DOI: 10.24996/ijs.2021.62.7.24

ISSN: 0067-2904

Semi-T-Hollow Modules and Semi-T-Lifting Modules

Alaa A. Elewi
College of Science, University of Baghdad, Baghdad, Iraq

Received: 16/8/2020 Accepted: 29/9/2020

Abstract

Let R be an associative ring with identity and let W be a unitary left R-module. Let T be a non-zero submodule of W.We say that W is a semi-T- hollow module if for every submodule K of W such that $T \nsubseteq K$ is a semi-T- small submodule ($K \ll_{S-T} W$). In addition, we say that W is a semi-T- lifting module if for every submodule X of W, there exists a direct summand F of W and $H \ll_{S-T} W$ such that $X = F + H$.

The main purpose of this work was to develop the properties of these classes of module.

Keywords: Small submodules, T- small submodules, T- hollow modules, T- lifting modules.2010. **Mathematics Subject Classification**: 16D99, 16L99.

مقاسات شبه المجوفة من النمط -T ومقاسات شبه الرفع من النمط – T

الاء عباس عليوي
قسم الرياضيات ، كلية العلوم ، جامعة بغداد ، بغداد ، العراق

الخلاصة

لتكن R حلقة تجميعيه ذات عنصر محايد وليكن Wمقاس احادي ايسر و Tمقاس جزئي غير صفري من W.نقول عن W انه مقاس شبه اجوف من النمط Tاذا كان لكل مقاس جزئي Kمن W بحيث $T \nsubseteq K$يكون مقاس جزئي شبه صغير من النمط T.

بالاضافه الى هذا نقول عن W هو مقاس شبه رفع من النمط T اذا كان لكل مقاس جزئي من W يوجد جمع مباشر Fمن Wو $H \ll_{S-T} W$ بحيث ان $X = F + H$.

الغرض الرئيسي من هذا العمل هو تطوير خصائص هذين الصنفين من المقاسات.

1.INTRODUCTION

In this work, R is a ring with identity and every R- module is a unitary left R- module. Recall that a submodule N of W is T- small in an R- module W denoted by $N \ll_T W$, in case for any submodule X of W, $T \subseteq X + N$ implies that $T \subseteq X$ [1]. AL-Redeeni and AL-Bahrani [2] introduced the concepts of T-hollow and T- Lifting modules as: Let T be a submodule of a non-zero module W, we say that W is a T-hollow module, if for every submodule $K$of W such that $T \nsubseteq K$, is a T-small submodule of W. W is said to be T lifting module (where T is a submodule of W), if for each submodule X of W, there exists a direct summand F of W and $H \ll_T W$ such that $X = F + H$. Elewi [3] introduced the concept of semi-T-small:

*Email: alaa.abbas.math@gmail.com

Let T be a submodule of a module W. A submodule N of a module W is called semi-T-small in W, denoted by $N \ll_{S-T} W$, in case for any submodule X of W, $T \subseteq X + N$ implies that $T \subseteq X + Rad(W)$.

In this work we introduce the following concepts:
Let W be a non-zero module and let T be a submodule W. We say that W is semi-T-hollow module if every submodule K of W such that $T \nsubseteq K$ is a semi-T-small submodule of W. We say that W a is semi-T- lifting module, if for each submodule X of W, there exists a direct summand F of W and $H \ll_{S-T} W$ such that $X = F + H$.
The main goal of this work is to develop the properties of these concepts.

2. Semi-*T*-Hollow modules

In this section, we present the concept semi-T-hollow modules as a generalization of T-hollow modules.

Definition 2.1:

Let W be a non-zero module and T be a submodule of W. We say that W is a semi-T-hollow module if every submodule K of W such that $T \nsubseteq K$ is a semi-T- small submodule of W.

Remarks and Examples 2.2

1. Every T- hollow module is semi-T-hollow.
2. The converse of (1) is not true, for example: consider $\mathbb{Q}$ as $\mathbb{Z}$-module. Let $T = 2\mathbb{Z}$, $N = n\mathbb{Z}$ such that $n \neq 2$, thus $T \nsubseteq N$. Therefore every submodule $X = m\mathbb{Z}$ of $\mathbb{Q}$ $(m \neq n)$ such that $T \subseteq N + X$. Since $Rad\ \mathbb{Q} = \mathbb{Q}$, thus $T \subseteq N + Rad(\mathbb{Q}) = N + \mathbb{Q} = \mathbb{Q}$. Thus $\mathbb{Q}$ as $\mathbb{Z}$ –module is semi –T-hollow. But $\mathbb{Q}$ is not T- hollow [2, Example.(2.2.2) (1), p40].
3. If W is a non-zero R- module such that $Rad\ (W) = W$, then W is a semi- T- hollow for every submodule T of W.
4. $\mathbb{Z}_{12}$ as $\mathbb{Z}$- module is not semi-T-hollow, since if we take $T =< \bar{2} >$, $K =< \bar{4} >$, it is clear that $< \bar{2} > \nsubseteq < \bar{4} >$. Now T==$< \bar{2} > \subseteq < \bar{3} > + < \bar{4} > = \mathbb{Z}_{12}$, but $< \bar{3} > + Rad\ t_{12} =< \bar{3} > + < \bar{6} > =< \bar{3} >$ and hence T$\nsubseteq \bar{3} > + Rad(\mathbb{Z}_{12})$. Thus $\mathbb{Z}_{12}$ as $\mathbb{Z}$-module is semi-T-hollow.
5. Consider $\mathbb{Z}_6$ as $\mathbb{Z}$- module and let $T = \{\bar{0}, \bar{3}\}$, thus by [2, Example (2.2.2) (2), p40] $\mathbb{Z}_6$ is T-hollow module. Every semisimple module is T-hollow, and hence it is semi-T- hollow. By (1), $\mathbb{Z}_6$ as $\mathbb{Z}$- module is a semi-T- hollow module.
6. If W is an R- module such that $Rad\ (W) = 0$, then W is T-hollow module if and only if W is semi-T- hollow module. In fact, let T, $K \leq W$ such that $T \nsubseteq K$ and $T \subseteq X + K$, where $X \leq W$. Since W is semi-T-hollow, then $T \subseteq X + Rad(W)$. But $Rad\ (W) = 0$, thus $T \subseteq X$, i.e. W is T-hollow module.

Proposition (2.3):

Let T be a submodule of a non-zero module W with $Rad(W) = 0$. If W is a semi-T- hollow module, then every non-zero submodule N of W such that $T \subseteq N$ is a semi-T-hollow-module.

Proof: Let N be a submodule of W such that $T \subseteq N$. Let L be a submodule of N such that $T \nsubseteq L$. Since W is a semi –T- hollow module, thus $L \ll_{S-T} W$. Then by [3], $L \ll_{S-T} N$, therefore N is a semi-T-hollow module.

Proposition (2.4):

Let W be a semi-T- hollow module, and let $f: W \rightarrow \acute{W}$ be an epimorphism where $\acute{W}$ is a non-zero module. Then $\acute{W}$ is a semi-$f(T)$-hollow module.

Proof: Assume that W is a semi-T-hollow module and let $f: W \rightarrow \acute{W}$ be an epimorphism. Let $\acute{N}$ be a submodule of $\acute{W}$ such that $f(T) \nsubseteq N$. Claim that $N \ll_{S-f(T)} \acute{W}$. To show that, let $f(T) \subseteq \acute{N} + X$, for some $X \leq \acute{W}$. Then $f^{-1}(f(T)) \subseteq f^{-1}(\acute{N} + X)$. Therefore $T + Kerf \subseteq f^{-1}(\acute{N}) + f^{-1}(\acute{X})$. Thus $T \subseteq f^{-1}(\acute{N}) + f^{-1}(X)$ on $T \nsubseteq f^{-1}(\acute{N})$. But W is semi-T-hollow. Therefore $f^{-1}(\acute{N}) \ll_{S-T} W$. We can easily show that $T \subseteq f^{-1}(X) + Rad(W)$. So $f(T) \subseteq X + f(Rad(W)) \subseteq X + Rad(\acute{W})$ [4, Theorem (9.1.4), p.214]

3. Semi-*T*-lifting modules

In this section, we introduce the concept semi-T-hollow module and illustrate it by some examples. We also give some basic properties. We start by this definition which is given in [2, Definition (2.2.1), p40].

Definition (3.1)

Let W be an R- module and let T be a submodule of W. We say that W is a T-lifting module if for every submodule X of W, there exists a direct summand F of W and $H \ll_{S-T} W$ such that $X = F + H$.
We introduce the following definition:

Definition (3.2)

Let T be a submodule of an R –module W. We say that W is a semi-T-lifting module if for every submodule X of W, there exists a direct summand F of W and $H \ll_{S-T} W$ such that $X = F + H$.

Remarks and Example (3.3)

1. Every T- lifting module is a semi-T-lifting module. In fact, let W be a T-lifting module, where T is a submodule of W and let X be a submodule of W. Thus by assumption, there exists a direct summand F of W and $H \ll_T W$ such that $X = F + H$. Thus $H \ll_{S-T} W$ by [3], and W is semi-T-lifting module.
2. A semi-T-lifting module needs not to be T-lifting module, for example $\mathbb{Z}_8$ as $\mathbb{Z}$-module is not $\{\bar{0}, \bar{4}\}$-lifting [2,Remark.and Example (2.3.3) (2),p44]. But one can easily show that $\mathbb{Z}_8$as $\mathbb{Z}$-module is a semi-$\{\bar{0}, \bar{4}\}$-lifting module, where $Rad(\mathbb{Z}_8) = \{\bar{0}, \bar{4}\}$.
3. A lifting module needs not be a semi-T-lifting module. For example $\mathbb{Z}_8$ as $\mathbb{Z}$-module is not a semi-$\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ lifting module. To show that, assume not and let $X = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$, then there exists a direct summand F of $\mathbb{Z}_8$ and $H \ll_{S-X} \mathbb{Z}_8$ such that $X = F + H$. Since $\mathbb{Z}_8$ is indecomposable, thus $F = 0$ and $H = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$. But X is not semi- X-small in $\mathbb{Z}_8$. Thus $\mathbb{Z}_8$ is not semi X lifting module. One can easily show that $\mathbb{Z}_8$ as $\mathbb{Z}$-module is a lifting module.
4. The converse of (1) is true if $Rad\ (W) = 0$, i.e. if W is an R- module with $Rad(W) = 0$, then W is T-lifting module if and only if W is semi-T-lifting module
5. On can easily show that every semisimple module W is semi-T-lifting for every submodule T of .
6. It is clear that every semi-T-hollow module is semi-T-lifting module.

Proposition (3.4)

Let W be a semi-T-lifting module. Then every submodule N of W such that $T \subseteq N$ is also semi-T-lifting module.

Proof: Assume that W is semi-T-lifting. Let N be a submodule of W such that $T \subseteq N$. Let X be a submodule of N. Since W is semi-T-lifting, then $X = F + H$, where F is a direct summand of W and $H \ll_{S-T} H$. Clearly, H is a direct summand of N. Since $T \subseteq N$, then $H \ll_{S-T} N$ by [3].

Recall that a submodule X of a module M is called projective invariant, if for every $P = P^2 \in End\ (M)$, $P(X) \leq X$ [5].

It is clear that if each submodule X of W, there exists a decomposition $W = F \oplus \acute{F}$ such that $F \subseteq X$ and $X \cap F \ll_{S-T} W$, then W is a semi-T-lifting module. It is natural to ask if the convers is true.

Proposition (3.5)

Let T be a submodule of a module W. If W is a semi-T-lifting module, then for each submodule X of W, there exists a decomposition $W = F \oplus \acute{F}$ such that $F \subseteq X$ and $X \cap F \ll_{S-T} W$.

Proof: Assume that W is semi-T-lifting module and every semi-T-small of W is projective invariant. Let X be a submodule of W, then $X = F + H$, where F is a direct summand of W, and $H \ll_{S-T} W$. Now, by Modular Law, $X = F \oplus X \cap \acute{F}$. Let $P: W \to \acute{F}$ be the projection map,

$$P(H) = P(F + H) = P\left(F \oplus \left(X \cap \acute{F}\right)\right) = X \cap \acute{F}.$$

Since H is semi-T-small in W, then, by our assumption, H is projective invariant. So $P(H) = (X \cap \acute{F}) \leq H$. Thus $X \cap \acute{F} \ll_{S-T} W$, by [3]

Theorem (3.6)

Let T be a submodule of a module W, then the following statements are the same:

1. For each submodule N of W, there exists a decomposition $W = L \oplus \acute{L}$, where $L \leq N$ and $N \cap \acute{L} \ll_{S-T} W$.
2. For each submodule N of W, there exists $f \in End(W)$ such that $f^2 = f$, $f(W) \subseteq X$ and $(I - f)(N) \ll_{S-T} W$.

Proof: (1)⇒(2): Let N be a submodule of W, thus by assumption, there exists a decomposition $W = L \oplus \acute{L}$ such that $L \leq N$ and $N \cap \acute{L} \ll_{S-T} W$. Let $f: W \to L$ be the projection map. Clearly, $f^2 = f$ and $W = f(W) \oplus (I-f)(W)$, $f(W) \subseteq N$. Now, we have $(I-f)(N) = N \cap (I-f)(N) = N \cap \acute{L} \ll_{S-T} W$.
(2)⇒(1): Let N be a submodule of W, then there exists $f \in End(W)$ such that $f^2 = f$, $f(W) \subseteq N$ and $(I-f)(N) \ll_{S-T} W$. It is clear that $W = f(W) \oplus (I-f)(W)$. Suppose that $L = f(W)$ and $\acute{L} = (I-f)(W)$. Then $N \cap \acute{L} = N \cap (I-f)(W)$. Claim that $N \cap (I-f)(W) = (I-f)(N)$. To show that, let $x = (I-f)(y) \in N \cap (I-f)(W)$. Since $(I-f)^2 = (I-f)$, then $x = (I-f^2)(y) = (I-f)(x) \in (I-f)(N)$. Now, let $x = (I-f)(y) \in (I-f)(x)$, $y \in N$, then $x \in (I-f)(W)$, $x = y - f(y) \in N$. Thus $x \in N \cap (I-f)(W)$. Now, $(I-f)(N) = N \cap (I-f)(W) = N \cap \acute{L} \ll_{S-T} W$.

The following proposition gives a characterisation of a semi-T-lifting module W when every semi-T-small submodule of W is projective invariant.
From Proposition (3.5) and Theorem (3.6) we get:

Proposition (3.7):
Let T be a submodule of a module W such that every semi-T-small submodule of W is projective invariant. Then the following statements are the same:
1. W is a semi-T-lifting module.
2. For each submodule N of W, there exists a decomposition $W = L \oplus \acute{L}$ such that $L \leq N$ and $N \cap \acute{L} \ll_{S-T} W$.
3. For each submodule N of W, there exists $f \in End(W)$ such that $f^2 = f$, $f(W) \subseteq N$ and $(I-f)(N) \ll_{S-T} W$.

Proposition (3.8):
Let W be a semi-T-lifting module and let A be a submodule of W, such that for every direct summand N of W, $\frac{N+A}{A}$ is a direct summand of $\frac{W}{A}$. Then $\frac{W}{A}$ is a semi-$\frac{T+A}{A}$-lifting module.

Proof Let $\frac{X}{A}$ be a submodule of $\frac{W}{A}$. Since W is semi-T-lifting, then $X = N + H$, where N is a direct summand of W and $H \ll_{S-T} W$. Hence $\frac{X}{A} = \frac{N+H}{A} + \frac{H+A}{A}$. By our assumption, $\frac{N+A}{A}$ is a direct summand of $\frac{W}{A}$. To show that $\frac{H+A}{A} \ll_{S-\frac{T+A}{A}} \frac{W}{A}$, let $\frac{K}{A} \leq \frac{W}{A}$ such that $\frac{T+A}{A} \subseteq \frac{H+A}{A} + \frac{K}{A} = \frac{H+A+K}{A}$, then $T \subseteq T + A \subseteq H + A + K$. Since $H \ll_{S-T} W$, thus $T \subseteq A + K + Rad(W)$ and hence $\frac{T+A}{A} \subseteq \frac{K}{A} + \frac{Rad(W)}{A} \subseteq \frac{K}{A} + Rad\left(\frac{W}{A}\right)$, thus $\frac{W}{A}$ is semi - $\frac{T+A}{A}$ – lifting module.
Recall that W is called a distributive module if for every submodules A, B and C of W, $A + (B \cap C) = (A+B) \cap (A+C)$ and $A \cap (B+C) = (A \cap B) + (A \cap C)$, see[6] .

Corollary (3.9)
Let W be a semi-T-lifting and distributive module and A be a submodule of W. Then $\frac{W}{A}$ is a semi-$\frac{T+A}{A}$-lifting module.

Proof. Let N be a direct summand of W, then $W = N \oplus \acute{N}$, for some submodule $\acute{N}$ of W. Thus $\frac{W}{A} = \frac{N+\acute{N}}{A} = \frac{N+A}{A} + \frac{N\acute{+}A}{A}$. Since W is distributive, then $(N+A) \cap (\acute{N}+A) = \left((N+A) \cap \acute{N}\right) + \left((N+A) \cap A\right) = A$. Hence, $\frac{W}{A} = \frac{N+A}{A} \oplus \frac{N\acute{+}A}{A}$. Therefore, by Prop (3.8), $\frac{W}{A}$ is semi-$\frac{T+A}{A}$-lifting module.

Lemma (3.10) [7, Lemma 5.4]
Let $W = W_1 \oplus W_2$ be an R-module. Then $\frac{W}{A} = \frac{W_1+A}{A} \oplus \frac{W_2+A}{A}$ for every fully invariant submodule A of W.

Corollary (3.11)
Let W be a semi-T-lifting module. If A is a fully invariant of W, then $\frac{W}{A}$ is a semi-$\frac{T+A}{A}$-lifting module.

Proof: It is clear by Proposition (3.8) and Lemma (3.10).

Proposition (3.12)
Let $W = W_1 \oplus W_2$ be a module such that $R = AnnW_1 + AnnW_2$. If W_1 is semi-T_1-lifting and W_2 is semi-T_2-lifting, then $W = W_1 \oplus W_2$ is a semi-$T_1 \oplus T_2$-lifting module.

Proof : Let X be a submodule of a module W. Since $R = AnnW_1 + AnnW_2$, then by the same way of the proof of [8, proposition 4.2, chaphter1], $X = X_1 \oplus X_2$, where $X_1 \leq W_1$ and $W_2 \leq W_2$. By W_1 is semi-T_1-lifting and W_2 is semi-T_2-lifting module, thus $X_1 = F_1 + H_1$, where $F_1 \leq\oplus W_1$ and $H_1 \ll_{S-T_1} W_1$, and $X_2 = F_2 + H_2$ where $F_2 \leq\oplus W_2$ and $H_2 \ll_{S-T_2} W_2$.

Hence $X = X_1 \oplus X_2 = (F_1 \oplus F_2) + (H_1 \oplus H_2)$. One can easily show that $(F_1 \oplus F_2)$ is a direct summand of W. By [3] we have $H_1 \oplus H_2 \ll_{S-T_1 \oplus T_2} W$. Thus $W_1 \oplus W_2$ is a semi-$T_1 \oplus T_2$-lifting module.

Proposition (3.14):

Let $W = \oplus_{i \in I} W_i$ be a fully stable module and $T = \oplus_{i \in I} T_i$ where $T_i \leq W_i$, for every $i \in I$. If W_i is semi-T_i-lifting module, for each $i \in I$, then W is a semi-$\oplus_{i \in I} T$-lifting module.

Proof : Let X be a submodule of W. For each $i \in I$, one can easily show that $X = \oplus_{i \in I}(X \cap W_i)$. Since $X \cap W_i \subseteq W_i$ and W_i is semi-T_i-lifting, then $X \cap W_i = F_i + H_i$, where $F_i \leq\oplus W_i$ and $H_i \ll_{S-T_i} W_i$. Therefore $\oplus(X \cap W_i) = \oplus_{i \in I} F + \oplus_{i \in I} H_i$. One can easily show that $\oplus_{i \in I} F_i \leq \oplus_{i \in I} W_i$. Thus by [3], $\oplus H_i \ll_{S-\oplus T_i} W$. Therefore W is a semi-$\oplus T_i$-lifting module.

Proposition (3.15)

Let W be a finitely generated, faithfull and multiplication module. Then W is semi-T-lifting module if and only if R is semi-$[T:W]$-lifting module.

Proof: Assume that W is a semi-T-lifting module and let I be an ideal of R. Since W is a semi-T-lifting module, then there exist $F \leq\oplus W$ and $H \ll_{S-T} W$ such that $IW = F + H$. Since W is a multiplication module, then there exists ideals J and K of R such that $F = JW$ and $H = KW$. Hence $IW = (J + K)W$. But W is finitely generated, faithful and multiplication module, thus by [8] W is a cancellation module. Therefore $I = J + K$. Claim that $J \leq\oplus R$. To show that, let $W = F \oplus \acute{F}$, where $\acute{F} \leq W$ such that $\acute{F} = \acute{J}W$, for some ideal $\acute{J}$ of R. Hence $RW = W = JW \oplus \acute{J}W = (J + \acute{J})W$. But W is a cancellation module, therefore $R = J + \acute{J}$.

To show that $J \cap \acute{J} = 0$, since W is a finitely generated, faithful multiplication module, then $0 = JW \cap \acute{J}W = (J \cap \acute{J})W$ and hence $J \cap \acute{J} = 0$. Thus $J \leq\oplus R$. By [3], $K \ll_{S-[T:W]} R$. Hence R is semi$-[T:W]-$lifting module.

Conversely, assume that R be a semi-$[T:W]$-lifting and let X be a submodule of W. Since W is multiplication module, then there exists an ideal I of R such that $X = IW$. Then there exist $J \leq\oplus R$ and $K \ll_{S-[T:W]} R$ such that $I = J + K$. Hence $IW = JW + KW$. Claim that $JW \leq\oplus W$. To show that, let $R = J \oplus \acute{J}$, for some ideal $\acute{J}$ of R, hence $W = RW = (J + \acute{J})W = JW + \acute{J}W$. Since W is a finitely generated, faithful and multiplication module, then $JW \cap \acute{J}W = (J \cap \acute{J})W = 0W = 0$. Thus $JW \leq\oplus W$, by [3], and $KW \ll_{S-T} W$. Therefore W is a semi-T-lifting module.

References

1. Beyranvand, R and Moradi F. **2015**. Small submodule with respect to an arbitrary submodule. *Journal of Algebra and Related Topics,* **3**(2): 43-51.
2. AL-Redeeni, H.S. **2017**. On $(G^* -)$T-lifting modules and $T-H$ supplemented Module. Ms. Thesis, Department of Mathematic, college of science, Baghdad University, Iraq.
3. Elewi, A.A. **2020**. Semi-T-small submodule, *Iraqi journal of science*, accepted.
4. Kasch.F.**1982**.*Modules and ring*. London, Academic Press.
5. Wisbauer, R. **1991**. *Foundations of module and ring theory.* Philadelphia. Gordon and Brwach.
6. Erdogdu, V.**1987**. Distributive Module. *Can. Math. Bull,* **30**: 248-254.
7. Orhan, N., Tutuncu, D.K. and Tribak, R. **2007**.*On Hollow Modules*. Tutuncu, **11**(20): 545-568.
8. Abbas, M.S. **1991**. On fully stable modules. Ph.D. Thesis, Department of Mathematic, college of science, Baghdad University, Iraq.