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On Small Primary Modules

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Abstract:
Let R be a commutative ring with an identity and X be a unitary R-module. We say that a non-zero submodule P of X is small primary if for each $a \in R$, $x \in X, (x) \ll X$ with $ax \in P.$ Then either $x \in P$ or $a \in \sqrt{[P:X]}$ and an R-module X is a small primary if $\sqrt{ann\, X} = \sqrt{ann\, P}$ for each proper submodule P small in X. We provided and demonstrated some of the characterizations and features of these types of submodules (modules).

Keywords: Primary submodules, Primary modules, Small submodules, Small primary submodule, Small primary modules.

حول المقاسات الابتدائية الصغيرة

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الخلاصة:
لتكن R حلقة ابدالية ذات عنصرمحايد، وليكن X مقاسا احاديا معرفا على R. نقول ان المقاس الجزئي الغير صفري P من X هو مقاسا جزئيا ابتدائيا صغيرا اذا كان a ينتمي الى R، x ينتمي الى X، (x) صغيرا في X و ax ينتمي الى N فانه يؤدي الى اما x ينتمي الى P او a ينتمي الى $\sqrt{[P:X]}$ والمقاس X هو مقاسا ابتدائيا صغيرا اذا كان $\sqrt{annX} = \sqrt{annP}$ لكل P مقاسا جزئيا فعليا صغيرا من X . لقد اعطينا وبرهنا بعض خصائص ومميزات هذا النوع من المقاسات الجزئية (المقاسات) في هذا البحث .

1. Introduction

A non-zero submodule P of X is called primary if whenever $a \in R$ and $m \in X$ with $am \in P$ implies that $a \in \sqrt{[P:X]}$ or $\in P$. Also, X is called primary if $\sqrt{ann\, X} = \sqrt{ann\, P}$ for each proper submodule P of X [1]. These two concepts were generalized by many researchers [2, 3, 4]. As for this research, we present and study a generalization of the concepts of small primary submodule and small primary module as follows; We call a submodule P of X as a small primary submodule if whenever $a \in R, m \in X, (m)$ is small in X and $am \in P$, then either $m \in P$ or $a \in \sqrt{[P:X]}$, and X is a small primary module if $\sqrt{ann\, X} = \sqrt{ann\, P}$ for each proper submodule P small in X, where "a submodule P of X is called small (notationally, $P \ll X$) if $P + W = X$ for all submodules W of X implies $W = X$" [5]. This research consists of two parts; in the first part, we present the definition of small primary submodules and discuss some of their relationships with some types of the previously studied submodules and gave the conditions of equivalence between them. We also gave and demonstrated some of the characteristics and features of this type of submodules. In the second part, we present a definition of small primary modules and study and demonstrate some of their properties in detail.

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2- Small Primary Submodules

Definition (2.1): i) A non-zero submodule P of R-module X is called small primary iff whenever $a \in R$, $m \in X$ and $(m) \ll X$ such that $am \in P$, then either $m \in P$ or $a \in \sqrt{[P:X]}$.

ii) A proper ideal A of R is small primary if A is a small primary submodule of $an\, R$-module R.

Remark (2.2)

1- Every primary submodule is small primary . But the converse is not true; for example: Let $X = Z$ be a Z-module, then each non-zero submodule P of Z is small primary. Since if $ax \in P$ with $a \in Z$ and $(x) \ll Z$. But (0) is the only small submodule in Z, so $x = 0$. Hence $x = 0 \in P$. However, if we take $N = 30Z$, then it is clear that P is not primary.

2- Suppose that X is an R-module and let A be an ideal of R with $A \subsetneq annX$. Then P is small primary R-submodule of X iff P is a small primary R/A – submodule of X.

Proof: Let $\bar{a} \in R/A,\ m \in X$ with $(m) \ll X$, and $\bar{a}m \in P$. But $\bar{a}m = am$. Therefore , we achieve the result.

3- Let X be a hollow R-module, then every small primary submodule P of a module X is primary submodule, where " An R-module X is called a hollow module if every non-zero submodule of X is small in X" [6].

Proof: Suppose that $am \in P$, where $a \in R,\ m \in X$. But X is hollow, so (m) $\ll X$. Since P is a small primary in X, hence either $a \in \sqrt{[P:X]}$ or $m \in P$. Therefore, P is a primary submodule in X.

4- If C is a small prime submodule of an R-module X, then C is a small primary submodule in X where "A proper submodule C of an R-module X is called small prime iff whenever $a \in R$, $x \in X$ with $(x) \ll X$ such that $ax \in C$ implies either $x \in C$ or $a \in [C:X]$" [7].

Proof: Let $a \in R$, $(m) \ll X$ with $am \in C$. Hence either $m \in C$ or $a \in [C:X]$. But $[C:X] \subsetneq \sqrt{[C:X]}$. So either $m \in C$ or $a \in \sqrt{[C:X]}$. Hence C is small primary. But the converse is not true; for example: Let $X = Z_4$ be a Z-module, then ($\bar{0}$) is small primary since it is primary by [1] . But ($\bar{0}$) is not small prime, by [7] .

5- If $[P:X]$ is a semiprime ideal of R, then P is a small primary submodule iff it is a small prime.

Proof: Since $[P:X]$ is a semiprime, so $[P:X] = \sqrt{[P:X]}$. Hence the result follows easily.

6- If $W < P < X$ and P is a small primary of X , then W needs not to be small primary, as the following example shows:

Consider that $X = Z_{24}$ as a Z –module , $P = (\bar{6})$ is small primary since P is small prime by [7]. However $W = (\overline{12})$ is not small primary submodule of X , since $(\bar{6}) \ll Z_{24}$ and $(\overline{12}) = 2.\ \bar{6} \in W$, but $2 \notin \sqrt{[W:X]} = \sqrt{12Z}$ and $\bar{6} \notin W$.

7- $(\bar{0})$ is not small primary in Z_{32} , since $4.\ (\bar{8}) = (\bar{0})$ and $(\bar{8}) \ll Z_{32}$. But $(\bar{8}) \notin (\bar{0})$ and $4 \notin \sqrt{ann\, Z_{32}} = \sqrt{32Z} = 32Z$.

Theorem (2.3): Suppose that P is a non-zero submodule of a module X.Then, the followings are equivalent:

i. A submodule P is small primary.

ii. $\forall\ a \in R, W \ll X$ such that $aW \subsetneq P$, implies either $W \subsetneq P$ or $a \in \sqrt{[P:X]}$.

Proof: (i) $\longrightarrow$ (ii): Let $aW \subsetneq P$. Suppose that $W \nsubseteq P$, then $\exists, w \in W$ such that $w \notin P$. Hence $(w) \ll X$, since $w \in W$ and $W \ll X$ by [8]. Now $aw \in P$. But P is small primary submodule of X and $w \notin P$, hence $a \in \sqrt{[P:X]}$.

(ii) $\longrightarrow$(i): Let $a \in R$, $y \in X$ and $(y) \ll X$ such that $ay \in P$. Then $< a >< y > \subsetneq P$. So either $< y > \subsetneq P$ or $a \in \sqrt{[P:X]}$ by (ii). Thus, either $y \in P$ or $a \in \sqrt{[P:X]}$. Hence P is small primary .

Now, we can give the following result.

Theorem(2.4): Let P be a non-zero submodule of an R- module X.Then, the followings are equivalent:

1. A submodule P is small primary .
2. $(P:_X A)$ is a small primary submodule of X , $\forall$, $A \subsetneq R$ such that $AX \nsubseteq P$.
3. $(P:_X (a))$ is a small primary submodule of X, $\forall$, $a \in R$ such that $aX \nsubseteq P$.

Proof: (1) ⟶(2): Let ax ∈ (P:$_X$ A) and (x) ≪ P; that is a(x) ⊊ (P:$_X$ A), then aAx ⊊ P. Since (x) ≪ X, then (Ax) ≪ X. But P is small primary, so (Ax) ⊊ P or a ∈ $\sqrt{[P:X]}$ by theorem (2.3). But (Ax) ⊊ P implies that AX ⊊ P, which is a contradiction. So a ∈ $\sqrt{[P:X]}$ and hence a^n X ⊊ P for some $n \in Z_+$. But P ⊊ (P:$_X$ A) and hence a^n X ⊊ (P:$_X$ A). It follows that a^n ϵ [(P:$_X$ A): X] . Hence (P:$_X$ A) is a small primary.
(2) ⟶(3): It is clear.
(3) ⟶(1): By taking a = 1, so it follows easily.
Proposition(2.5): Let $\alpha: X \longrightarrow Y$ be an R-epimorphism. If P is small primary submodule of a module Y, then α^{-1}(P) is small primary submodule of X.
Proof: To prove that α^{-1}(P) is a non-zero submodule of X, suppose that α^{-1}(P) = X, then α(X) ⊊ P, which is a contradiction to the assumption. Let a ∈R, m ∈ X such that (m) ≪ X and am ∈α^{-1}(P). Hence $a\, \alpha\,(m) \in P$. But (m) ≪ X, so α (m) ≪ Y by [8], and as P is a small pimary of Y, then either α(m) ∈ P or a^n Y ⊊ P for some n ∈Z_+ . If α (m) ∈ P, then m∈α^{-1}(P) . If a^n Y ⊊ P, then a^n α (X) ⊊ P since α(X) = Y. This implies that a^n X ⊊ α^{-1}(P) for some n ∈Z_+. Therefore α^{-1}(P) is small primary.
Proposition (2.6): Suppose that X is an R-module, S is a multiplicative subset of R, and P is a small primary of X. Then Ps is a small primary submodule of X_S .
Proof: Suppose that $a\,/\,s$ ∈ R_S and x / t ∈ X_S with ax / st ∈ P_S such that (x/t) << X_S . So ∃ u∈S such that uax ∈ P. But (x/t) ≪ X_S , so (x) ≪ X by [7]. So (ux) << X. Since P is small primary of X, then either ux ∈ P or $(a)^n$∈ [P:X] for some n ∈Z_+.Therefore either ux /ut = x/t∈ P_S or $(a / s)^n$ ∈$[P:X]_S$ ⊊ [P_S : X_S] for some n∈Z_+ . Therefore, P_S is a small primary submodule of X_S.
Remark (2.7): If W is a small primary submodule of X, then [W : X] is not a primary ideal of R. For example: $X = Z_{24}$ as a Z –module , $W = (\bar{6})$ is small primary. But 6Z = [W:X] is not primary ideal of Z.
Proposition (2.8): Let P be a non-zero submodule of R- module X. If P is a small primary submodule of X, then [P:X] is a small primary ideal of R.
Proof: Suppose that uv∈[P:X] where u, v∈ R such that (v) << R. Suppose that v ∉ [P:X]. Now for any x∈ X, define α_x : R⟶X by α(x) = ax. So it is clear that this function is well-defined and is a homomorphism. Since (v) << R, so for any x∈ X we get (vx) << X ……. (1). But v ∉ [P:X], so ∃ m∈ X such that vm∉P. But uvm∈P. Also by (1), (vm) << X. Since P is a small primary submodule of X, so either vm∈N or u^n ∈ [P:X] for some n ∈Z_+. If u^n ∈ [P:X], then we are done. If vm ∈P then this contradicts our assumption.
Remark (2.9): If [P:X] is a small primary ideal of R, so it is not necessary that P is a small primary submodule of X. For example: $X = Z_{24}$ as a Z –module , $P = (\overline{12})$ is not small primary see (2.2,6). But [P:X] = 12Z which is small primary ideal of Z.

Recall that an R-module X is called a mulitplication if for each submodule P of X there is an ideal A of R such that P =AX [9].
Proposition (2.10): Let P be a non-zero submodule of a faithful finietly generated mulitplication R-module X. Then P is a small primary submodule of X if [P:X] is a small primary ideal of R.
Proof:. Let ax ∈ P where a ∈ R , y ∈ X such that (y) ≪ X. But X is a finietly generated faithful mulitplication module, so (y) =AX and A<< R [8]. It follows that $aAX \subsetneq P$, then a A ⊊ [P:X]. But [P:X] is small primary, so either A⊊ [P:X] or a ∈$\sqrt{[P:X]}$. Hence AX ⊊ P or a ∈$\sqrt{[P:X]}$, so (y) ⊊ P or a ∈$\sqrt{[P:X]}$. Thus y ∈ P or a ∈$\sqrt{[P:X]}$.
Propostion (2.11): Let P and C be small primary submodules of a module X and $\sqrt{[P:X]}$ =$\sqrt{[C:X]}$. Then $P \cap C$ is a small primary submodule of X.
Proof: Suppose that $a \in R, m \in X$ and $(m) \ll X$ such that $am \in P\ \cap C$. Then $am\ \in P$ and $am \in C$. Therefore either $m \in P$ or $a \in \sqrt{[P:X]}$ and either $m \in W$ or $a \in \sqrt{[C:X]}$. Hence either ($m \in P$ and $m \in C$) or ($a \in \sqrt{[P:X]} = \sqrt{[C:X]}$), which implies that either $m \in P \cap C$ or $a \in \sqrt{[P \cap C:X]}$. Hence $P \cap C$ is small primary.
Proposition (2.12): Let X_1 ,X_2 be two R-modules and let $X = X_1 \oplus X_2$. If $P = P_1 \oplus P_2$ is a small primary submodule of X, then P_1 and P_2 are small primary of X_1 and X_2, respectively.
Proof: Suppose that $a \in R,\ x \in X_1, (x) \ll X_1$ such that $ax \in P_1$, then $a(x, 0) \in P_1 \oplus P_2$. But $(x) \ll X_1$ and (0) ≪ X_2 , so $(x, 0) \ll X_1 \oplus X_2$ by [8]. But $P_1 \oplus P_2$ is a small primary submodule of X. Hence

either $(x, 0) \in P_1 \oplus P_2$ or $a^n \in [P_1 \oplus P_2 : X_1 \oplus X_2] = [P_1 : X_1] \cap [P_2 : X_2]$ for some $n \in Z_+$. Thus, either $x \in P_1$ or $a^n \in [P_1 : X_1]$ for some $n \in Z_+$. Therefore P_1 is a small primary of X_1.
By a similar proof , P_2 is a small primary of X_2.

3- Small Primary Modules

Definition (3.1) :i) An R-module X is called small primary iff $\sqrt{ann\, X} = \sqrt{ann\, P}$, $\forall\ 0 \neq P \ll X$.
ii) A ring R is a small primary ring iff $\sqrt{ann\, A} = 0$, $\forall\ 0 \neq A \ll R$.

Remark (3.2)

1- If X is a primary R-module , then X is small primary . But the converse is not true; for example : Z_6 as a Z –module is small primary but not primary.
2- Let X be a hollow small primary R-module, then X is primary.
3- Every small prime R-module is small primary, but the converse is not true in general; for example : Z_4 as a Z –module is small primary but not small prime, by [7].

Theorem (3.3): Suppose that X is a module, then X is small primary iff $\sqrt{ann\, X} = \sqrt{ann\,(x)}$, $\forall\ 0 \neq x \in X$ and $(x) \ll X$.
Proof: $\Longrightarrow$ It is clear.
$\Longleftarrow$ Let $0 \neq P \ll X$ and $a \in \sqrt{annP}$. Then $a^n x = 0$, $\forall x \in P$ and for some $n \in Z_+$ hence $a \in \sqrt{ann(x)}$, $\forall x \in P$. Since $P \ll X$ and (x) $\subsetneq P$, so $(x) \ll X$ [8]. Hence $\sqrt{ann\, X} = \sqrt{ann\,(x)}$. But $\sqrt{annP} \subsetneq \sqrt{ann(x)}$, therefore $\sqrt{annP} \subsetneq \sqrt{annX}$. Hence $\sqrt{ann\, X} = \sqrt{ann\, P}$ and so X is small primary.

Theorem (3.4): Suppose that X is a module. Then X is small primary iff (0) is a small primary submodule of X .
Proof: $\Longrightarrow$ Suppose that $a \in R, m \in X$ with $(m) \ll X$ such that $am = 0$. If $m \neq 0$, so $a \in \sqrt{ann(m)}$ and hence $a \in \sqrt{annX}$ (since X is small primary). So $a \in \sqrt{[0 : X]}$. If $m = 0$, so $m \in (0)$. Hence (0) is a small primary submodule of X.
$\Longleftarrow$ Suppose that $0 \neq P \ll X$ and let $a \in \sqrt{ann\, P}$. Then $am = 0, \forall m \in P$. Hence $am \in (0)$. Let $m \neq 0$, so $a \in \sqrt{[0 : X]} = \sqrt{ann\, X}$. Then $\sqrt{ann\, P} \subsetneq \sqrt{ann\, X}$, therefore $\sqrt{ann\, X} = \sqrt{ann\, P}$. Thus X is small primary.

Corollary (3.5): A non-zero submodule P of a module X is a small primary submodule iff X/P is a small primary R-module.

Corollary (3.6): Suppose that X is a module . Then the followings are equivalent:
a- A module X is small primary.
b- $\sqrt{ann\, X} = \sqrt{ann\,(x)}$, $\forall, 0 \neq x \in X$ and $(x) \ll X$.
c- (0) is small primary.

Proposition (3.7): If X is a small primary R-module, then $annP$ is a primary ideal of R, ,$\forall, 0 \neq P \ll X$.
Proof: Let $u, v \in R$ such that $uv \in annP$ and $0 \neq P \ll X$. Suppose that $v \notin annP$, so $vx \neq 0$ for some $x \in P$, and since $uv \in annP$, implies that $uvx = 0$. But (vx) is a submodule of P and $P \ll X$, implies that $(vx) \ll X$ [8]. On the other hand , X is small primary , so (0) is a small primary of X. Then u $\in \sqrt{ann\, X}$. But $\sqrt{ann\, X} = \sqrt{ann\, P}$, hence u $\in \sqrt{ann\, P}$. Thus, $annP$ is a primary ideal in R.

Proposition (3.8): If X is a small primary R-module, then a non-zroe submodule is a small primary R-module.
Proof: Supose that $P \neq 0$ is a submodule of X. Suppose that $0 \neq W \ll P$. So $W \ll X$ [8]. Hence $\sqrt{ann\, X} = \sqrt{ann\, W}$. But $\sqrt{annX} \subsetneq \sqrt{ann\, P}$, so $\sqrt{annW} \subsetneq \sqrt{annP}$. Hence $\sqrt{ann\, P} = \sqrt{ann\, W}$ and therefore P is small primary.

The following example shows that the converse is not true : Let $X = Z_6$ be a Z-module, then Z_6 is a small primary Z-module. While Z_{12} as a Z-module is not a small primary Z-module. Since $(\bar{6}) \ll Z_{12}$ but $\sqrt{annZ_{12}} = \sqrt{12Z} \neq \sqrt{ann(\bar{6})} = 2Z$.

Proposition (3.9): If $J(Y)$ is a direct summand small primary of an R-module Y and $\sqrt{ann\, Y} = \sqrt{ann\, J(Y)}$, then Y is a small primary R-module, where $J(Y)$ is the Jacobson radical of Y .
Proof: Supose that $0 \neq x \in Y$ and (x) $\ll Y$. Then $x \in J(Y)$, so $(x) \ll J(Y)$, [8]. Therefore $\sqrt{ann\, J(Y)} = \sqrt{ann\,(x)}$. But $\sqrt{annY} = \sqrt{ann\, J(Y)}$, so $\sqrt{ann\, Y} = \sqrt{ann\,(x)}$ and therefore Y is small primary.

Theorem (3.10): Suppose that $X = X_1 \oplus X_2$ is an R-module and $annX_1 + annX_2 = R$. Then X is a small primary R-module iff X_1 and X_2 are small primary R-modules.
Proof: ⇒) Let $0 \neq P \ll X$. Since $annX_1 + annX_2 = R$, so $P = P_1 \oplus P_2$ where P_1 and P_2 are submodules of X_1 and X_2, respectively [10]. But $P \ll X$, so $P_1 \ll X_1$ and $P_2 \ll X_2$ [8]. Now, $\sqrt{annP} = \sqrt{ann\,(P_1 \oplus P_2)} = \sqrt{annP_1 \cap annP_2} = \sqrt{annX_1 \cap annX_2}$ (since X_1 and X_2 are small primary). Hence $\sqrt{annP} = \sqrt{ann\,(X_1 \oplus X_2)} = \sqrt{ann\,X}$. Therefore, X is small primary.
⇐) It followos directly by (3. 8).
Theorem (3.11): Suppose that $X \cong Y$. Then X is small primary if and only if Y is small primary.
Proof: Let X be small primary. Since $X \cong Y$, so there exists $\alpha: X \to Y$ that is an R-isomorphism. Assume that $0 \neq P \ll Y$. Hence $\alpha^{-1}(P) \ll X$ and $\alpha^{-1}(P) \neq 0$ [8]. So $\sqrt{ann\,X} = \sqrt{ann\,\alpha^{-1}(P)}$. But $X \cong Y$ implies that $\sqrt{ann\,X} = \sqrt{ann\,Y}$, [11]. Thus $\sqrt{ann\,Y} = \sqrt{ann\,\alpha^{-1}(P)}$. But it is easily that $\sqrt{ann\,P} = \sqrt{ann\,\alpha^{-1}(P)}$, which completes the proof.
Proposition (3.12): If $\alpha: X \to Y$ is an R-homomorphism and Y is small primary such that $\sqrt{annX} = \sqrt{annY}$, then X is small primary.
Proof: Let $a \in R$ such that $a \in \sqrt{ann\,P}$ and $0 \neq P \ll X$. Then $a^nP = 0$, for some $n \in Z_+$ so $\alpha(a^nP) = a^n\alpha(P) = 0$ implies that $a \in \sqrt{ann\,\alpha(P)}$. But $P \ll X$, so $\alpha(P) \ll Y$ [8]. Since Y is small primary, hence $a \in \sqrt{annY}$. But $\sqrt{annX} = \sqrt{annY}$ so $a \in \sqrt{annX}$ and hence $\sqrt{annP} \subseteq \sqrt{annX}$. Therefore $\sqrt{annP} = \sqrt{ann\,X}$. Thus X is small primary.
Corollary (3.13): Suppose that P is a submodule of R-module X and $\sqrt{annX} = \sqrt{[P{:}X]}$. If X/P is small primary, then X is small primary.
Corollary (3.14): If P is a small primary submodule of R-module X and $\sqrt{annX} = \sqrt{[P{:}X]}$, then X is small primary.
Recall that an R-module M is called coprime if $annX = annX/P$ for every proper submodule P of X [12] .
Corollary (3.15): If X is a coprime R-module, P is a submodule of X, and X/P is small primary, then X is small primary.
Proposition (3.16): Let U be a submodule of an R-module X. If X/U is small primary, so $\sqrt{[U{:}W]} = \sqrt{[U{:}X]}, \forall$ W ≪ X and $W \supsetneq U$.
Proof: Let $0 \neq W \ll X$ and $W \supsetneq U$. Hence $W/U \ll X/U$, [8]. But X/U is small primary, so $\sqrt{ann\,X/U} = \sqrt{annW/U}$. Therefore $\sqrt{[U{:}W]} = \sqrt{[U{:}X]}$.
Corollary (3.17): If U is a small primary submodule of an R-module X, then $\sqrt{[U{:}W]} = \sqrt{[U{:}X]}$, ∀, W ≪ X and $W \supsetneq U$.
Proposition (3.18): If U is a small submodule of an R-module X and $\sqrt{[U{:}W]} = \sqrt{[U{:}X]}$, ∀ W ≪ X and $W \supsetneq U$, then X/U is small primary.
Proof: Let U, W be two submodules of X and $W \supsetneq U$ such that $W/U << X/U$. Then $W \ll X$ [8]. Therefore $\sqrt{[U{:}W]} = \sqrt{[U{:}X]}$, so $\sqrt{ann\,X/U} = \sqrt{ann\,W/U}$. Hence X/P is small primary.
Corollary (3. 19): Supose that U is a small submodule of R-module X. Then $\sqrt{[U{:}W]} = \sqrt{[U{:}X]}$, ∀ W ≪ X and $W \supsetneq U$ iff X/U is a small primary in X.
Corollary (3.20): Supose that U is a small submodule of an R-module X. Then $\sqrt{[U{:}W]} = \sqrt{[U{:}X]}$, ∀ W ≪ X and $W \supsetneq U$ iff u is a small primary in X.
Corollary (3. 21): Supose that U is a submodule of a hollow R-module X. Then $\sqrt{[U{:}W]} = \sqrt{[U{:}X]}$, ∀ W ≪ X and $W \supsetneq U$ iff X/U is a small primary in X.
Theorem (3.22): Let X be a finietly generated R-module. Then X is a small primary R −module iff X_S is a small primary R_S −module, where S is a multiplicatively closed subset of R.
Proof:⇒) Let $u/v \in R_S$, $x/y \in X_S$ such that $u/v\,.\,x/y = 0_S$, and suppose that $0_S \neq x/y \ll X_S$. So $(x) \ll X$ [7]. Then for each $s \in S, sx \neq 0$. On the other hand , $ux/vy = 0_S$, so $\exists t \in S$ such that $tux = u(tx) = 0$. But $(tx) \neq 0$ is a submodule of (x) and $(x) \ll X$, which implies that $0 \neq (tx) \ll X$ [8]. On the other hand , X is small primary , so (0) is a small primary of X. Then $u^n \in annX$ for some

$n \in Z_+$, therefore $(u/v)^n = \frac{u^n}{v^n} \in (annX)_S$. But X is finietly generated, so $(annX)_S = annX_S$, [13]. Hence $(u/v)^n \in annX_S$. Thus, $(0)_S$ is a small primary R_S −module.
⇐) It follows similarly.

Theorem (3.23): Let X be a multiplication finietly generated faithful R-module. Then X is a small primary R −module iff R is a small primary ring.

Proof: ⇒): Suppose that $0 \neq A$ is a small ideal of R. But X is a multiplication finietly generated faithful, so AX is a small submodule of X and 0 ≠ AX. Since X is small primary and faithful, then 0 = $\sqrt{annX} = \sqrt{annAX}$. But $\sqrt{annI} \subseteq \sqrt{annAX}$, therefore $\sqrt{annA}$ = 0. Hence R is a small primary ring.

⇐): Suppose that $0 \neq P \ll X$. So $[P:X] \ll R$ [8]. But X is a multiplication, so $P = [P:X]X$ [9] . Hence $[P:X] \neq 0$. But R is a small primary ring, so $\sqrt{ann[P:X]}$ =0. Since X is faithful , so $\sqrt{ann[P:X]X} = \sqrt{ann[P:X]}$ and hence $\sqrt{annX}$ = 0 . Thus, $\sqrt{annX} = \sqrt{annP}$. Therefore X is small primary .

Corollary (3.24): Let X be a multiplication cyclic faithful R-module.Then X is a small primary R −module iff R is a small primary ring.

Recall that an R-module X is called a scalar module if $\forall, \varphi \in End(X); \varphi \neq 0, \exists a \in R, a \neq 0$ such that $\varphi(x) = ax \; \forall x \in X$ [14].

Proposition(3.25): Suppose that X is a finietly generated multiplication R-module, then X is a small primary R −module iff X is a small primary $\mathrm{S-module}$ (where $End(X) = S$).

Proof:⟹) Let $0 \neq P$ be a small S-submodule of X . Then $0 \neq P$ is a small R-submodule of X. Assume that $\exists\ \alpha \in S, \alpha \in \sqrt{ann_S P}$ and $\alpha \notin \sqrt{ann_S X}$. Since X is a multiplication finietly generated, hence X is a scalar R-module [14]. Hence $\alpha(m) = am, \forall m \in X$. Thus, $\alpha^n(P) = a^n P = 0$ and so $a \in \sqrt{ann\,P} = \sqrt{annX}$. Hence $a^n X = 0$, so $\alpha^n(X) = 0$, which is a contradiction. Therefore $\sqrt{ann_S P} = \sqrt{ann_S X}$. Thus, X is a small primary $\mathrm{S-module}$.

⟸) Suppose that $0 \neq P \ll X$ and $\sqrt{annP} \subsetneq \sqrt{annX}$, so $\exists a \in \sqrt{annP}$ and $a \notin \sqrt{annX}$. Thus, $a^n X \neq 0$ for some $n \in Z_+$. Define $\alpha: X \to X$ by $\alpha(x) = ax, \forall x \in X$. Clearly, $0 \neq \alpha$ is R-homomorphism and well-defined. Since $\alpha^n(P) = a^n P = 0$, so $\alpha \in \sqrt{ann_S P} = \sqrt{ann_S X}$ (since X is a small primary S-module). Hence $\alpha^n(X) = 0$, so $\alpha = 0$, which is a contradication. Thus, $\sqrt{annP} = \sqrt{annX}$ and so X is a small primary R-maodule.

Proposition(3.26): If X is a scalar R-module and $annX$ is a prime ideal of R, then $End(X) = S$ is a small primary ring.

Proof: Since X is a scalar R-module and $annX$ is a prime, so $End(X)$=S is a small prime ring, by [7]. Hence $End(X) = S$ is a small primary ring, by (3.2, (5)).

Theorem (3.27): Let X be a scalar faithful R-module . Then $End\,(X) = S$ is a small primary ring iff R is a small primary ring.

Proof: Since X is a scalar, then $\frac{R}{annX} \cong S$ [15]. But X is faithful, so R $\cong S$. Therefore R is a small primary ring iff $End\,(X) = S$ is a small primary ring.

Theorem (3.28): The followingss are equivalent for a multiplication faithful finietly generated R-module

a- A module X is small primary.
c- $A\ ring\ \ R$ is small primary .
c- $End\,(X) = S$ is a small primary ring.

Proof:

(1) ⟺ (2); by Theorem (3.23).
(2) ⟺ (3); since X is a multiplication finietly generated, then X is a scalar, by [14]. Hence, by Theorem (3.27), the result follows.

References

1. LU. C. P. **1984**. M-radicals of submodules in modules, *Math japon*, **34**: 211-219.
2. Abdul-Alkalik, J. A. **2019**. I-Nearly Primary Sumodules, *Iraqi Journal of Science*. **60**(11): 2468-2472.
3. Al-Mothafar, N. S and Husain, A. T. **2016**. Z- Primary Sumodules, *Iraqi Journal of Science*. **Special Issue**(Part A): 163-167.
4. Athab, I. A. **2018**. NS-Primary Submodules, *Iraqi Journal of Science*. **59**(1B): 404-407.
5. Kash, F. **1982**. *Modules and Rings*, Academic Press. London.
6. Fleury, P. **1974**, Hollow Modules and Local Endomorphism Rings, *Pac. J.Math.*, **53**(2): 379-385.
7. Selman, M. L. **2012**. Small Prime Modules and Small Prime Submodules, *Journal of Al-Nahrain Universtiy*. **15**(4): 191-199.
8. Athab , I. A. **2004**. Some Generalization of Projective Modules, Ph. D. Thesis, College of Science, University of Baghded.
9. Smith, P. F. **1988**. Some Remarks on Multiplication Modules, *Arch. Math.*, **50**: 223-235.
10. Abbas, M.S., Al-Hosainy , A. M. A. **2012**. Fully Dual-Stable Modules, *Archives Des Sciences*, **65**(12): 643-651.
11. Khalaf, H. Y., **2007**. Semimaximal Submodules, Ph. D. Thesis, College of Education Ibn-Al-Haitham, University of Baghded.
12. Bican, L., Jambor, P., Kepka, T. and Nemec, P. **1980**. Prime and coprime Modules, *Fundamenta Mathematicae*, **107**(1): 33-45.
13. Larsen, M. D. and McCarlthy, P. J. **1971**. *Multiplicative theory of ideals*, New York: Academic Press.
14. Shihap, B. N. **2004**. Scalar Reflexive Modules, Ph. D.Thesis, University of Baghded.
15. Mohamed, y. Wisbauer, R. and Zhou, Y. **2002**. Ikeda-Nakayama Modules, *Beitrage zur Algebra and und Geometrie*, **43**(1): 1-9.