S-K-nonsingular module relative to M_2, hence $\rho_1 \circ \psi_1(M_1) \ll M_1$ and $\rho_2 \circ \psi_1(M_1) \ll M_2$.

Similarly $\rho_1 \circ \psi_2(M_2) \ll M_1$ and $\rho_2 \circ \psi_2(M_2) \ll M_2$, where ρ_1 and ρ_2 are the natural projections from M_1 onto M_1 and M_2, $\psi_2 = \psi\big|_{M_2}: M_2 \mapsto M$. Thus $\psi(M) = \sum_{i=1,2} (\rho_j \circ \psi_i)(M_i) \ll M_1 \oplus M_2 = M$. Therefore, M is S-K-nonsingular.

Proposition 2.11: Let M and M' be two R-modules and $f \in Hom(M, M')$ such that f is onto. If M is S-K-nonsingular relative to M', then M' is S-K-nonsingular.

Proof: Let $g \in End(M')$ and $Kerg \leq_{ess} M'$. Then $g \circ f \in Hom(M, M')$ and $Ker(g \circ f) \leq_{ess} M$. But M is S-K-nonsingular relative to M', so $(g \circ f)(M) \ll M'$, that is $g(M') \ll M'$. Thus, M' is S-K-nonsingular.

Proposition 2.12: Let M be S-K-nonsingular quasi-injective. Then for each $N \leq_{ess} M, N$ is S-K-nonsingular relative to M.

Proof: Let $f \in Hom(N, M)$with $Ker(f) \leq_{ess} N$. As $N \leq_{ess} M$, then $Ker(f) \leq_{ess} M$. Since M is quasi-injective, then there exists $g \in End(M)$ such that $g \circ i = f$, where i is an inclusion mapping from N into M. But it is clear that $Ker(f) \leq Kerg$, so $Kerg \leq_{ess} M$ and by S-K-nonsingularity of M , $g(M) \ll M$. It follows that $g \circ i(N) = g(N) \leq g(M)$ and $g(N) \ll M$. Besides that, $f(N) = g(N)$ so that $f(N) \ll M$ and N is S-K-nonsingular relative to M.

Corollary 2.13: Let M be an R-module. If $\bar{M}$ (quasi-injective hull of M) is S-K-nonsingular, then M is S-K-nonsingular relative to $\bar{M}$.

Theorem 2.14: Let M be a S-K-nonsingular R-module such that $RadM \ll M$. Then M is S-K-nonsingular relative to the ring R_R.

Proof: Let $f \in Hom(M, R)$ and $(f) \leq_{ess} M$. Suppose that $f(M) + J = R$ for some ideal J of R. Hence $1 = f(x) + j$ for some $x \in M$ and $j \in J$. Now, for any $m \in M$, define $g_m: R \mapsto M$ by $g_m(r) = rm$, for each $r \in R$ g, as a well-defined homomorphism. It follows that $g_m \circ f \in End(M)$. Since $Ker(g_m \circ f) \supseteq Ker(f)$, then $Ker(g_m \circ f) \leq_{ess} M$. Hence $(g_m \circ f)(M) \ll M$, since M is S-K-nonsingular, and, hence, for each $m \in M$, $mf(M) \ll M$. This implies that $\sum_{m \in M} mf(M) \leq RadM \ll M$. Hence $Mf(M) \ll M$ and so $Mf(x) \ll M$. But $1 = f(x) + j$, so that $M = Mf(x) + Mj$. It follows that $Mj = M$, hence $x = y$j for some $y \in M$ and so $1 = f(yj) + j = f(y)j + j \in J$. Thus $J = R$ and $f(M) \ll R$.

3. Additional features of S-K-nonsingular modules

Remark 3.1: For an R-module M, if M is S-K-nonsingular , then N≤M. Then, this shows that M/N is not necessarily S-K-nonsingular, as in the following example.

Example 3.2: $M = Z\oplus Z$, as the Z-module is an S-K-nonsingular module. Let $N = (\bar{0})\oplus(\bar{2}) \leq M$. Then $\frac{M}{N} \simeq Z\oplus Z_2$, which is not S-K-nonsingular, since if $f \in End(M)$ then $f(x, \bar{y}) = (0, \bar{x})$, $Ker(f) = 2Z\oplus Z_2 \leq_{ess} M$. But $f(M) = (0)\oplus Z_2 \not\ll M$.

Proposition 3.3: Let M be an S-K-nonsingular module such that $\frac{M}{K}$ is projective for each $K \leq_{ess} M$. Then $\frac{M}{N}$ is S-K-nonsingular for each $N \leq M$.

Proof: Let $\frac{U}{N} \leq_{ess} \frac{M}{N}$ and $f \in Hom(\frac{M/N}{U/N}, \frac{M}{N})$. Since $\frac{U}{N} \leq_{ess} \frac{M}{N}$, then $U \leq_{ess} M$. On the other hand, $Hom(\frac{M/N}{U/N}, \frac{M}{N}) \simeq Hom(\frac{M}{U}, \frac{M}{N})$ that is $f \in Hom(\frac{M}{U}, \frac{M}{N})$, but $\frac{M}{U}$ is projective, so there exist $g \in Hom(\frac{M}{U}, M)$ and $\pi \circ g = f$ where π is the natural projection from M to $\frac{M}{N}$. Also, $g\left(\frac{M}{U}\right) \ll M$ by Proposition 1.3, so that $(\pi \circ g)\left(\frac{M}{U}\right) \ll \frac{M}{N}$ and hence $f\left(\frac{M}{U}\right) \ll \frac{M}{N}$. Therefore, $\frac{M}{N}$ is S-K-nonsingular.

It is known that if M is an R-module, such that $E(M)$ (the injective hull of M) is K-nonsingular, then M is K-nonsingular [12, Proposition 2.18]. However, the S-K-nonsingular of $E(M)$ is not inherited by M (see example 3.4). Also, if M is K-nonsingular, then $E(M)$ is not necessarily K-nonsingular [5, Example 2.19].

Examples 3.4

1- By Example 2.3, $M = Z\oplus Z_2$, as the Z-module is not S-K-nonsingular. $E(M) = Q\oplus Z_{2^\infty}$. Since $Hom(Q, Z_{2^\infty}) = 0$ and $Hom(Z_{2^\infty}, Q) = 0$, then $S = End(M) = \begin{pmatrix} EndQ & 0 \\ 0 & Z_{2^\infty} \end{pmatrix}$. Assume that $f \in S$, hence $f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$ where $f_1 \in End(Q), \in f_2 End(Z_{2^\infty})$, and $Ker(f) \leq_{ess} E(M)$. But $Ker(f) = Kerf_1 \oplus Kerf_2$, so $Kerf_1 \leq_{ess} Q$ and $Kerf_2 \leq_{ess} Z_{2^\infty}$. Since Q and Z_{2^∞} are S-K-nonsingular modules, then $f_1(Q) \ll Q$ and $f_2(Z_{2^\infty})$ <<Z_{2^∞}). Hence $f(M) = f_1(Q)\oplus f_2(Z_{2^\infty}) \ll Q\oplus Z_{2^\infty} = E(M)$, so that $E(M)$ is S-K-nonsingular.

2- Let $M = Z_{p^\infty}\oplus Z_P$ as a Z-module that is not S-K-nonsingular. $End(M) = \begin{pmatrix} End(Z_{p^\infty}) & Hom(Z_P, Z_{p^\infty}) \\ 0 & Z_P \end{pmatrix}$. Let $= \begin{pmatrix} P^2 & 0 \\ 0 & 0 \end{pmatrix} \in End(M)$ $Ker\varphi \simeq Z_{p^2}\oplus Z_P \leq_{ess} M$, but $\varphi(M) = Z_{p^\infty}\oplus 0 \not\ll M$. Also, $E(M) = Z_{p^\infty}\oplus Z_{p^\infty}$ is S-K-nonsingular.

Now we ask if M S-K-nonsingular. Then $E(M)$ a S-K-nonsingular. However we have the following:

Remark 3.5

1- Let M be a nonsingular (hence M is S- K-nonsingular). Then $E(M)$ is -K-nonsingular.

Proof: Since M nonsingular, then this implies that $E(M)$ is nonsingular. Hence $E(M)$ is S- K-nonsingular.

2- Let M be a polyform extending module. Then $\overline{M}$ (quasi-injective hull of M) is S- K-nonsingular.

Proof: By [4, Proposition 2.4.22], $\overline{M}\oplus M$ is a Rickart module, so it is S- K-nonsingular by Remarks and Examples 1.2(3). Hence $\overline{M}$ is S- K-nonsingular by Proposition 2.1.

3- Let R be a polyform ring. Then $E(R)$ is an S-K-nonsingular R-module.

Proof: R is polyform, implies R is nonsingular by [5, Proposition 2.7]. Hence $E(R)$ is nonsingular and so (R) is S-K-nonsingular.

4- Let M be a prime R-module. Then $\overline{M}$ is S-K-nonsingular.

Proof: Since M is prime, then $\overline{M}$ is polyform. Hence $\overline{M}$ is S-K-nonsingular by Remarks and Examples 1.2(9).

Recall that an R-module M is multiplication if, for each $N \leq M$, there exists an ideal I of R such that $N = MI$ [3].

Theorem 3.6: Let M be a finitely generated faithful multiplication R-module. Then M is S-K-nonsingular if and only if R is S-K-nonsingular, where R is a commutative ring .

Proof: $\Rightarrow$ Let $f \in End(R)$ with $Ker(f) \leq_{ess} R$. Then there exists $r \in R$ such that $f(a) = ar$ for each $a \in R$. Hence $Ker(f) = ann_R(r) \leq_{ess} R$. Define $g: M \mapsto M$ by $g(m) = mr$ for each $m \in M$. g is well-defined homomorphism and $Kerg = ann_M(r)$. But , since M is a faithful multiplication finitely generated module, then $ann_M(r) = Mann_R(r)$ and hence $ann_M(r) \leq_{ess} M$ [3, Theorem 2.13], that is $Kerg \leq_{ess} M$. But M is S-K-nonsingular, so that $g(M) = Mr \ll M$. It follows that $< r >= f(R) \ll R$ [12, Proposition 1.1.8]. Thus R is S-K-nonsingular.

$\Leftarrow$ Let $f \in End(M)$ with $Ker(f) \leq_{ess} M$. Since M is a finitely generated multiplication, then there exists $r \in R$ such that $f(m) = mr$, for each $m \in M$.

Define $g: R \mapsto R$ by $g(a) = ra$ for each $a \in R$. Then $Kerg = ann_R(r)$ and $g(R) =< r >$. But $Ker(f) = ann_M(r) = Mann_R(r) \leq_{ess} M$, which implies that $Kerg = ann_R(r) \leq_{ess} M$ [3, Theorem 2.13]. It follows that $g(R) =< r > << R$ (since R is S-K-nonsingular). Thus $f(M) = Mr \ll M$ [1. Proposition 1.1.8] and so M is S-K-nonsingular.

Corollary 3.7: Let M be a faithful finitely generated multiplication R-module (where R is a commutative ring) . Then the followings are equivalent:

1- M is a S- K-nonsingular module;
2- R is a S- K-nonsingular ring;
3- $End(M)$ is a S- K-nonsingular ring;
4- For each $N \leq_{ess} M$, $annN \ll R$;
5- For each $I \leq_{ess} R$, $annI \ll R$.

Proof: (1)$\Leftrightarrow$(2) It follows by Theorem 3.6.

(2) $\Leftrightarrow$ (3) Since M is a finitely generated faitful multiplication module, then $End(M) \simeq \frac{R}{ann(M)} \simeq R$ and so the result is obtained.

(2) $\Leftrightarrow$ (5) It follows by Theorem 2.11.

(5) $\Rightarrow$ (4) Let $N \leq_{ess} M$. Since M is a faithful multiplication, then $N = MI$ for some essential ideal I of R [11, Theorem 2.13]. Also, $ann_R N = ann_R I$ because M is a faithful multiplication. By (5), $ann_R I \ll R$, hence $ann_R N \ll R$.

(4) $\Rightarrow$ (5) Let $I \leq_{ess} R$. Then $N = MI \leq_{ess} M$ [11, Theorem 2.13]. By (4), $ann_R N \ll R$. But M is a faithful multiplication, so $ann_R N = ann_R I$. Thus $ann_R I \ll R$.

Notice that $M = Z \oplus Z$ as Z-module is not a multiplication module and it is not S-K-nonsingular, but the ring Z is S-K-nonsingular.

Corollary 3.8: If M is a local faithful R-module, then R is a S- K-nonsingular.

Proof: Since M is local faithful, then M is hollow and cyclic. Hence M is S- K-nonsingular (by Remarks and Examples 1.2(2)) and, by Theorem 3.6, R is S- K-nonsingular.

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