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Simultaneous Approximation by a New Sequence of Integral Type Based on Two Parameters

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Abstract

This paper introduces a generalization sequence of positive and linear operators of integral type based on two parameters to improve the order of approximation. First, the simultaneous approximation is studied and a Voronovskaja-type asymptotic formula is introduced. Next, an error of the estimation in the simultaneous approximation is found. Finally, a numerical example to approximate a test function and its first derivative of this function is given for some values of the parameters.

Keywords: Positive and linear operators, Simultaneous approximation, Voronovskaja-type asymptotic formula, order of approximation and Korovkin's theorem.

تقريب متعدد لمتتابعة جديدة من نمط تكامل معتمدة على معلمتين

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الخلاصه

هذا البحث قدم تعميما لمتتابعة من مؤثرات موجبة وخطية من نمط تكامل معتمدا على معلمتين لتحسين رتبة التقريب. أولا، درس التقريب المتعدد وقدم صيغة مشابهة لفرونوفسكي. ثم، وجد تخمين لخطأ التقريب المتعدد.

أخيرا، اعطي مثالا عدديا لتقريب دالة اختبار ومشتقتها الأولى لقيم اختيارية لبعض المعلمات.

1. Introduction

Szasz, in 1950, defined and studied the sequence of operators $Z_n(f;x)$ [1]:

$$Z_n(f(t);x) = \frac{1}{e^{nx}} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right). \quad (1)$$

where, $x \in [0,\infty)$ and $n \in N = \{1,2,\dots\}$.

After that, several researchers modified many sequences of operators [2, 3, 4, 5, 6].

Rempulska *et al.*, in 2009, studied another modification for the sequence (1) in polynomials weighted space [7]:

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$$Z_{n,r}(f(t);x)=\frac{1}{A_r(nx)}\sum_{k=0}^{\infty}\frac{(nx)^{rk}}{(rk)!}f\left(\frac{rk}{n}\right), \quad (2)$$

for $n,r\in N$, where $A_r(x)=\sum_{k=0}^{\infty}\frac{x^{rk}}{(rk)!}$.

Clearly, $A_1(x)=e^x$, i.e. $Z_{n,1}(f;x)=Z_n(f;x)$ and $A_2(x)=\cosh(x)$.

After that, Mohammad and Abbod [8] introduced a general form for a sequence of positive and linear operators based on four parameters. Also, Mohammad and Muslim [9] studied the simultaneous approximation of a new sequence of integral type operators with parameter δ_0.

Recently, in 2019, Mohammad and Hassan defined and studied a new sequence of positive and linear operators of integral type [10], as follows:

$$M_{n,r}(f;y)=\frac{1}{A_r(ny)}\int_0^y A_r'(n\tau)f(\tau)d\tau, \quad (3)$$

where $n,r\in N, y\in[0,\infty)$ and $f\in C_\alpha[0,\infty)$, such that $C_\alpha[0,\infty)=\{f\in C[0,\infty):|f(\tau)|=O(e^{\alpha\tau}),\text{for some }\alpha>0\}$ and $\|f\|_{C_\alpha[0,\infty)}=\sup_{\tau\in[0,\infty)}|f(\tau)|e^{-\alpha\tau}$.

In our present work, we generalize the sequence in (3) for $f\in C_\alpha[0,\infty), n,r,s\in N$ and $y\in[0,\infty)$, as:

$$M_{n,r,s}(f;y)=\frac{1}{A_r(ny)}\int_0^y A_r'(n\tau)f(y+(\tau-y)^s)d\tau \quad (4)$$

such that $M_{n,r,1}(f;y)=M_{n,r}(f;y)$. We denote $A_{r,i}(y)=\sum_{l=0}^{\infty}\frac{y^{rl+i}}{(rl+i)!}, i\in N^0=N\cup\{0\}$, and $A_{r,0}(y)=A_r(y)$.

Then, we study the properties of approximation for the sequence (4). Also, we discuss Voronovskaja formula and the error estimate by using the modulus of continuity. Besides, we give a numerical example to approximate the test function $f(\tau)=\sin(10\tau)$ and its first derivative.

2.Preliminary Results

In this section, we present some preliminary results which are used in the main results.

Lemma 1 [7,10] Let $n,r\in N$ and $i\in N^0$, we have:

1. $\lim_{n\to\infty}\frac{A_{r,i}(ny)}{A_r(ny)}=1;$
2. $\lim_{n\to\infty}\frac{A_r^{(i)}(ny)}{n^iA_r(ny)}=1.$

By Lemma 2 for $y\in[0,\infty)$, we get:

$(i)\ M_{n,r,s}(1;y)=\left(1-\frac{1}{A_r(ny)}\right)\to 1\ as\ n\to\infty;$

$(ii)\ M_{n,r,s}(\tau;y)=y\left(1-\frac{1}{A_r(ny)}\right)-\frac{(-y)^s}{A_r(ny)}+\frac{(-1)^s s!}{n^s}\frac{A_{r,s}(ny)}{A_r(ny)}\to y\text{ as }n\to\infty;$

$$(iii)\ M_{n,r,s}(\tau^2;y)=y^2\left(1-\frac{1}{A_r(ny)}\right)-\frac{(-1)^s2y}{A_r(ny)}+\left\{y^s-\frac{s!}{n^s}A_{r,s}(ny)\right\}$$
$$-\frac{1}{A_r(ny)}\left\{y^{2s}-\frac{(2s)!}{n^{2s}}A_{r,2s}(ny)\right\}\to y^2\text{ as }n\to\infty.$$

Proof. We can easily prove this lemma by using direct computation. So, by applying the theorem of Korovkin [11], we get that:

$$\lim_{n\to\infty}M_{n,r,s}(f(\tau);y)=f(y).$$

Definition 3. Let $k\in N^0$, then we define the k-th order moment $T_{n,k,r}^s(y)$ for the sequence $M_{n,r,s}(f(\tau);y)$, which is given as follows:

$$T_{n,k,r}^s(y)=M_{n,r,s}((\tau-y)^k;y)=\frac{1}{A_r(ny)}\int_0^y A_r'(n\tau)(\tau-y)^{ks}d\tau \quad (5)$$

Lemma 4. For the k-th order moment $T_{n,k,r}^s(y)$, we have:

$(i)\ T_{n,0,r}^s(y)=1-\frac{1}{A_r(ny)};$

$(ii)\ T_{n,1,r}^s(y)=\frac{(-1)^s}{A_r(ny)}\left\{\frac{s!}{n^s}A_{r,s}(ny)-y^s\right\};$

$(iii)\ T^s_{n,2,r}(y)=\frac{1}{A_r(ny)}\left\{\frac{(2s)!}{n^{2s}}A_{r,2s}(ny)-y^{2s}\right\}$;

$(iv)\ T^s_{n,k,r}(y)=\frac{(-1)^{ks}}{A_r(ny)}\left\{\frac{(ks)!}{n^{ks}}A_{r,ks}(ny)-y^{ks}\right\}, k\geq 1.$

Further, we have

(1) $T^s_{n,k,r}(y)$ is a polynomial in y which does not exceed ks, for sufficiently large n.

(2) for each $y\in[0,\infty), T^s_{n,k,r}(y)=O(n^{-ks})$.

Proof. Using lemma 2 and the direct evaluations, the proof of this lemma is immediate

Lemma 5. Let $k\geq 1$, then we obtain:

$$M_{n,r,s}(\tau^k;y)=\left(1-\frac{1}{A_r(ny)}\right)y^k+\frac{(-1)^s k}{A_r(ny)}\left(\frac{s!\,A_{r,s}(ny)}{n^s}-y^s\right)y^{k-1}+O(n^{-2s}).$$

Proof. The proof is immediate by using lemma 2 and direct computation.

Lemma 6. For δ, $\alpha>0$ and $[a,b]\subset(0,\infty)$, we get:

$\sup_{y\in[a,b]}\left|\int_{y-\tau\geq\delta}\frac{A'_r(n\tau)}{A_r(n\tau)}e^{\alpha\tau}d\tau\right|=O(n^{-\lambda})$, for $\lambda>0$.

Proof. We can get the proof of this lemma by using the expansion of Taylor, (2), and Schwarz inequality.

Lemma 7 [12]

1) Let $v\in N^0$ and assume that g and h have v-times derivative of x, then:

$$\frac{d^v}{dx^v}(gh(x))=\sum_{i=0}^{v}\binom{v}{i}\frac{d^{v-i}}{dx^{v-i}}(g(x))\frac{d^i}{dx^i}(h(x)).$$

2) Let $g(x)$ be a real or complex-valued and has v-times derivative, then we obtain that:

$$\frac{d^v}{dx^v}\left(\frac{1}{g(x)}\right)=\sum_{l=0}^{v}(-1)^l\binom{v+1}{l+1}\frac{1}{(g(x))^{l+1}}\frac{d^v}{dx^v}(g(x))^l.$$

Lemma 8 [13]. Suppose that the function $F(y)$ is defined by:

$F(y)=\int_{a(y)}^{b(y)}f(y,z)dz$. Then, its derivative is given by:

$$F'(y)=b'(y)f(y,b(y))-a'(y)f(y,a(y))+\int_{a(y)}^{b(y)}\frac{\partial}{\partial y}f(y,z)dz.$$

3.Main Results

This section introduces the main results for the sequence $M^{(v)}_{n,r,s}(f(\tau);y)$, beginning with the following:

For $v\in N$, we have $M^{(v)}_{n,r,s}(f(\tau);y)\to f^{(v)}(y), as\ n\to\infty, v\in N.$

Theorem 1. Let $v\in N, f\in C_\alpha[0,\infty)$ and if f has $(v+1)$ derivatives at a point $y\in(0,\infty)$, then: $\lim_{n\to\infty}M^{(v)}_{n,r,s}(f(\tau);y)\to f^{(v)}(y)$

(6)

Further, if f^{v+1} exists and is continuous on $(a-\eta,b+\eta)\subset(0,\infty)$, $\eta>0$, then the limit (6) holds uniformly on $[a,b]$.

Proof. The expansion of Taylor to the function f is given by:

$f(\tau)=\sum_{i=0}^{v}\frac{f^{(i)}(y)}{i!}(\tau-y)^i+\left(\frac{f^{(v+1)}(\xi)}{(v+1)!}\right)(\tau-y)^{v+1}$, where $\xi\in(\tau,y)$

By operating the sequence $M_{n,r,s}$, and then derivative v-times, we get that:

$$M^{(v)}_{n,r,s}(f(\tau);y)=\sum_{i=0}^{v}\frac{f^{(i)}(y)}{i!}M^{(v)}_{n,r,s}((\tau-y)^i;y)+\frac{f^{(v+1)}(\xi)}{(v+1)!}M^{(v)}_{n,r,s}((\tau-y)^{v+1};y)$$

$:=I_1+I_2$

$$I_1=\sum_{i=0}^{v}\frac{f^{(i)}(y)}{i!}M^{(v)}_{n,r,s}((\tau-y)^i;y)$$

$$= \sum_{i=0}^{v} \frac{f^{(i)}(y)}{i!} \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} y^{i-j} M_{n,r,s}^{(v)}(\tau^j; y)$$
$$= \frac{f^{(v)}(y)}{v!} M_{n,r,s}^{(v)}(\tau^v; y).$$
when $j < v$, $M_{n,r,s}^{(v)}(\tau^j; y) \to 0$ as $n \to \infty$
$$I_1 = \frac{f^{(v)}(y)}{v!} \frac{d^v}{dy^v} \left\{ \left(1 - \frac{1}{A_r(ny)}\right) y^v + \frac{(-1)^s v}{A_r(ny)} \left(\frac{s!\, A_{r,s}(ny)}{n^s} - y^s \right) y^{v-1} + O(n^{-2s}) \right\}$$
Using lemmas.1, 5, and 7, we get:
$$I_1 = \frac{f^v(y)}{v!} \left\{ v! - \frac{d^v}{dy^v} \frac{y^v}{A_r(ny)} + \frac{(-1)^s v s!}{n^s} \frac{d^v}{dy^v} \frac{A_{r,s}(ny)}{A_r(ny)} - (-1)^s v \frac{d^v}{dy^v} \frac{y^{v+s-1}}{A_r(ny)} \right\} + O(n^{-2s})$$
$$\to f^{(v)}(x) \ as\ n \to \infty.$$
$$I_2 = \frac{f^{(v+1)}(\xi)}{(v+1)!} M_{n,r,s}^{(v)}((\tau - y)^{(v+1)}; y)$$
$$= \frac{f^{(v+1)}(\xi)}{(v+1)!} \frac{d^v}{dx^v} \left\{ \frac{1}{A_r(ny)} \int_0^y A'_r(n\tau)(\tau - y)^{(v+1)s} d\tau \right\}$$
Using lemmas 7 and 8, we have:
$$I_2 = \frac{f^{(v+1)}(\xi)}{(v+1)!} \sum_{l=0}^{v} \binom{v}{l} \frac{d^{v-l}}{dy^{v-l}} \left(\frac{1}{A_r(ny)} \right) \frac{d^l}{dy^l} \left(\int_0^y A'_r(n\tau)(\tau - y)^{(v+1)s} d\tau \right)$$
$$= \frac{f^{(v+1)}(\xi)}{(v+1)!} \frac{d^v}{dy^v} \left(\frac{1}{A_r(ny)} \right) \int_0^y A'_r(n\tau)(\tau - y)^{(v+1)s} d\tau$$

$$+ \frac{f^{(v+1)}(\xi)}{(v+1)!} \sum_{l=1}^{v} \binom{v}{l} \frac{d^{v-l}}{dy^{v-l}} \left(\frac{1}{A_r(ny)} \right) \frac{d^l}{dy^l} \left(\int_0^y A'_r(n\tau)(\tau - y)^{(v+1)s} d\tau \right)$$
$$= \Sigma_1 + \Sigma_2$$
$$|\Sigma_1| \le \left\| \frac{f^{(v+1)}(\xi)}{(v+1)!} \right\| \sum_{l=0}^{v} \binom{v+1}{l+1} \left| \frac{1}{(A_r(ny))^{l+1}} \frac{d^v}{dy^v} (A_r(ny))^l \right| \times \left(\int_0^y A'_r(n\tau)|\tau - y|^{(v+1)s} d\tau \right)$$
Applying Schwarz inequality for integration, we obtain:
$$|\Sigma_1| \le \left\| \frac{f^{(v+1)}(\xi)}{(v+1)!} \right\| \sum_{l=0}^{v} \binom{v+1}{l+1} \left| \frac{1}{(A_r(ny))^l} \frac{d^v}{dy^v} (A'_r(ny))^l \right|$$
$$\times \left[\frac{1}{A_r(ny)} \int_0^y A'_r(n\tau)\, d\tau \right]^{1/2} \left[\frac{1}{A_r(ny)} \int_0^y A'_r(n\tau)(\tau - y)^{2(v+1)s}\, d\tau \right]^{1/2}$$
$$= O(1) n^v\, O(n^{-(v+1)s})$$
$= O(n^{v-(v+1)s})$ =o (1) as $n \to \infty$.
Lemmas 7 and 8 give the following:
$$\Sigma_2 = \frac{f^{(v+1)}(\xi)}{(v+1)!} \sum_{l=1}^{v} \binom{v}{l} \frac{d^{v-l}}{dy^{v-l}} \left(\frac{1}{A_r(ny)} \right) \frac{d^l}{dy^l} \left(\int_0^y A'_r(n\tau)(\tau - y)^{(v+1)s} d\tau \right)$$
$$|\Sigma_2| \le \left\| \frac{f^{(v+1)}(\xi)}{(v+1)!} \right\| \sum_{l=1}^{v} \binom{v}{l} \sum_{l_1=0}^{v-l} \binom{v-l+1}{l_1+1} \left| \frac{1}{(A_r(ny))^{l_1+1}} \frac{d^{v-l}}{dy^{v-l}} (A_r(ny))^{l_1} \right|$$

$$\times\left(\int_0^y A'_r(n\tau)\left|\frac{d^l}{dy^l}(\tau-y)^{(v+1)s}\right|d\tau\right)$$
$$= C\left\{\left[\frac{v^2(v-1)}{2A_r^2(ny)}\frac{d^{v-1}}{dy^{v-1}}A_r(ny)+\cdots\right.\right.$$
$$\left.\left.+\frac{(-1)^v(v+1)s((v+1)s-1)\ldots((v+1)s-v+1)}{A_r(ny)}\left(\int_0^y A'_r(n\tau)\left|(\tau-y)^{(v+1)s-v}\right|d\tau\right)\right)\right\}$$

Hence,

$\Sigma_2 = CO(n^{v-(v+1)s})$

$\quad = o(1)$ as $n\to\infty$.

Since $\Sigma_1 \to 0$ and $\Sigma_2 \to 0$ as $n\to\infty$, then $I_2\to 0$ as $n\to\infty$.

Theorem 2. Suppose that $v\in N, f\in C_\alpha[0,\infty)$, and f has $(v+3)$-times at a point $y\in(0,\infty)$, then:

$$\lim_{n\to\infty} n^s\left(M^{(v)}_{n,r,s}(f(\tau);y)-f^{(v)}(y)\right)=(-1)^s s!\, f^{(v+1)}(y) \qquad (7)$$

Further, if f^{v+1} exists and is continuous on $(a-\eta,b+\eta)\subset(0,\infty)$, $\eta>0$, this limit holds uniformly on $[a,b]$.

Proof. By using Taylor's formula for the function f, we get:

$$f(\tau)=\sum_{i=0}^{v+2}\frac{f^{(i)}(y)}{i!}(\tau-y)^i+\frac{f^{(v+3)}(\xi)}{(v+3)!}(\tau-y)^{v+3},$$
$$M^{(v)}_{n,r,s}(f(\tau);y)=\sum_{i=0}^{v+2}\frac{f^{(i)}(y)}{i!}M^{(v)}_{n,r,s}\big((\tau-y)^i;y\big)$$
$$+\frac{f^{(v+3)}(\xi)}{(v+3)!}M^{(v)}_{n,r,s}((\tau-y))^{v+3};y)$$
$$:=\mathcal{H}_1+\mathcal{H}_{2.},$$

Using the same technique of Theorem 1, we obtain:

$\mathcal{H}_2\to 0$ as $n\to\infty$.

Since, $M^v_{n,r,s}\big((\tau-y)^i;y\big)\to 0$ as $i<v$, then

$$\mathcal{H}_1=\sum_{i=v}^{v+2}\frac{f^{(i)}(y)}{i!}M^{(v)}_{n,r,s}\big((\tau-y)^i;y\big)$$
$$=\frac{f^{(v)}(y)}{v!}M^{(v)}_{n,r,s}(\tau^v;y)+\frac{f^{(v+1)}(y)}{(v+1)!}\left\{(-y)(v+1)M^{(v)}_{n,r,s}(\tau^v;y)+M^{(v)}_{n,r,s}(\tau^{v+1};y)\right\}$$
$$+\frac{f^{(v+2)}(y)}{(v+2)!}\left\{\frac{(v+1)(v+2)}{2}y^2M^{(v)}_{n,r,s}(\tau^v;y)-y(v+2)M^{(v)}_{n,r,s}(\tau^{v+1};y)+M^{(v)}_{n,r,s}(\tau^{v+2};y)\right\}$$

By using lemmas 5 and 7, we get:

$$\mathcal{H}_1=\frac{f^{(v)}(y)}{v!}\left\{v!-\frac{d^v}{dy^v}\frac{y^v}{A_r(ny)}+\frac{(-1)^s s!\,v}{n^s}\sum_{l=1}^{v}\binom{v}{l}\frac{d^{v-l}}{dy^{v-l}}y^{v-1}\frac{d^l}{dy^l}\frac{A_{r,s}(ny)}{A_r(ny)}\right.$$
$$\left.-(-1)^s v\frac{d^v}{dy^v}\frac{y^{v+s-1}}{A_r(ny)}\right\}$$

$$+\frac{f^{(v+2)}(y)}{(v+2)!}\Bigg\{-\frac{(v+1)(v+2)}{2}y^2\frac{d^v}{dy^v}\frac{y^v}{A_r(ny)}$$
$$+\frac{(-1)^s v(v+1)(v+2)s!}{2n^s}y^2\sum_{l=1}^{v}\binom{v}{l}\frac{d^{v-l}}{dy^{v-l}}y^{v-1}\frac{d^l}{dy^l}\frac{A_{r,s}(ny)}{A_r(ny)}$$
$$-\frac{(-1)^s v(v+1)(v+2)}{2}y^2\frac{d^v}{dy^v}\frac{y^{v+s}}{A_r(ny)}+(v+2)y\frac{d^v}{dy^v}\frac{y^{v+1}}{A_r(ny)}$$
$$-\frac{(-1)^s v(v+1)(v+2)s!}{n^s}y\sum_{l=1}^{v}\binom{v}{l}\frac{d^{v-l}}{dy^{v-l}}y^{v}\frac{d^l}{dy^l}\frac{A_{r,s}(ny)}{A_r(ny)}$$
$$+(-1)^s(v+1)(v+2)y\frac{d^v}{dy^v}\frac{y^{v+s}}{A_r(ny)}-\frac{d^v}{dy^v}\frac{y^{v+2}}{A_r(ny)}$$
$$+\frac{(-1)^s(v+2)s!}{n^s}\sum_{l=1}^{v}\binom{v}{l}\frac{d^{v-l}}{dy^{v-l}}y^{v-1}\frac{d^l}{dy^l}\frac{A_{r,s}(ny)}{A_r(ny)}+(-1)^s(v+2)\frac{d^v}{dy^v}\frac{y^{v+s+1}}{A_r(ny)}\Bigg\}$$
$$+O(n^{-2s})$$

Now, using lemmas 1:

$$\lim_{n\to\infty}n^s\left(M_{n,r,s}^{(v)}(f(t);x)-f^{(v)}(x)\right)=(-1)^s s!\,f^{(v+1)}(x)$$

Theorem 3. Suppose that $f\in C_\alpha[0,\infty)$, for some $\alpha>0$ and $\le p\le v+2$. It has $p+1$-times derivative and is continuous on $(a-\eta,b+\eta)\subset(0,\infty)$, $\eta>0$, then:

$$\left\|M_{n,r,s}^{(v)}(f(t);x)-f^{(v)}(x)\right\|_{C[a,b]}\le K_1 n^{-s}\sum_{i=v}^{p}\left\|f^{(i)}\right\|_{C[a,b]}$$
$$+K_2 n^{\frac{-1}{2}}\omega_{f^{(p+1)}}\left(n^{\frac{-1}{2}}\right)+O(n^{-2s}), \qquad (8)$$

for sufficiently large n, where K_1,K_2 are constants independent of f and n.

Proof. By a finite Taylor expansion of f, we have

$$f(\tau)=\sum_{j=0}^{p}\frac{f^{(j)}(y)}{j!}(\tau-y)^j+\frac{f^{(p+1)}(\xi)-f^{(p+1)}(y)}{(p+1)!}(\tau-y)^{p+1}\chi(\tau)+h(\tau,y)\big(1-\chi(\tau)\big),$$

where $\xi\in(\tau,y)$ and $\chi(\tau)$ is the characteristic function on the interval $(a-\eta,b+\eta)$.
Let $\tau\in(a-\eta,b+\eta)$ and $\in(0,\infty)$, then we get that:

$$f(\tau)=\sum_{j=0}^{p}\frac{f^{(j)}(y)}{j!}(\tau-y)^j+\frac{f^{(p+1)}(\xi)-f^{(p+1)}(y)}{(p+1)!}(\tau-y)^{p+1}.$$

We define $h(\tau,y)=f(\tau)-\sum_{j=0}^{p}\frac{f^{(j)}(y)}{j!}(\tau-y)^j$, $\tau\in[0,y]\backslash(a-\eta,b+\eta)$, and $y\in[a,b]$,
then $\frac{\partial^v}{\partial y^v}h(\tau,y)=\sum_{j=0}^{p}\frac{f^{(j)}(y)}{j!}(\tau-y)^j$

$$M_{n,r,s}^{(v)}(f(\tau);y)-f^{(v)}(y)=\left[\sum_{j=0}^{p}\frac{f^{(j)}(y)}{j!}M_{n,r,s}^{(v)}\big((\tau-y)^j;y\big)-f^{(v)}(y)\right]$$
$$+M_{n,r,s}^{(v)}\left(\frac{f^{(p+1)}(\xi)-f^{(p+1)}(y)}{(p+1)!}(\tau-y)^{p+1}\chi(\tau);y\right)+M_{n,r,s}^{(v)}\big(h(\tau,y)\big(1-\chi(\tau)\big);y\big)$$
$$:=E_1+E_2+E_3.$$

By using lemmas 4 and 5, we obtain:

$$E_1=\sum_{j=0}^{p}\frac{f^{(j)}(y)}{j!}\sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}y^{j-i}M_{n,r,s}^{(v)}\big(t^i;y\big)-f^{(v)}(y)$$
$$=\sum_{j=v}^{p}\frac{f^{(j)}(y)}{j!}\sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}y^{j-i}\frac{d^v}{dy^v}\left[\left(1-\frac{1}{A_r(ny)}\right)y^i+\left(\frac{(-1)^s iA_{r,s}(ny)}{n^sA_r(ny)}\right)y^{i-1}+O(n^{-2s})\right]$$
$$-f^{(v)}(y).$$

Consequently,

$$\|E_1\|_{C[a,b]} \le K_1 n^{-s}\left[\sum_{j=v}^{p}\|f^{(j)}\|_{C[a,b]}\right] + O(n^{-2s}), \text{uniformly on } [a,b].$$

Now,

$$|E_2| \le M_{n,r,s}^{(v)}\left(\frac{|f^{(p+1)}(\xi)-f^{(p+1)}(y)|}{(p+1)!}|\tau-y|^{p+1}\chi(\tau);y\right)$$

$$\le \frac{\omega_{f^{(p+1)}}(\delta)}{(p+1)!}M_{n,r,s}^{(v)}\left(\left(1+\frac{|\tau-y|}{\delta}\right)|\tau-y|^{p+1};y\right)$$

$$\le \frac{\omega_{f^{(p)}}(\delta)}{(p+1)!}\left|\frac{d^v}{dy^v}\left[\frac{1}{A_r(ny)}\int_0^y A_r'(n\tau)\left((\tau-y)^{(p+1)s}+\frac{1}{\delta}(\tau-y)^{(p+2)s}\right)d\tau\right]\right|, \delta>0. \quad (9)$$

By using lemmas 1 and 7 with $l_1 = 0,1,2,\cdots$, we have:

$$\left|\frac{d^v}{dy^v}\left[\frac{1}{A_r(ny)}\int_0^y A_r'(n\tau)\,(\tau-y))^{(l_1+1)s}d\tau\right]\right|$$

$$=\sum_{l=0}^{v}\binom{v}{l}\left|\frac{d^{v-l}}{dy^{v-l}}\left(\frac{1}{A_r(ny)}\right)\right|\left|\frac{d^l}{dy^l}\left(\int_0^y A_r'(n\tau)(\tau-y)^{(l_1+1)s}d\tau\right)\right|$$

$$=\left|\frac{d^v}{dy^v}\left(\frac{1}{A_r(ny)}\right)\right|\int_0^y A_r'(n\tau)|\tau-y)|^{(l_1+1)s}d\tau$$

$$+\sum_{l=1}^{v}\binom{v}{l}\left|\frac{d^{v-l}}{dy^{v-l}}\left(\frac{1}{A_r(ny)}\right)\right|\left|\frac{d^l}{dy^l}\left(\int_0^y A_r'(n\tau)(\tau-y))^{(l_1+1)s}d\tau\right)\right|$$

$$=I_1+I_2.$$

By applying the same manner in theorem 1(Σ_1, Σ_2), we get:

$I_1 = O\left(n^{v-(l_1+1)s}\right)$ and $I_2 = O\left(n^{v-(v+1)s}\right)$.

By choosing $\delta = \frac{1}{n^s}$ and applying (9), we are led to:

$$\|E_2\|_{C[a,b]} \le \frac{\omega_{f^{(p+1)}}(n^{-s})}{(p+1)!}\left[O\left(n^{(v-(p+1)s)}\right)+n^s O(n^{(v-(p+2)s)})+O\left(n^{v-(v+1)s}\right)\right],$$

$$\le \left(K_2 n^{-(v-(p+1)s)}\right)\omega_{f^{(p+1)}}(n^{-s}).$$

Since $\tau \in [0,y]\backslash(a-\eta,b+\eta)$, there is $\delta>0$ such that $y-\tau\ge\delta$ for all $y\in[a,b]$. Then

$$|E_3| \le \left|\frac{d^v}{dy^v}\left[\frac{1}{A_r(ny)}\int_0^y A_r'(n\tau)\left(h(\tau,y)\left(1-\chi(\tau)\right)\right)d\tau\right]\right|$$

$$\le \left|\frac{d^v}{dy^v}\left[\frac{1}{A_r(ny)}\int_{y-\tau\ge\delta} A_r'(n\tau)\,h(\tau,y)d\tau\right]\right|$$

$$=\left|\frac{d^v}{dy^v}\left(\frac{1}{A_r(ny)}\right)\right|\int_{y-\tau\ge\delta} A_r'(n\tau)\,h(\tau,y)d\tau$$

$$+\sum_{l=1}^{v}\binom{v}{l}\left|\frac{d^{v-l}}{dy^{v-l}}\left(\frac{1}{A_r(ny)}\right)\right|\left|\frac{d^l}{dy^l}\left(\int_{y-\tau\ge\delta} A_r'(n\tau)\,h(\tau,y)d\tau\right)\right|$$

$$=J_1+J_2$$

For $y-\tau\ge\delta$, there exists a constant $K>0$ such that $|h(\tau,y)|\le Ke^{\alpha\tau}$ and $\left|\frac{\partial^v}{\partial y^v}h(\tau,y)\right|\le Ke^{\alpha\tau}$. By applying Lemma 6, we obtain $|E_3| = O\left(n^{-\lambda}\right), \lambda>0$.

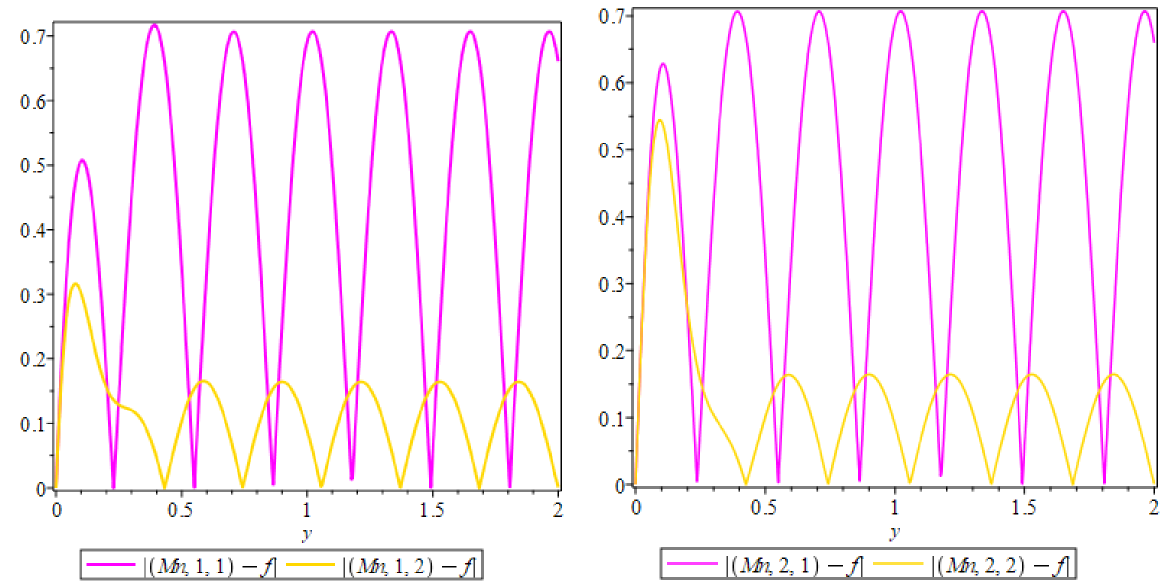
Then, the proof is complete.

4. Numerical Example

Here, we offer a numerical example for the sequence $M_{n,r,s}(f;y)$ $at\ s = 1,2$ and the first derivative, by using the function $f(\tau)=\sin(10\tau)$, $\tau\in[0,2]$. We compare the error of the test function and the first derivative on the interval [0,2], and we discuss the results in figures 1 and 2 for some values of $n = 10$ and $r = 1,2$.

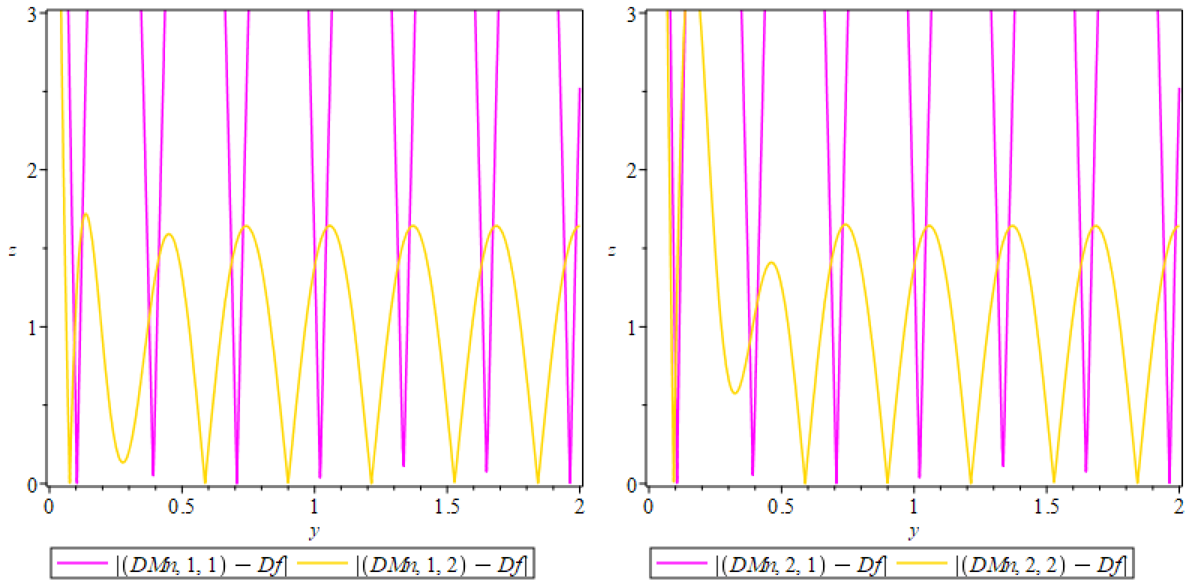
Example

For $n = 10$ and $r = 1,2$ respectively, the comparison between the error functions of the sequences $M_{n,r,s}(f;y)$ at s=1,2 converge to the test function $f(\tau) = \sin(10\tau), \tau \in [0,2]$ when $s = 1$ (Fuchsia) and $s = 2$ (Gold).



(a) (b)

Figure1-The error function for the test function $f(\tau)$, $E_n(y) = \left|M_{n,r,s}(f;y) - f(y)\right|$, when $n = 10, s = 1,2,$, for $r = 1$ in (a) and $r = 2$ in (b).



(c) (d)

Figure 2- The error function for the one derivative of the test function $f(\tau)$, $E_n(y) = \left|\frac{d}{dy}M_{n,r,s}(f;y) - \frac{d}{dy}f(y)\right|$, when $n = 10, s = 1,2,$, for $r = 1$ in (c) and $r = 2$ in (d).

5. Conclusions

From the above numerical example, we observed that the error of approximation becomes lower whenever s becomes higher for a fixed n. This fact is clear from the graphs of the error functions demonstrated in figures (1) and (2). Hence, the sequence $M_{n,r,s}(f;y)$ is more appropriate than the other sequences applyied to the functions in the space $C_\alpha[0,\infty)$.

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