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# The Dynamics of Biological Models with Optimal Harvesting

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**Abstract**

This paper aims to introduce a concept of an equilibrium point of a dynamical system which will call it almost global asymptotically stable. We also propose and analyze a prey-predator model with a suggested function growth in prey species. Firstly the existence and local stability of all its equilibria are studied. After that the model is extended to an optimal control problem to obtain an optimal harvesting strategy. The discrete time version of Pontryagin's maximum principle is applied to solve the optimality problem. The characterization of the optimal harvesting variable and the adjoint variables are derived. Finally these theoretical results are demonstrated with numerical simulations.

**Key words**: Global asymptotically stable, discrete-time predator-prey system, optimal harvesting

# ديناميكية الانظمة البايولوجية مع الحصاد الامثل

**صادق ال ناصر**
**قسم الرياضيات, كلية العلوم, جامعة بغداد, بغداد, العراق**

**الخلاصة :**

هدف هذا البحث هو تقديم مفهوم الاستقرار الكلي التقريبي للنقطة المتزنة . كذلك تمت دراسة السلوك الديناميكي لنماذج احيائية متقطعة مع وجود دالة نمو مقترحة حيث تم ايجاد نقاط الاتزان جميعها وتم دراسة الاستقرار المحلي لكل نقاط الاتزان . تم دراسة الحصاد بنسبة ثابتة ثم توسيع النموذج الى مسالة سيطرة مثلى واستعمل مبدا Pontryagin الاعظم للحصول على الحل الامثل للنظام عدديا . تم تاكيد النتائج النظرية باعطاء امثلة عددية .

## 1-Introduction

The theory of mathematical models plays an important role for studying populations behavior. These models can be described in continuous time case or in discrete time case by a system of ordinary differential equations or a system of
deference equations respectively, this description depends on the study problem.
Discrete time systems are suitable for populations that reproduce at specific times each month or year or each circle, this can be seen in many insects populations, marine fish, and plants. There are variety of studies in the literature that analyzed and investigated the dynamical behavior of this kind of models [1-4],and the references therein.
The essential concept in mathematical modeling is the stability of an equilibrium
point so that an equilibrium point is called globally asymptotically stable if the solution approaches to this equilibrium regardless of the initial condition, while it is called locally asymptotically stable if

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there exists a neighborhood of this equilibrium such that from every initial condition within this neighborhood the solution approaches to it. Some authors in the literatures [5-12] put some assumptions or conditions for example in some fisheries models they consider the net reproduction number is greater than one that means they excluded the origin point from the state space of their models or they did not consider the extinct of one or more member species in their analysis in order to prove the unique positive equilibrium point in their models is global asymptotically stable. All that assumptions or conditions are needed to avoid the violation of the well known fact , if an equilibrium point of a system is global asymptotically stable then it must to be the unique equilibrium point of that system[13]. Note that all biological interpretations of that assumptions or conditions are well set and are well done. So that in this study we will introduce a definition of an equilibrium point of a system which we call it almost global asymptotically stable, simply means an equilibrium point is global asymptotically stable in the interior domain of the state space of that system as well as biological model, prey predator models with suggested growth function in prey species is investigated in discrete time case.

A system of difference equations may have and show a rich and more complicated dynamical behaviors even for a simple one dimensional system. For example the logistic equation which is very known equation, its equilibria can vary from stability behavior to chaotic behavior[14-16]. Many researchers investigated and analyzed different kind of two or more than two dimensional models in ecology [17- 19], they derived conditions for local and global stability of solutions as well as the existence of periodic solutions [20, 21]. Some authors have discussed and assumed that the life of populations have two stages immature and adults. However stage-structured or age structured models are considered in the literatures [22-24].

The general form of the suggested model is given by :

$$x_{t+1} = \frac{r x_t^m}{(1+ke^{bx_t})^n} - a y_t h(x_t) \qquad 1.1)$$
$$y_{t+1} = -c y_t + d y_t h(x_t)$$

The continuous time version of model (1.1) can be written in the following form

$$x' = \frac{r x(t)^m}{\left(1+ke^{bx(t)}\right)^n} - a y(t) h(x(t)) \qquad (1.2)$$
$$y' = -c_t y(t) + d y(t) h(x(t))$$

Where $x(t)$, $y(t)$and $h(x(t))$ are the prey population density, the predator population density and the predator functional response at time $t$ respectively. The parameters $r, c\ , a,$ and $d$ are model parameters supposing only positive values. These parameters are the intrinsic growth rate of prey species, the mortality rate of predators species, the maximum per capita killing rate ,and the conversion rate predator respectively .While $b, k, m$ and $n$ are positive constants.

This paper is organized as following: In section 2 we will discuss the model (1.1 ) with absent the predator species and when $n = m = b = 1$. In section 3 the prey-predator discrete time system is analyzed and all behavior of its equilibria are investigated. In section 4 the model is extended to an optimal control problem. The discrete time version of Pontryagin's maximum principle is applied to solve the optimality problem. In section 5 numerical simulations is provided to confirm the theoretical results. Finally conclusions are given.

**2- Single species model**

**Definition 1**

Consider the following nonlinear discrete dynamical system $x_{t+1} = f(\ x_t)$ where $f: D \subset R^n \to R^n$. An equilibrium point $x_e$that means $f(x_e) = x_e$ is said to be almost global asymptotically stable in $D$ if it is global asymptotically stable in $D - \partial D,$ where $\partial D$ is the boundary of $D$.

For the continuous time case the definition will be as follows:

Consider the following nonlinear continuous dynamical system $x' = f(x(t))$ where $f: D \subset R^n \to R^n$. An equilibrium point $x_e$that means $f(x_e) = 0$ is said to be almost global asymptotically stable in $D$ if it is global asymptotically stable in $D - \partial D$.

**Remarks:**

1- It is clear that every global asymptotically stable is almost global asymptotically stable. The converse is not true. In [7], the trivial equilibrium point always exists so that the positive equilibrium point is almost global asymptotically stable.

2- It is clear that if $x_e$ is almost global asymptotically stable then it is local asymptotically stable, however the converse is not true. For example :
Consider the following system

$$x_{t+1}=s^2x_t(1-x_t)(1-sx_t+sx_t^2)$$

Where $s$ is constant parameter. If $s>3$the system has two positive equilibria ,namely$x^*=\frac{1+s-\sqrt{(1-s)^2-4}}{2s}$ $and\ y^*=\frac{1+s+\sqrt{(1-s)^2-4}}{2s}$ If $s=3.1$ then $x^*=$0.558 and $y^*=$0.7646. The point $x^*=$0.558 is locally stable which is not almost global asymptotically stable point.

Now we will investigate the dynamics of single species model of (1.1) in the absent of the predator species and n=m=b=1. Thus the model will be as follows

$$x_{t+1}=\frac{rx_t}{1+ke^{x_t}} \quad (2.1)$$

The model(2.1) has two equilibria , namely, the trivial equilibrium point $x_1=0$ and the unique positive po$x_2=\ln(\frac{r-1}{k})$ . The trivial equilibrium point always exists, while the positive equilibrium exists when $r>k+1$. The following Theorem gives the behavior of its equilibria.

**Theorem 1** For the model (2.1) we have :

1- The trivial equilibrium point, $x_0=0$ is locally stable (sink) point if and only if $r<k+1$ and it is unstable (source) point if and only if $r>k+1$ ,while it is non-hyperbolic point if and only if $r=k+1$ .

2- The equilibrium point $x_2=\ln(\frac{r-1}{k})$ is locally stable (sink) point if $k\in((r-1)e^{-\frac{2r}{r-1}},(r-1))$It is unstable (source) point if and only if $k<(r-1)e^{-\frac{2r}{r-1}}$ ,while it is non-hyperbolic point if and only if $k=(r-1)e^{-\frac{2r}{r-1}}$

Proof:
It is clear that $f'(x_1)=\frac{r}{k+1}$ so that the results in 1 can be easily obtained.
To prove 2, one can easily see that $f'(x_2)=\frac{r-rx_2+x_2}{r}$ then $|f'(x_2)|<1$ if and only if $(r-1)e^{-\frac{2r}{r-1}}<k<(r-1)$ , and the results can be got.

Now we will consider a situation that population is exposition to harvest by a constant rate harvesting which is proportional to the respective population size therefore the model (2.1) including the harvesting will be as follows :

$$x_{t+1}=\frac{rx_t}{1+ke^{x_t}}-\ hx_t \quad (2.2)$$

Where $h$ is a positive constant representing the intensity of removing due to hunting or removal. It is obvious that one cannot remove more than the population density therefore $h\le h_{max}<1$, $h_{max}$ is the maximum removing amount.

The model(2.2) has also two equilibria , the trivial equilibrium point $x_0$which always exists, and the unique positive equilibrium $x_h=\ln(\frac{r-(1+h)}{k(1+h)})$ exists only when $\frac{r-(1+h)}{k(1+h)}>1$ .

Next Theorem describes the behavior of the equilibria of model(2.2).

**Theorem 2** For the model (2.2),the equilibria, $x_0$ , and $x_h$are

1- The equilibrium point $x_0$ is locally stable(sink) point if $r<(k+1)(h+1)$ and it is unstable(source) if $r>(k+1)(h+1)$ , while it is non-hyperbolic point if $r=(k+1)(h+1)$.

2- The equilibrium point $x_h=\ln(\frac{r-(1+h)}{k(1+h)})$ is locally stable(sink) point if $k\in(\frac{(r-(1+h))e^{-\frac{2r}{m}}}{1+h},\frac{r-(1+h)}{1+h})$and it is unstable(source) point if $k<\frac{(r-(1+h))e^{-\frac{2r}{m}}}{1+h}$, while it is non-hyperbolic point if $k=\frac{(r-(1+h))e^{-\frac{2r}{m}}}{1+h}$, where $m=(1+h)(r-(1+h))$

**Proof:**

It is clear that $f'(x) = \frac{r(1+ke^x)-rke^x}{r+ke^{x^2}} - h$, then $f'(x_0) = \frac{r}{1+k} - h$ and $x_0$ is locally stable if $|f'(x_0)| = \left|\frac{r}{1+k} - h\right| < 1$. Therefore this holds if $r < (k+1)(h+1)$, and the results of (1) are directly got.

It is clear that $f'(x_h) = \frac{r - x_h m}{r}$, and let $k \in (\frac{(r-(1+h))e^{-\frac{2r}{m}}}{1+h}, \frac{r-(1+h)}{1+h})$ then $\frac{(r-(1+h))e^{-\frac{2r}{m}}}{1+h} < k < \frac{r-(1+h)}{1+h}$ this gives $1 < \frac{r-(1+h)}{k(1+h)} < e^{\frac{2r}{m}}$. Therefore $0 < x_h < \frac{2r}{m}$, and $-2r < -x_h m < 0$, so that $\left|f'(x_h) = \frac{r-x_h m}{r}\right| < 1$. Therefore all results of (2) can be directly obtained.

**3-Two Species Model, Prey-Predator Model**

In this section we will study in details the dynamics of the two species model discrete time case of model (1.1) with $n = m = b = 1$. Thus the system can be written as

$$x_{t+1} = \frac{rx_t}{1+ke^{x_t}} - ay_t x_t \qquad (3.1)$$

$$y_{t+1} = -cy_t + dy_t x_t$$

The all parameter $a, r, c, d$ and $k$ are defined the same as before. By solving the following algebraic equation one can get all equilibrium points of the model(3.1 ):

$$\text{x}= \frac{rx}{1+ke^x} - ayx \qquad (3.2)$$

$$y = -cy + dxy$$

Therefore we have the following Theorem.

**Theorem 3**

For all parameters values the equilibrium points of the model (3.1) are

1- The trivial equilibrium point $e_0 = (0,0)$ always exists.

2- The boundary equilibrium point $e_1 = (\ln\left(\frac{r-1}{k}\right), 0)$ exists only when $r > 1 + k$.

3- The unique positive equilibrium point $e_2 = (x^*, y^*) = (\frac{1+c}{d}, \frac{r-\left(1+ke^{x^*}\right)}{a(1+ke^{x^*})})$ $r > 1 + ke^{x^*}$

In order to investigate the dynamic behavior of the model(3.1 ) one has to compute the general Jacobian matrix of the model (3.1) at point (x,y). This is given by:

$$J(x,y) = \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix}$$

Where $j_{11} = \frac{r+rke^x - rkxe^x}{(1+ke^x)^2} - ay, j_{12} = -ax, j_{21} = dy$, and $j_{22} = dx - c$.

Next theorems give the local stability of $e_0$ and $e_1$ respectively.

**Theorem 4**

For the model (3-1), the equilibrium point $e_0$ is

1- Locally stable (sink) point if $r < k + 1$ and $c < 1$. It is unstable (source) point if $r > k + 1$ , and $c > 1$ ,while it is non-hyperbolic point if $r = k + 1$ or $c = 1$.

2- Saddle point if $r > k + 1$ and $c < 1$or $r < k + 1$ and $c > 1$ .

**proof**

It is clear that the Jacobian matrix at the point $e_0$ is

$J_{e_0} = \begin{bmatrix} \frac{r}{r+k} & 0 \\ 0 & -c \end{bmatrix}$ so that the eigenvalues of $J_{e_0}$ are $\lambda_1 = \frac{r}{r+k}$ and $\lambda_2 = -c$ Thus $|\lambda_1| < 1$ $(|\lambda_1| > 1)$ if and only if $r < k + 1 (r > k + 1)$ and $|\lambda_2| < (|\lambda_2| > 1)$ if and only if $c < 1 (c > 1)$ as well as $|\lambda_1| = 1$ or $|\lambda_2| = 1$ if and only if $r = 1 + k$ or $c = 1$ Therefore the proof is finished.

**Theorem 5**

For model (3.1) the equilibrium point $e_1$ has the following:

1- The equilibrium point $e_1$ is locally stable(sink) point if $k \in I_1 \cap I_2$ .
2-The equilibrium point $e_1$ is unstable (source) if $k \in I_3 \cap I_5$ , or $k \in (\max\{(r-1),(r-1)e^{-\frac{c-1}{d}}\},\infty)$.
3- The equilibrium point $e_1$ is saddle point if one of the following holds:
a)- $k \in I_1 \cap I_5$
b)- $k \in I_1 \cap I_4$
c)- $k \in I_3 \cap I_2$
4- The equilibrium point $e_1$ is non-hyperbolic point if
$k = (r-1)e^{-\frac{2r}{r-1}}$, or $k = (r-1)e^{-\frac{c+1}{d}}$, or $k = (r-1)e^{-\frac{c-1}{d}}$ .

where $I_1 = ((r-1)e^{-\frac{2r}{r-1}},(r-1))$, $I_2 = ((r-1)e^{-\frac{c+1}{d}},(r-1)e^{-\frac{c-1}{d}})$ $I_3 = (0,(r-1)e^{-\frac{2c}{r-1}})$ , $I_4 = ((r-1)e^{-\frac{c-1}{d}},(r-1))$ and $I_5 = (0,(r-1)e^{-\frac{c+1}{d}})$
**proof:**
The Jacobian matrix at the point $e_1$ is

$$J_{e_1} = \begin{bmatrix} \frac{r-(r-1)\ln\left(\frac{r-1}{k}\right)}{r} & -a\ln\left(\frac{r-1}{k}\right) \\ 0 & d\ln\left(\frac{r-1}{k}\right)-c \end{bmatrix}$$

then the eigenvalues of $J_{e_1}$ are $\lambda_1 = \frac{r-(r-1)\ln\left(\frac{r-1}{k}\right)}{r}$ and $\lambda_2 = d\ln\left(\frac{r-1}{k}\right)-c$ then $|\lambda_1|<1$ if and only if $\left|\frac{r-(r-1)\ln\left(\frac{r-1}{k}\right)}{r}\right|<1$. Hence $|\lambda_1|<1$ if and only if $-r < r-(r-1)\ln\left(\frac{r-1}{k}\right) < r$ . Therefore $|\lambda_1|<1$ if and only if $\frac{-2r}{(r-1)} < \ln\left(\frac{k}{r-1}\right) < 0,$ *and* $(r-1)e^{-\frac{2r}{r-1}} < k < (r-1)$ . *Now* $|\lambda_2|<1$ if and only if $\left|d\ln\left(\frac{r-1}{k}\right)-c\right|<1$ . Therefore $|\lambda_2|<1$ if and only if $(r-1)e^{-\frac{c+1}{d}} < k < (r-1)e^{-\frac{c-1}{d}}$. Therefore all results can be directly obtained.
In order to discuss the dynamic behavior of the unique positive equilibrium, the next lemma is needed which is proved in [24].
**Lemma 1** [24]
Let $F(\lambda) = \lambda^2 + P\lambda + Q$ and suppose that $F(1) > 0$, and $\lambda_1, \lambda_2$ are the roots of F then :
1- $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $Q < 1$.
2- $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$) if and only if $F(-1) < 0$.
3- $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $Q > 1$.
4- $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $F(-1) = 0$ and $P \neq 0, 2$.
Proof: See[24].
**Theorem 6**
For the unique positive equilibrium point $e_2$ of the model (3.1 ) we have :
1- The equilibrium point $e_2$ is locally asymptotically stable (sink) point if and only if $d > N$ and $r \in S$.
2- The equilibrium point $e_2$ is unstable (source) point if and only if $d > N$ and $r > Max\left\{\frac{M_2}{M_1},\frac{N_2}{N_1},k_1\right\}$
3- The equilibrium point $e_2$ is saddle point if and only if $d > N$ and $k_1 < r < \frac{M_2}{M_1}$.
4- The equilibrium point $e_2$ is non-hyperbolic point if these conditions are hold:
i) $d \neq \frac{2ke^{x^*}}{k_1}$
ii) $r = \frac{M_2}{M_1}$

iii) $r \neq \frac{2k_1^2}{k\,x^* e^{x^*}}$ or $r \neq \frac{4k_1^2}{k\,x^* e^{x^*}}$

Where $k_1 = 1 + ke^{x^*}$, $N = \frac{2ke^{x^*}}{k_1}$, $S = \left(Max\left\{k_1, \frac{M_2}{M_1}\right\}, \frac{N_2}{N_1}\right)$, $M_1 = dk_1 - 2k\,x^* e^{x^*}$, $M_2 = dx^* k_1^2 - 4k_1^2$ $N_1 = dk_1 - k\,e^{x^*}$, and $N_2 = dk_1^2$.

**Proof:**

We will apply Lemma 1 then the Jacobian matrix at the unique positive equilibrium point is given by

$$J_{e_2} = \begin{bmatrix} \frac{rk_1 - rx^* k\,e^{x^*}}{k_1^2} - ay^* & -ax^* \\ dy^* & 1 \end{bmatrix}$$

So that the characteristic polynomial of $J_{e_2}$ *is* $F(\lambda) = \lambda^2 + P\lambda + Q$

Where $P = a\,y^* + \frac{rx^* ke^{x^*}}{k_1^2} - \frac{r}{k_1} - 1$ and $Q = \frac{r}{k_1} - \frac{rx^* ke^{x^*}}{k_1^2} - a\,y^* + ad\,x^*\,y^*$.

It is easy to see that $F(1) = ad\,x^*\,y^*$ hence $F(1) > 0$. We want to prove $F(-1) > 0$ and $Q < 1$. So that $F(-1) = 1 - P + Q > 0$

if and only if $2 - 2ay^* + \frac{2r}{k_1} - \frac{2rx^* ke^{x^*}}{k_1^2} + ad\,x^*\,y^* > 0$, since $ay^* = \frac{r}{k_1} - 1$ then $F(-1) > 0$ if and only if $4 - \frac{2r}{k_1} + \frac{2r}{k_1} - \frac{2rx^* ke^{x^*}}{k_1^2} + \frac{drx^*}{k_1} - d\,x^* > 0$. Hence $4k_1^2 - 2rx^* ke^{x^*} + rdx^* k_1 + dx^* k_1^2 > 0$ and this gives $r\left(dx^* k_1 - 2x^* ke^{x^*}\right) > dx^* k_1^2 - 4k_1^2$. Therefore if we have $d > N$ and $r \in S$ then $F(-1) > 0$.

Now $Q = \frac{r}{k_1} - \frac{rx^* ke^{x^*}}{k_1^2} - a\,y^* + ad\,x^*\,y^* < 1$ if and only if $-rx^* ke^{x^*} + rdx^* k_1 - dx^* k_1^2 < 0$. Then $Q < 1$ if and only if $r(dx^* k_1 - x^* ke^{x^*}) < dx^* k_1^2$. Therefore if we have $r < \frac{N_2}{N_1}$ with $d > N$, then $Q < 1$ According to the lemma 1 the proof (1), (2), (3), and (4) can be got.

**4-An optimal harvesting approach :**

In this section we will extend the model(1.1) to an optimal control problem and will discuss the optimal harvesting management of renewable resources. We assume that the population is harvested or removed with the harvesting rate $h_t$, which represents our control variable. For the single species the model (2.1) including the harvesting effect becomes :

$$x_{t+1} = \frac{rx_t}{1 + ke^{x_t}} - h_t x_t$$

The $x_t$ $r$, and $k$ are defined as before. In this problem the free terminal value problem is discussed and the terminal time $T$ is specified. The aim is to maximize the following objective functional

$$J(h) = \sum_{t=0}^{T-1} c_1\,h_t x_t + \frac{c_2}{2} h_t^2$$

Where $c_1 h_t x_t$ represents the amount of money that one has to obtain, and $c_2 h_t^2$ is the cost of catching and supporting the animal. $c_1$ and $c_2$ are positive constants. The control variable is subject to the constraint

$0 \leq h_t \leq h_{max} < 1$

Now according to the discrete version of Pontryagin's maximum principle [25] the Hamiltonian functional for this problem is given by

$$H_t = \sum_{t=0}^{T-1} c_1\,h_t x_t - c_2 h_t^2 + \lambda_{t+1}\left(\frac{rx_t}{1 + ke^{x_t}} - h_t x_t\right)$$

Where $\lambda_t = c_1 h_t + \frac{r + ke^{x_t} - rx_t ke^{x_t}}{(1 + ke^{x_t})^2} - h_t$ is the adjoint variable or shadow price [26]. Then the characterization of the optimal control solution is

$$h_t^* = \begin{cases} 0 \quad if \quad \frac{c_1 x_{t-} \lambda_{t+1} x_t}{2c_2} \le 0 \\ \frac{c_1 x_{t-} \lambda_{t+1} x_t}{2c_2} \quad 0 < \frac{c_1 x_{t-} \lambda_{t+1} x_t}{2c_2} < h_{max} \\ h_{max} \quad if \quad h_{max} < \frac{c_1 x_{t-} \lambda_{t+1} x_t}{2c_2} \end{cases}$$

In order to extend the two species model (3.1) to an optimal control problem, the model (3.1)with control harvesting variable will become as follows:

$$x_{t+1} = \frac{r x_t}{1 + k e^{x_t}} - a y_t x_t - h_t x_t \qquad (4.1)$$

$$y_{t+1} = -c y_t + d y_t x_t$$

Our aim in this problem is to get an optimal harvesting amount for that we will maximize the following cost functional

$J(h_t) = \sum_{t=0}^{T-1} c_1 h_t x_t - c_2 h_t^2.$

subject to the state equations (4.1) with control constraint

$0 \le h_t \le h_{max} < 1$

All terms and parameters are as before . So that the Hamiltonian functional will be as the following:

$$H_t = \sum_{t=0}^{T-1} c_1 h_t x_t - c_2 h_t^2 + \lambda_{1,t+1} \left( \frac{r x_t}{1 + k e^{x_t}} - h_t x_t \right) + \lambda_{2,t+1}(-c y_t + d y_t x_t)$$

Where $\lambda_{2,t}$ and $\lambda_{2,t}$are the adjoint functions that satisfy :

$$\lambda_{1,t} = c_1 h_t + \lambda_{1,t+1}\left(\frac{r + k e^{x_t} - r x_t k e^{x_t}}{(1 + k e^{x_t})^2} - a y_t - h_t\right) + \lambda_{2,t+1} d y_t$$

$$\lambda_{2,t+1} = \lambda_{1,t+1}(-a\, x_t) + \lambda_{1,t+1}(-c + d x_t)$$

Furthermore the characterization of the optimal harvesting solution $h_t^*$ satisfies:

$$h_t^* = \begin{cases} 0 \quad if \quad \frac{c_1 x_{t-} \lambda_{1,t+1} x_t}{2c_2} \le 0 \\ \frac{c_1 x_{t-} \lambda_{1,t+1} x_t}{2c_2} \quad 0 < \frac{c_1 x_{t-} \lambda_{1,t+1} x_t}{2c_2} < h_{max} \\ h_{max} \quad if \quad h_{max} < \frac{c_1 x_{t-} \lambda_{1,t+1} x_t}{2c_2} \end{cases}$$

An iterative method is used to get the optimal control with corresponding optimal state solutions of the above optimal control problems at time t by maximizing the Hamiltonian functional at that t numerically.

## 5- Numerical Simulations

In this section we will confirm the theoretical findings numerically for various set of parameters. For the local stability of the equilibrium point $e_0$ of the model (3.1), we choose this set of values $a = 0.1, r = 09., k = 0.01, d = 1.2, c = 0.01$ and $(x_0, y_0) = (0.3, 0.01)$ so that by (1) in theorem(3) the point is sink. For the equilibrium point $e_1$ this set of values $a = 0.1, r = 1.9., k = 0.6, d = 2, c = 0.2$ and $(x_0, y_0) = (0.9, 0.4)$ are used. Then according to the condition (1) in theorem (5) the point is sink. Figures- 1-2 show the locally stability of $e_0$ ,and $e_1$ respectively. For the unique positive equilibrium point $e_2$ the set of values $a = 0.1, r = 5., k = 2, d = 3, c = 0.61$ and $(x_0, y_0) = (0.53, 1.9)$ are chosen. According to (1) in theorem (6). The local stability of $e_2$ is illustrated in Figure- 3. Other sets of values can be chosen to show the local stability of $e_0$, $e_1$ ,and $e_2$ .

For the optimal control problem an iterative numerical method in [20] is used to determine the optimal solutions with corresponding state solutions. For the control problem of single species we choose this set of values of parameters $r = 1.99, k = 0.8, c_1 = 0.1, c_2 = 0.01, x_0 = 0.1$ and $T = 80$ so that the total optimal objective functional $J_{opt}$ is found equal to 0.0412. In Figure- 4 the prey population density with control , without control and with constant harvesting is plotted. Figure- 5 illustrates the control variable as a function of time.

For the optimal control problem of two species model we choose this set of values of parameters $a = 0.1, r = 5.2, k = 2.1, c = 0.5, d = 2.9, c_1 = 0.025, c_2 = 0.08, x_0 = 0.1$ $(x_0, y_0) = (0.5,0.8)$ and $T = 80$. So that the total optimal objective functional $J_{opt}$ is found equal to 0.0491. Figures-6-7 illustrate the prey population and the predator population with control ,without control and with constant control respectively, and Figure- 8 shows the control variable as a function of time. Finally, Table- 1 contains the total optimal objective functional and other different total harvesting amount strategies of both control problems by using the same values of the parameters in each problem.

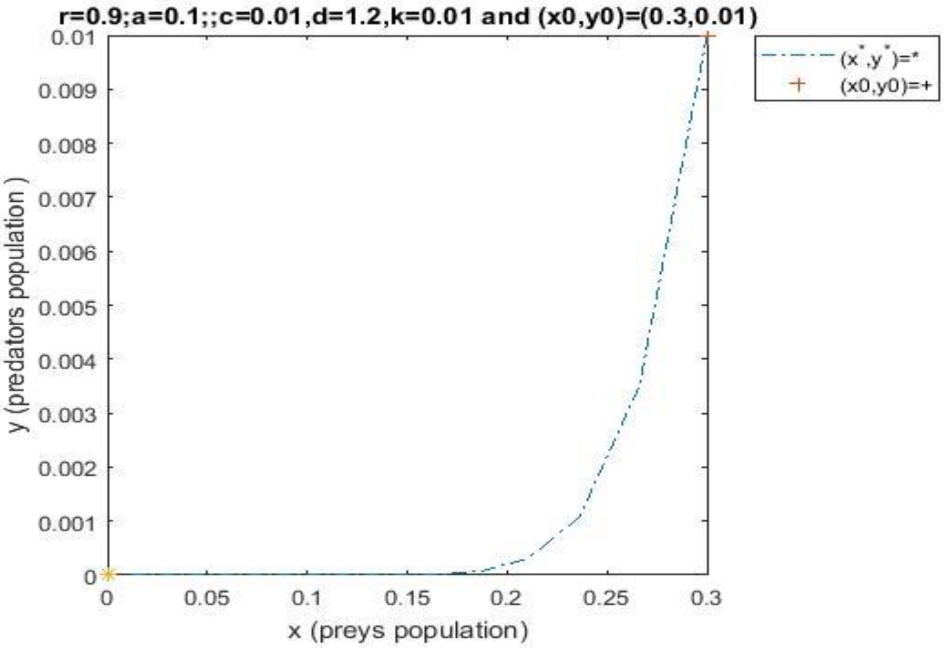


**Figure 1-** This figure shows the local stability of the equilibrium point $e_0$ of the model (3.1).

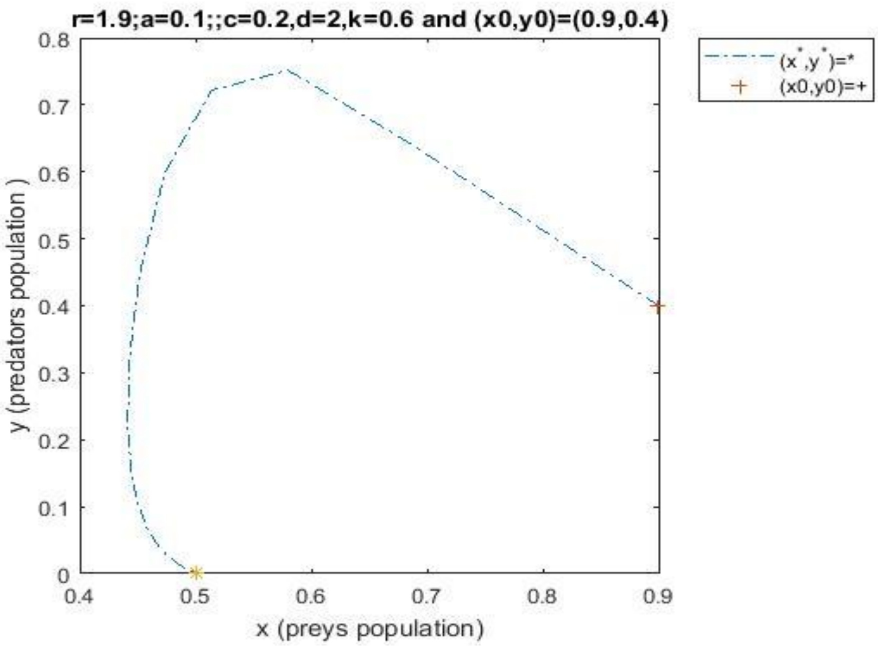


**Figure 2-** This figure shows the local stability of the equilibrium point $e_1$ of the model (3.1).

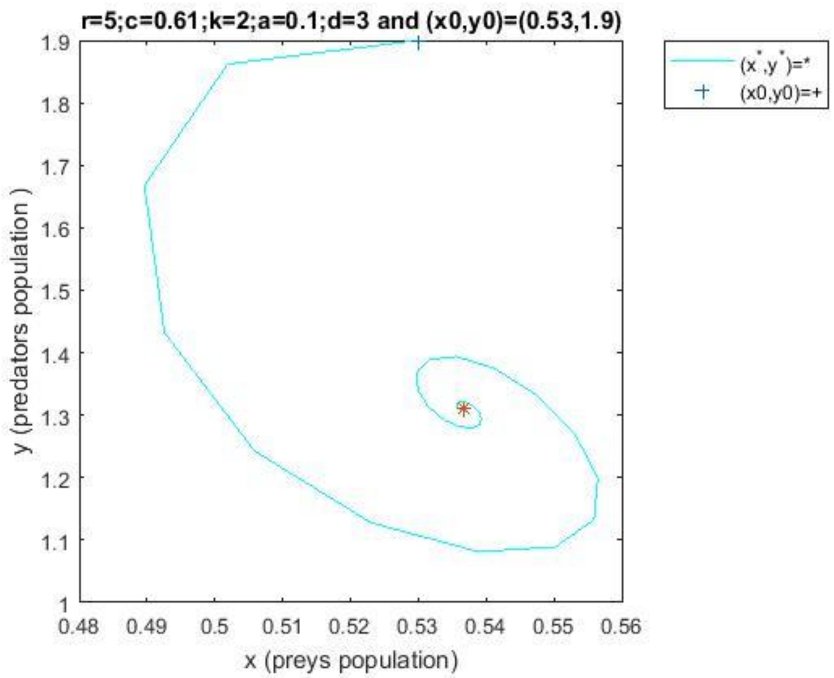


**Figure 3-**This figure shows the local stability of the unique equilibrium point $e_2$ of the model (3.1).

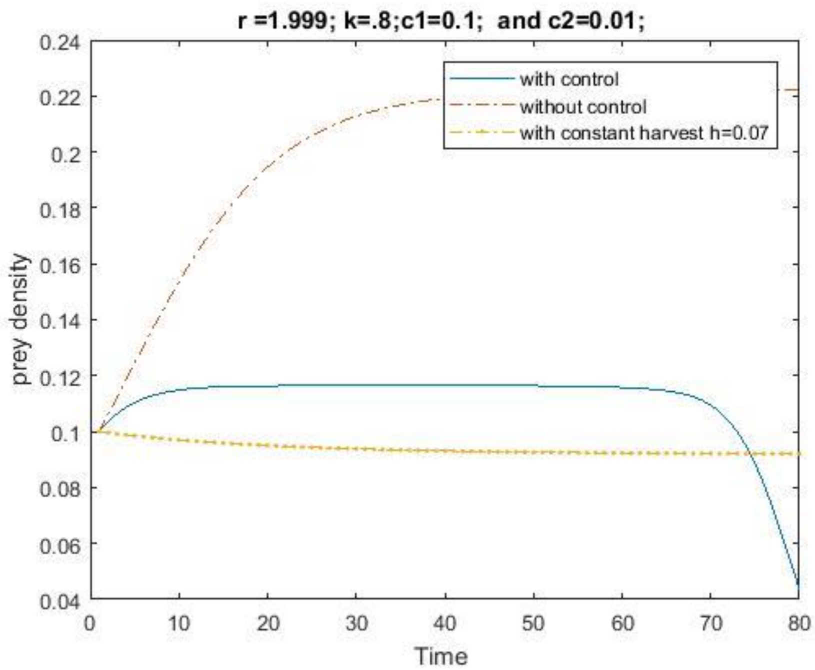


**Figure 4-** This figure shows the effect of different harvesting variables on the prey density in model (4.1). All values of parameters are the same in all cases.

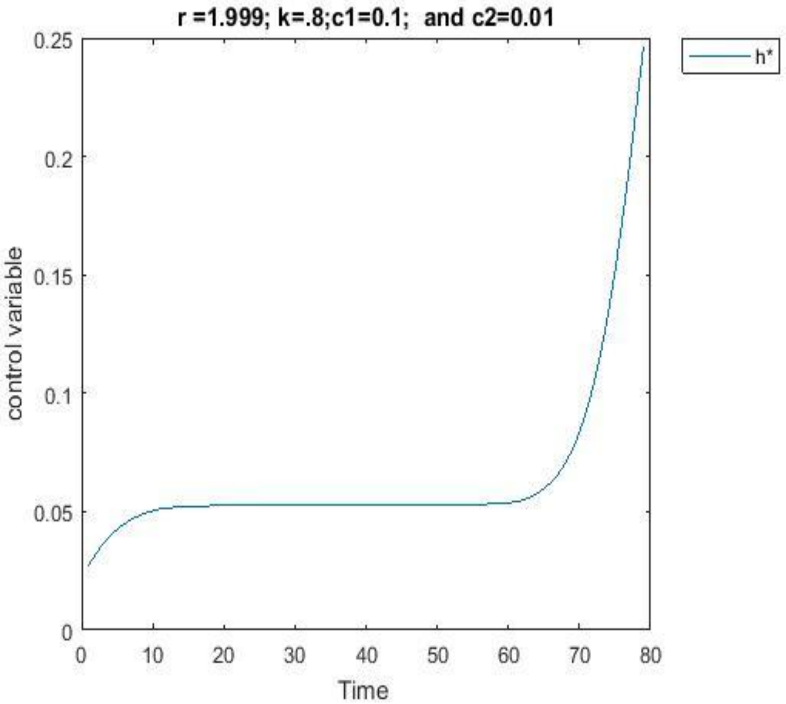


**Figure 5-** The optimal control variable of single species problem is illustrated function of time.

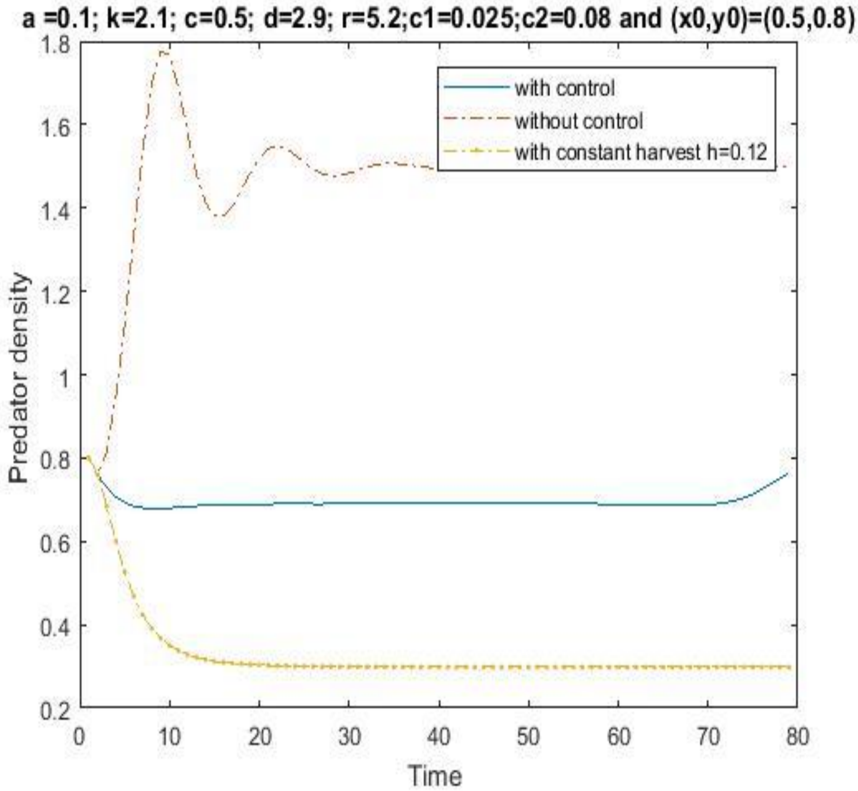


**Figure 6-** This figure shows the affect of harvesting variable on the predator density in model (4.3). All values of parameters are the same

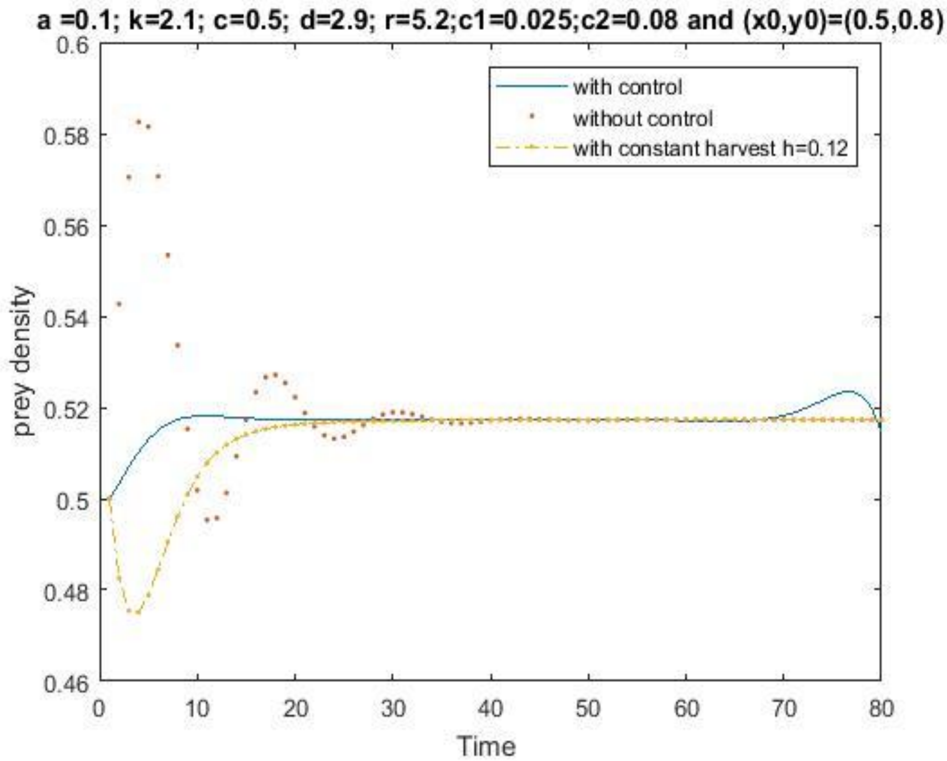


**Figure 7-** This figure shows the affect of different harvesting variables on the prey density in model (4.3). All values of parameters are the same in all cases.

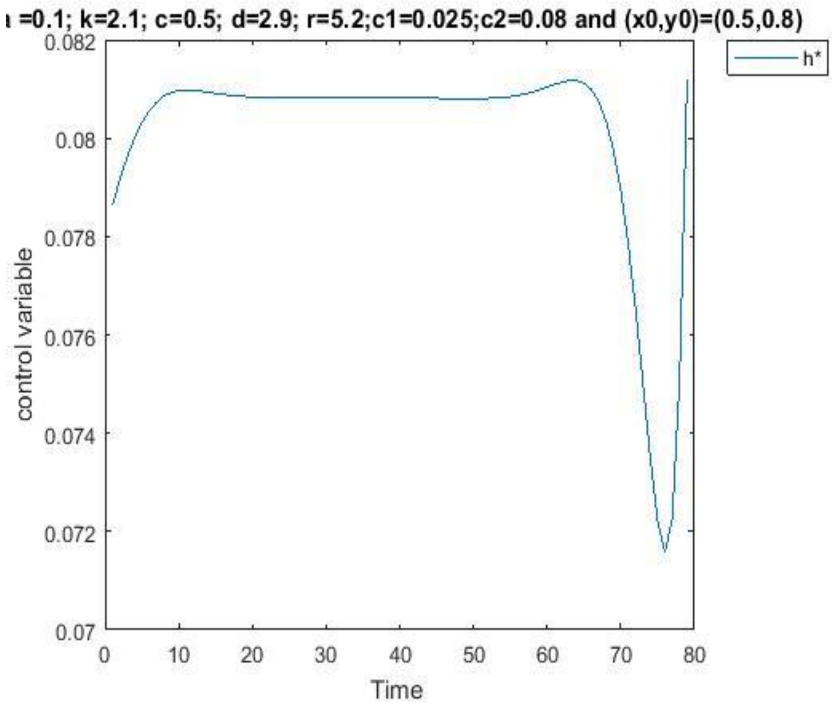


**Figure 8-**The optimal control solution of two species problem is plotted as function of time.

**Table 1-** This table indecates the results of the optimal harvesting amount with different constant harvesting.

| single species model | | Two species model | |
|---|---|---|---|
| The harvesting variable | The objective functional (J) | The harvesting variable | The objective functional (J) |
| $h_t = h^*$ | $J_{opt}$ =0.0491 | $h_t = h^*$ | $J_{opt} = 0.04121$ |
| $h_t$ =0.065 | J =0.0449 | $h_t$ =0.12 | $J = 0.0306$ |
| $h_t = 0.06$ | $J = 0.04524$ | $h_t$ =0.1 | $J$ =0.0386 |
| $h_t$ =0.058 | J =0.04522 | $h_t$ =0.09 | $J$ =0.04052 |
| $h_t$ =0.055 | J =0.0450 | $h_t$ =0.08 | $J = 0.04118$ |
| $h_t$ =0.05 | J =0.0442 | $h_t$ =0.07 | $J$ =0.04054 |

## 6- Conclusions and Discussion:

In this section the definition of almost global asymptotically stable of an equilibrium point has been introduced. A discrete time biological model in one dimension as well as in two dimensions has been investigated and analyzed. A modified growth function is also studied. We shall summarize the obtained results as following

i- It is found that the one dimension model(2.1) has two equilibria ,while the two dimension prey predator model(3.1) has three equilibrium points. In both models the trivial (vanishing) equilibrium point always exists without any restriction. Under certain conditions the axial point and the unique equilibrium point dynamics in two dimension model are exist.

ii- It is shown that the local stability of the trivial equilibrium point only when the value of $r$ , the growth rate, is less than $k + 1$.The local stability around the other possible equilibrium points are studied in details.

iii- the model (2.1) and (3.1) are extended to an optimal control problem. The optimal control solutions are found by using the Pontaygin's principle maximum. Table- 1 shows the results and the total optimal objective functional which is $J_{opt} = 0.0491$ *and* $J_{opt} = 0.04121$ in one dimension model and two dimension model respectively. Other different total harvesting amount strategies of both control problems is given .

iv- In order to confirm the findings results we represent the numerical results in form time series. Figures show the behavior of the local stability as well as the optimal control solution clearly.

v- The dynamics behavior of the continuous time model as well as the other values of $n, m$ and $b$ can study in future work.

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