On Small Primary Modules

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Abstract:
Let \( R \) be a commutative ring with an identity and \( X \) be a unitary \( R \)-module. We say that a non-zero submodule \( P \) of \( X \) is small primary if for each \( a \in R, x \in X \), \( x \ll X \) with \( ax \in P \), then either \( x \in P \) or \( a \in \sqrt{[P:X]} \) and an \( R \)-module \( X \) is a small primary if \( \sqrt{ann X} = \sqrt{ann P} \) for each proper submodule \( P \) small in \( X \). We provided and demonstrated some of the characterizations and features of these types of submodules (modules).

Keywords: Primary submodules, Primary modules, Small submodules, Small primary submodule, Small primary modules.

1. Introduction
A non-zero submodule \( P \) of \( X \) is called primary if whenever \( a \in R \) and \( m \in X \) with \( am \in P \) implies that \( a \in \sqrt{[P:X]} \) or \( \in P \). Also, \( X \) is called primary if \( \sqrt{ann X} = \sqrt{ann P} \) for each proper submodule \( P \) of \( X \) [1]. These two concepts were generalized by many researchers [2, 3, 4]. As for this research, we present and study a generalization of the concepts of small primary submodule and small primary module as follows: We call a submodule \( P \) of \( X \) as a small primary submodule if whenever \( a \in R, m \in X, (m) \) is small in \( X \) and \( am \in P \), then either \( m \in P \) or \( a \in \sqrt{[P:X]} \), and \( X \) is a small primary module if \( \sqrt{ann X} = \sqrt{ann P} \) for each proper submodule \( P \) small in \( X \), where "a submodule \( P \) of \( X \) is called small (notationally, \( P \ll X \)) if \( P + W = X \) for all submodules \( W \) of \( X \) implies \( W = X \)" [5]. This research consists of two parts; in the first part, we present the definition of small primary submodules and discuss some of their relationships with some types of the previously studied submodules and gave the conditions of equivalence between them. We also gave and demonstrated some of the characteristics and features of this type of submodules. In the second part, we present a definition of small primary modules and study and demonstrate some of their properties in detail.

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2- Small Primary Submodules

Definition (2.1): i) A non-zero submodule \( P \) of \( R \)-module \( X \) is called small primary iff whenever \( a \in R, m \in X \) and \( (m) \ll X \) such that \( am \in P \), then either \( m \in P \) or \( a \in \sqrt{[P:X]} \).

ii) A proper ideal \( A \) of \( R \) is small primary if \( A \) is a small primary submodule of an \( R \)-module \( R \).

Remark (2.2)

1- Every primary submodule is small primary. But the converse is not true; for example: Let \( X = \mathbb{Z} \) be a \( \mathbb{Z} \)-module, then each non-zero submodule \( P \) of \( X \) is small primary. Since if \( ax \in P \) with \( a \in \mathbb{Z} \) and \( (x) \ll \mathbb{Z} \). But \( (0) \) is the only small submodule in \( \mathbb{Z} \), so \( x = 0 \). Hence \( x = 0 \in P \). However, if we take \( N = 30\mathbb{Z} \), then it is clear that \( P \) is not primary.

2- Suppose that \( X \) is an \( R \)-module and let \( A \) be an ideal of \( R \) with \( A \subseteq \text{ann} X \). Then \( P \) is small primary \( R \)-submodule of \( X \) iff \( P \) is small primary submodule of \( R/\text{ann} X \).

Proof: Let \( \bar{a} \in R/A, m \in X \) with \( (m) \ll X \), and \( \bar{a}m \in P \). But \( \bar{a}m = am \). Therefore, \( \bar{a} \) is small primary.

3- Let \( X \) be a hollow \( R \)-module, then every small primary submodule \( P \) of a module \( X \) is primary submodule, where "An \( R \)-module \( X \) is called a hollow module if every non-zero submodule of \( X \) is small in \( X \)" [6].

Proof: Suppose that \( X = Z_4 \) as a \( \mathbb{Z} \)-module, \( P = \langle 0 \rangle \) is small primary since it is primary by [1]. But \( \langle 0 \rangle \) is not small prime, by [7].

4- If \( C \) is a small prime submodule of an \( R \)-module \( X \), then \( C \) is a small primary submodule of \( X \) where "A proper submodule \( C \) of an \( R \)-module \( X \) is called small prime iff whenever \( a \in R, x \in X \) with \( (x) \ll X \) such that \( ax \in C \) implies either \( x \in C \) or \( a \in \sqrt{[C:X]} \)" [7].

5- \( X \) is a semiprime ideal of \( R \), then \( P \) is a small primary submodule iff it is a small prime.

Proof: Since \( [P:X] \) is a semiprime, so \( [P:X] = \sqrt{[P:X]} \). Hence the result follows easily.

6- If \( W \prec P \prec X \) and \( P \) is a small primary of \( X \), then \( W \) needs not to be small primary, as the following example shows:

Consider that \( X = Z_{32} \) as a \( \mathbb{Z} \)-module, \( P = \langle 0 \rangle \) is small primary since \( P \) is small prime by [7]. However \( W = \langle 12 \rangle \) is not small primary submodule of \( X \), since \( \langle 6 \rangle \ll Z_{32} \) and \( \langle 12 \rangle = 2. \langle 6 \rangle \in W \), but \( 2 \notin \sqrt{[W:X]} = \sqrt{12Z} \) and \( 6 \notin W \).

7- \( \langle 0 \rangle \) is not small primary in \( Z_{32} \), since \( \langle 4 \rangle = \langle 0 \rangle \) and \( \langle 8 \rangle \ll Z_{32} \). But \( \langle 8 \rangle \notin \langle 0 \rangle \) and \( 4 \notin \sqrt{\text{ann} Z_{32}} = \sqrt{32Z} \).

Theorem (2.3): Suppose that \( P \) is a non-zero submodule of a module \( X \). Then, the followings are equivalent:

i. A submodule \( P \) is small primary.

Proof: i) \( \Rightarrow \) ii): Let \( aW \subseteq P \). Suppose that \( W \nsubseteq P \), then \( \exists, w \in W \) such that \( w \nsubseteq P \). Hence \( w \ll X \), since \( w \in W \) and \( W \ll X \) by [8]. Now \( aw \in P \). But \( P \) is small primary submodule of \( X \) and \( w \nsubseteq P \), hence \( a \notin \sqrt{[P:X]} \).

ii) \( \Rightarrow \) i): Let \( a \in R, y \in X \) and \( (y) \ll X \) such that \( ay \in P \). Then \( a > y \nsubseteq P \). So either \( y \nsubseteq P \) or \( a \in \sqrt{[P:X]} \) by (ii). Thus, either \( y \in P \) or \( a \in \sqrt{[P:X]} \). Hence \( P \) is small primary.

Now, we can give the following result.

Theorem (2.4): Let \( P \) be a non-zero submodule of an \( R \)-module \( X \). Then, the followings are equivalent:

1. A submodule \( P \) is small primary.

2. \( P \subseteq X \) is a small primary submodule of \( X, \forall, A \subseteq R \) such that \( AX \nsubseteq P \).

3. \( P \subseteq X (a) \) is a small primary submodule of \( X, \forall, a \in R \) such that \( aX \nsubseteq P \).
Proof: (1) ——> (2): Let \( ax \in (P_X;A) \) and \( x \ll P \); that is \( a(x) \subseteq (P_X;A) \), then \( aAx \not\subseteq P \). Since \( x \ll X \), then \( (Ax) \ll X \). But \( P \) is small primary, so \( (Ax) \not\subseteq P \) or \( a \in \sqrt{[P;X]} \) by theorem (2.3). But \( (Ax) \subseteq P \) implies that \( AX \subseteq P \), which is a contradiction. So \( a \in \sqrt{[P;X]} \) and hence \( a^n X \not\subseteq P \) for some \( n \in Z_+ \).

But \( P \not\subseteq (P_X;A) \) and hence \( a^n X \not\subseteq (P_X;A) \). It follows that \( a^n \in [ (P_X;A); X ] \). Hence \( (P_X;A) \) is a small primary.

(2) ——> (3): It is clear.

(3) ——> (1): By taking \( a = 1 \), so it follows easily.

Proposition (2.5): Let \( \alpha : X \longrightarrow Y \) be an \( R \)-epimorphism. If \( P \) is small primary submodule of a module \( Y \), then \( \alpha^{-1}(P) \) is small primary submodule of \( X \).

Proof: To prove that \( \alpha^{-1}(P) \) is a non-zero submodule of \( X \), suppose that \( \alpha^{-1}(P) = X \), then \( \alpha(X) \subseteq P \), which is a contradiction to the assumption. Let \( a \in R \), \( m \in X \) such that \( m \ll X \) and \( am \epsilon \alpha^{-1}(P) \). Hence \( a \alpha(m) \epsilon P \). But \( (m) \ll X \), so \( \alpha(m) \ll Y \) by [8], and as \( P \) is a small primary of \( Y \), then either \( \alpha(m) \epsilon P \) or \( a^n Y \subseteq P \) for some \( n \in Z_+ \). If \( \alpha(m) \epsilon P \), then \( m \epsilon \alpha^{-1}(P) \). If \( a^n Y \subseteq P \), then \( a^n \alpha(X) \subseteq P \) since \( \alpha(X) = Y \). This implies that \( a^n X \subseteq \alpha^{-1}(P) \) for some \( n \in Z_+ \). Therefore \( \alpha^{-1}(P) \) is small primary.

Proposition (2.6): Suppose that \( X \) is an \( R \)-module, \( S \) is a multiplicative subset of \( R \), and \( P \) is a small primary of \( X \). Then \( Ps \) is a small primary submodule of \( X_S \).

Proof: Suppose that \( a/s \in R_S \) and \( x/t \in X_S \) with \( ax/st \in P_S \) such that \( (x/t) \ll X_S \). So \( \exists s \epsilon S \) such that \( uax \epsilon P \). But \( (x/t) \ll X_S \), \( (x) \ll X \) by [7]. So \( (ux) \ll X \). Since \( P \) is small primary of \( X \), then either \( ux \epsilon P \) or \( (a^n) \subseteq [P;X] \) for some \( n \in Z_+ \). Therefore either \( ux/ut = x/t \epsilon P_S \) or \( (a/s)^n \subseteq [P;X] \subseteq [P_S;X_S] \) for some \( n \epsilon Z_+ \). Therefore \( P_S \) is a small primary submodule of \( X_S \).

Remark (2.7): If \( W \) is a small primary submodule of \( X \), then \( [W:X] \) is not a primary ideal of \( R \). For example: \( X = Z_{24} \) as a \( Z \)-module, \( W = (6) \) is small primary. But \( 6Z = [W:X] \) is not primary ideal of \( Z \).

Proposition (2.8): Let \( L \) be a non-zero submodule of \( R \)-module \( X \). If \( P \) is a small primary submodule of \( X \), then \( [P;X] \) is a small primary ideal of \( R \).

Proof: Suppose that \( uv \epsilon [P;X] \) where \( u, v \epsilon R \) such that \( (v) \ll R \). Suppose that \( v \not\epsilon [P;X] \). Now for any \( x \epsilon X \), define \( \alpha_x : R \longrightarrow X \) by \( \alpha(x) = ax \). So it is clear that this function is well-defined and is a homomorphism. Since \( \alpha(x) \) is small primary of \( X \), then either \( ux \epsilon P \) or \( a^n Y \subseteq P \) for some \( n \epsilon Z_+ \). Therefore either \( ux/ut = x/t \epsilon P_S \) or \( (a/s)^n \subseteq [P;X] \subseteq [P_S;X_S] \) for some \( n \epsilon Z_+ \). Therefore \( P_S \) is a small primary submodule of \( X_S \).

Remark (2.9): If \( [P;X] \) is a small primary ideal of \( R \), so it is not necessary that \( P \) is a small primary submodule of \( X \). For example: \( X = Z_{24} \) as a \( Z \)-module, \( P = (12) \) is not small primary see (2.2,6). But \( [P;X] = 12Z \) which is small primary ideal of \( Z \).

Recall that an \( R \)-module \( X \) is called a mulitplication if for each submodule \( P \) of \( X \) there is an ideal \( A \) of \( R \) such that \( P = AX \) [9].

Proposition (2.10): Let \( P \) be a non-zero submodule of a faithful finitely generated multiplication \( R \)-module \( X \). Then \( P \) is a small primary submodule of \( X \) if \( [P;X] \) is a small primary ideal of \( R \).

Proof: Let \( ax \epsilon P \) where \( a \epsilon R \), \( y \epsilon X \) such that \( (y) \ll X \). But \( X \) is a finitely generated faithful multiplication module, so \( (y) = AX \) and \( A \ll R \). It follows that \( aAx \not\subseteq P \), then \( a \not\epsilon [P;X] \). But \( [P;X] \) is small primary, so either \( A_x \subseteq [P;X] \) or \( a \subseteq \sqrt{[P;X]} \). Hence \( AX \not\subseteq P \) or \( a \subseteq \sqrt{[P;X]} \). Thus \( y \epsilon P \) or \( a \subseteq \sqrt{[P;X]} \).

Proposition (2.11): Let \( P \) and \( C \) be small primary submodules of a module \( X \) and \( \sqrt{[P;X]} = \sqrt{[C;X]} \). Then \( P \cap C \) is a small primary submodule of \( X \).

Proof: Suppose that \( a \epsilon R, m \epsilon X \) and \( (m) \ll X \) such that \( am \epsilon P \cap C \). Then \( am \epsilon P \) and \( am \epsilon C \). Therefore either \( m \epsilon P \) or \( a \subseteq \sqrt{[P;X]} \) and either \( m \epsilon W \) or \( a \subseteq \sqrt{[C;X]} \). Hence either \( m \epsilon P \) and \( m \epsilon C \) or \( (a \subseteq \sqrt{[P;X]} = \sqrt{[C;X]} \) \), which implies that either \( m \epsilon P \cap C \) or \( a \subseteq \sqrt{[P \cap C;X]} \). Hence \( P \cap C \) is small primary.

Proposition (2.12): Let \( X_1, X_2 \) be two \( R \)-modules and let \( X = X_1 \oplus X_2 \). If \( P = P_1 \oplus P_2 \) is a small primary submodule of \( X \), then \( P_1 \) and \( P_2 \) are small primary of \( X_1 \) and \( X_2 \), respectively.

Proof: Suppose that \( a \epsilon R, x \epsilon X_1 \) such that \( ax \epsilon P_1 \), then \( a(x, 0) \epsilon P_1 \oplus P_2 \). But \( (x) \ll X_1 \) and \( (0) \ll X_2 \), so \( (x, 0) \ll X_1 \oplus X_2 \). But \( P_1 \oplus P_2 \) is a small primary submodule of \( X \). Hence
either \( (x, 0) \in P_1 \oplus P_2 \) or \( a^n \in [P_1 \oplus P_2; X_1 \oplus X_2] = [P_1; X_1] \cap [P_2; X_2] \) for some \( n \in \mathbb{Z}_+ \). Thus, either \( x \in P_1 \) or \( a^n \in [P_1; X_1] \) for some \( n \in \mathbb{Z}_+ \). Therefore \( P_1 \) is a small primary of \( X_1 \).

By a similar proof, \( P_2 \) is a small primary of \( X_2 \).

### 3- Small Primary Modules

**Definition (3.1):**

1. An \( R \)-module \( X \) is called small primary iff \( \sqrt{\text{ann} X} = \sqrt{\text{ann} P}, \forall \, 0 \neq P \ll X \).
2. A ring \( R \) is a small primary ring iff \( \sqrt{\text{ann} A} = 0, \forall \, 0 \neq A \ll R \).

**Remark (3.2):**

1. If \( X \) is a primary \( R \)-module, then \( X \) is small primary. But the converse is not true; for example: \( Z_6 \) as a \( Z \)-module is small primary but not primary.
2. Let \( X \) be a hollow small primary \( R \)-module, then \( X \) is primary.
3. Every small prime \( R \)-module is small primary, but the converse is not true in general; for example: \( Z_4 \) as a \( Z \)-module is small primary but not small prime, by [7].

**Theorem (3.3):** Suppose that \( X \) is a module, then \( X \) is small primary iff \( \sqrt{\text{ann} X} = \sqrt{\text{ann} (x)}, \forall \, 0 \neq x \in X \) and \((x) \ll X \).

**Proof:** It is clear.

\( \Rightarrow \) Let \( 0 \neq P \ll X \) and \( a \in \sqrt{\text{ann} P} \). Then \( a^n x = 0, \forall x \in P \) and for some \( n \in \mathbb{Z}_+ \), hence \( a \in \sqrt{\text{ann} (x)}, \forall x \in P \). Since \( P \ll X \) and \((x) \ll P \), so \((x) \ll X \) and \((0) \ll X \) [8]. Hence \( \sqrt{\text{ann} X} = \sqrt{\text{ann} (x)} \). But \( \sqrt{\text{ann} P} \subseteq \sqrt{\text{ann} X} \), therefore \( \sqrt{\text{ann} P} \subseteq \sqrt{\text{ann} X} \). Hence \( \sqrt{\text{ann} X} = \sqrt{\text{ann} P} \) and so \( X \) is small primary.

**Theorem (3.4):** Suppose that \( X \) is a module. Then \( X \) is small primary iff \( (0) \) is a small primary submodule of \( X \).

**Proof:** Suppose that \( a \in R, m \in X \) with \((m) \ll X \) such that \( am = 0 \). If \( m \neq 0 \), so \( a \in \sqrt{\text{ann} (m)} \) and hence \( a \in \sqrt{\text{ann} X} \) (since \( X \) is small primary). So \( a \in \sqrt{[0:X]} \). If \( m = 0 \), so \( m \in (0) \). Hence \( (0) \) is a small primary submodule of \( X \).

\( \Rightarrow \) Suppose that \( 0 \neq P \ll X \) and \( a \in \sqrt{\text{ann} P} \). Then \( am = 0, \forall m \in P \). Hence \( am \in (0) \). Let \( m \neq 0 \), so \( a \in \sqrt{[0:X]} \). Then \( \sqrt{\text{ann} P} \subseteq \sqrt{\text{ann} X} \), therefore \( \sqrt{\text{ann} X} = \sqrt{\text{ann} P} \). Thus \( X \) is small primary.

**Corollary (3.5):** A non-zero submodule \( P \) of a module \( X \) is a small primary submodule iff \( X/P \) is a small primary \( R \)-module.

**Corollary (3.6):** Suppose that \( X \) is a module. Then the followings are equivalent:

a- A module \( X \) is small primary.

b- \( \sqrt{\text{ann} X} = \sqrt{\text{ann} (x)}, \forall \, 0 \neq x \in X \) and \((x) \ll X \).

\( \Rightarrow \) If \( X \) is a small primary \( R \)-module, then \( \sqrt{\text{ann} P} \) is a primary ideal of \( R, \forall \, 0 \neq P \ll X \).

**Proposition (3.9):** If \( X \) is a direct summand small primary of an \( R \)-module \( Y \) and \( \sqrt{\text{ann} Y} = \sqrt{\text{ann} Y} \), then \( Y \) is a small primary \( R \)-module, where \( \sqrt{\text{ann} Y} \) is the Jacobson radical of \( Y \).

**Proof:** Suppose that \( P \neq 0 \) is a submodule of \( X \). Suppose that \( 0 \neq W \ll P \). So \( W \ll X \) [8]. Hence \( \sqrt{\text{ann} X} = \sqrt{\text{ann} W} \). But \( \sqrt{\text{ann} X} \subseteq \sqrt{\text{ann} P} \), so \( \sqrt{\text{ann} W} \subseteq \sqrt{\text{ann} P} \). Hence \( \sqrt{\text{ann} P} = \sqrt{\text{ann} W} \) and therefore \( P \) is small primary.

The following example shows that the converse is not true: Let \( X = Z_6 \) be a \( Z \)-module, then \( Z_6 \) is a small primary \( Z \)-module. While \( Z_{12} \) as a \( Z \)-module is not a small primary \( Z \)-module. Since \( (6) \ll Z_{12} \) but \( \sqrt{\text{ann} Z_{12}} = \sqrt{12Z} \neq \sqrt{\text{ann} (6)} = 2Z \).

**Proposition (3.9):** If \( Y \) is a direct summand small primary of an \( R \)-module \( Y \) and \( \sqrt{\text{ann} Y} = \sqrt{\text{ann} Y} \), then \( Y \) is a small primary \( R \)-module, where \( \sqrt{\text{ann} Y} \) is the Jacobson radical of \( Y \).

**Proof:** Suppose that \( 0 \neq x \in Y \) and \((x) \ll Y \). Then \( x \in \sqrt{\text{ann} Y} \), so \((x) \ll \sqrt{\text{ann} Y} \) [8]. Therefore \( \sqrt{\text{ann} Y} \) = \( \sqrt{\text{ann} X} \). But \( \sqrt{\text{ann} Y} = \sqrt{\text{ann} Y} \), so \( \sqrt{\text{ann} Y} = \sqrt{\text{ann} X} \) and therefore \( Y \) is small primary.
Theorem (3.10): Suppose that $X = X_1 \oplus X_2$ is an R-module and $\text{ann}X_1 + \text{ann}X_2 = R$. Then $X$ is a small primary $R$-module if $X_1$ and $X_2$ are small primary $R$-modules.

Proof: \( \Rightarrow \) Let \( 0 \neq P \ll X \). Since $\text{ann}X_1 + \text{ann}X_2 = R$, so $P = P_1 \oplus P_2$ where $P_1$ and $P_2$ are submodules of $X_1$ and $X_2$, respectively [10]. But $P \ll X$, so $P_1 \ll X_1$ and $P_2 \ll X_2$ [8]. Now, $\sqrt{\text{ann}P} = \sqrt{\text{ann}(P_1 \oplus P_2)} = \sqrt{\text{ann}P_1 \cap \text{ann}P_2} = \sqrt{\text{ann}X_1 \cap \text{ann}X_2}$ (since $X_1$ and $X_2$ are small primary). Hence $\sqrt{\text{ann}P} = \sqrt{\text{ann}(X_1 \oplus X_2)} = \sqrt{\text{ann}X}$. Therefore, $X$ is small primary.

$\Leftarrow$ It follows directly by (3.8).

Theorem (3.11): Suppose that $X \cong Y$. Then $X$ is small primary if and only if $Y$ is small primary.

Proof: Let $X$ be small primary. Since $X \cong Y$, there exists $\alpha: X \rightarrow Y$ that is an R-isomorphism. Assume that $0 \neq P \ll Y$. Hence $\alpha^{-1}(P) \ll X$ and $\alpha^{-1}(P) \neq 0$ [8]. So $\sqrt{\text{ann}X} = \sqrt{\alpha^{-1}(P)}$. But $X \cong Y$ implies that $\sqrt{\text{ann}X} = \sqrt{\text{ann}Y}$, [11]. Thus $\sqrt{\text{ann}Y} = \sqrt{\alpha^{-1}(P)}$. But it is easily that $\sqrt{\text{ann}P} = \sqrt{\alpha^{-1}(P)}$, which completes the proof.

Proposition (3.12): If $\alpha: X \rightarrow Y$ is an $R$-homomorphism and $Y$ is small primary such that $\sqrt{\text{ann}X} = \sqrt{\text{ann}Y}$, then $X$ is small primary.

Proof: Let $a \in R$ such that $a \in \sqrt{\text{ann}P}$ and $0 \neq P \ll X$. Then $a^nP = 0$, for some $n \in Z_+$ so $a(a^nP) = a^n\alpha(P) = 0$ implies that $a \in \sqrt{\text{ann}\alpha(P)}$. But $P \ll X$, so $\alpha(P) \ll Y$ [8]. Since $Y$ is small primary, hence $a \in \sqrt{\text{ann}Y}$. But $\sqrt{\text{ann}X} = \sqrt{\text{ann}Y}$ so $a \in \sqrt{\text{ann}X}$ and hence $\sqrt{\text{ann}P} \subseteq \sqrt{\text{ann}X}$. Therefore $\sqrt{\text{ann}P} = \sqrt{\alpha^{-1}(P)}$, which completes the proof.

Corollary (3.13): Suppose that $P$ is a submodule of $R$-module $X$ and $\sqrt{\text{ann}X} = \sqrt{(P:X)}$. If $X/P$ is small primary, then $X$ is small primary.

Corollary (3.14): If $P$ is a small primary submodule of $R$-module $X$ and $\sqrt{\text{ann}X} = \sqrt{(P:X)}$, then $X$ is small primary.

Recall that an $R$-module $M$ is called coprime if $\text{ann}X = \text{ann}X/P$ for every proper submodule $P$ of $X$ [12].

Corollary (3.15): If $X$ is a coprime $R$-module, $P$ is a submodule of $X$, and $X/P$ is small primary, then $X$ is small primary.

Proposition (3.16): Let $U$ be a submodule of an $R$-module $X$. If $X/U$ is small primary, so $\sqrt{(U:W)} = \sqrt{(U:X)}$, \( \forall \ W \subseteq X \) and $W \supseteq U$.

Proof: Let $0 \neq W \ll X$ and $W \supseteq U$. Hence $W/U \ll X/U$, [8]. But $X/U$ is small primary, so $\sqrt{\text{ann}(X/U)} = \sqrt{\text{ann}(W/U)}$. Therefore $\sqrt{(U:W)} = \sqrt{(U:X)}$.

Corollary (3.17): If $U$ is a small primary submodule of an $R$-module $X$, then $\sqrt{(U:W)} = \sqrt{(U:X)}$, \( \forall \ W \ll X \) and $W \supseteq U$.

Proposition (3.18): If $U$ is a small submodule of an $R$-module $X$ and $\sqrt{(U:W)} = \sqrt{(U:X)}$, \( \forall \ W \ll X \) and $W \supseteq U$, then $X/U$ is small primary.

Proof: Let $W$ be two submodules of $X$ and $W \supseteq U$ such that $W/U \ll X/U$. Then $W \ll X$ [8]. Therefore $\sqrt{(U:W)} = \sqrt{(U:X)}$, so $\sqrt{\text{ann}X/U} = \sqrt{\text{ann}W/U}$. Hence $X/P$ is small primary.

Corollary (3.19): Suppose that $U$ is a small submodule of $R$-module $X$. Then $\sqrt{(U:W)} = \sqrt{(U:X)}$, \( \forall \ W \ll X \) and $W \supseteq U$ if $U$ is a small primary in $X$.

Corollary (3.20): Suppose that $U$ is a small submodule of an $R$-module $X$. Then $\sqrt{(U:W)} = \sqrt{(U:X)}$, \( \forall \ W \ll X \)$ and $W \supseteq U$ if $U$ is a small primary in $X$.

Corollary (3.21): Suppose that $U$ is a submodule of a hollow $R$-module $X$. Then $\sqrt{(U:W)} = \sqrt{(U:X)}$, \( \forall \ W \ll X \) and $W \supseteq U$ if $U$ is a small primary in $X$.

Theorem (3.22): Let $X$ be a finitely generated $R$-module. Then $X$ is a small primary $R$-module if $X_0$ is a small primary $R_0$-module, where $S$ is a multiplicatively closed subset of $R$.

Proof: Let $u/v \in R_S$, $x/y \in X_S$ such that $u/v \cdot x/y = 0_S$, and suppose that $0_S \neq x/y \ll X_S$. So $(x) \ll X$ [7]. Then for each $s \in S$, $sx \neq 0$. On the other hand, $ux/uy = 0_S$, so $x \ll X_S$ such that $tux = u(tx) = 0$. But $(tx) \neq 0$ is a submodule of $(x)$ and $(x) \ll X$, which implies that $0 \neq (tx) \ll X$ [8]. On the other hand, $X$ is small primary, so $(0)$ is a small primary of $X$. Then $u^n \in \text{ann}X$ for some
n \in \mathbb{Z}_+, therefore (u/v)^n = \frac{u^n}{v^n} \in (annX)_S$. But X is finitely generated, so $(annX)_S = annX_S$, [13]. Hence $(u/v)^n \in annX_S$. Thus, $(0)_S$ is a small primary $R_S$-module.

$\Leftarrow$ It follows similarly.

**Theorem (3.23):** Let $X$ be a multiplication finitely generated faithful $R$-module. Then $X$ is a small primary $R$-module iff $R$ is a small primary ring.

**Proof:** Suppose that $0 \neq A$ is a small ideal of $R$. But $X$ is a multiplication finitely generated faithful, so $AX$ is a submodule of $X$ and $0 \neq AX$. Since $X$ is small primary and faithful, then $0 = \sqrt{annX} = \sqrt{annAX}$. But $\sqrt{annI} \subseteq \sqrt{annAX}$, therefore $\sqrt{annA} = 0$. Hence $R$ is a small primary ring.

$\Rightarrow$: Suppose that $0 \neq P \ll X$. So $[P:X] \ll R$ [8]. But $X$ is a multiplication, so $P = [P:X]X$ [9]. Hence $[P:X] \neq 0$. But $R$ is a small primary ring, so $\sqrt{ann[P:X]} = 0$. Since $X$ is faithful, so $\sqrt{ann[P:X]} = \sqrt{annX} = 0$. Thus, $\sqrt{annX} = \sqrt{annP}$. Therefore $X$ is small primary.

**Corollary (3.24):** Let $X$ be a multiplication cyclic faithful $R$-module. Then $X$ is a small primary $R$-module iff $R$ is a small primary ring.

Recall that an $R$-module $X$ is called a scalar module if $\forall \varphi \in End(X); \varphi \neq 0, \exists a \in R, a \neq 0$ such that $\varphi(x) = ax \forall x \in X$ [14].

**Proposition (3.25):** Suppose that $X$ is a finitely generated multiplication $R$-module, then $X$ is a small primary $R$-module iff $X$ is a small primary $S$-module (where $End(X) = S$).

**Proof:** Let $0 \neq P$ be a small $S$-submodule of $X$. Then $0 \neq P$ is a small $R$-submodule of $X$.

Assume that $\exists \alpha \in S, \alpha \in \sqrt{ann_P}P$ and $\alpha \notin \sqrt{ann_X}$. Since $X$ is a multiplication finitely generated, hence $X$ is a scalar $R$-module [14]. Hence $\alpha(m) = am, \forall m \in X$. Thus, $\alpha^n(P) = a^nP = 0$ and so $a \in \sqrt{ann_P}P = \sqrt{annX}$. Hence $a^nX = 0$, so $\alpha^n(X) = 0$, which is a contradiction. Therefore $\sqrt{ann_P}P = \sqrt{annX}$.

Similarly, suppose that $0 \neq P \ll X$ and $\sqrt{ann_P} \ll \sqrt{annX}$, so $\exists a \in \sqrt{ann_P}$ and $\alpha \notin \sqrt{annX}$. Thus, $a^nX \neq 0$ for some $n \in \mathbb{Z}_+$. Define $\alpha: X \rightarrow X$ by $\alpha(x) = ax, \forall x \in X$. Clearly, $0 \neq \alpha$ is $R$-homomorphism and well-defined. Since $a^n(P) = a^nX = 0$, so $a \in \sqrt{ann_P}P = \sqrt{ann_X}$ (since $X$ is a small primary $S$-module). Hence $\alpha^n(X) = 0$, so $\alpha = 0$, which is a contradiction. Thus, $\sqrt{ann_P} = \sqrt{annX}$ and so $X$ is a small primary $S$-module.

**Proposition (3.26):** If $X$ is a scalar $R$-module and $annX$ is a prime ideal of $R$, then $End(X) = S$ is a small primary ring.

**Proof:** Since $X$ is a scalar $R$-module and $annX$ is a prime, so $End(X) = S$ is a small prime ring, by [7]. Hence $End(X) = S$ is a small primary ring, by (3.2), (5).

**Theorem (3.27):** Let $X$ be a scalar faithful $R$-module. Then $End(X) = S$ is a small primary ring iff $R$ is a small primary ring.

**Proof:** Since $X$ is a scalar, then $R_{annX} \cong S$ [15]. But $X$ is faithful, so $R \cong S$. Therefore $R$ is a small primary ring iff $End(X) = S$ is a small primary ring.

**Theorem (3.28):** The followings are equivalent for a multiplication faithful finitely generated $R$-module

a- A module $X$ is small primary.

b- A ring $R$ is small primary.

c- $End(X) = S$ is a small primary ring.

**Proof:**

(1) $\Leftrightarrow$ (2); by Theorem (3.23).

(2) $\Leftrightarrow$ (3); since $X$ is a multiplication finitely generated, then $X$ is a scalar, by [14]. Hence, by Theorem (3.27), the result follows.
References