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# Simultaneous Identification of Thermal Conductivity and Heat Source in the Heat Equation

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**Abstract**

This paper presents a numerical solution to the inverse problem consisting of recovering time-dependent thermal conductivity and heat source coefficients in the one-dimensional parabolic heat equation. This mathematical formulation ensures that the inverse problem has a unique solution. However, the problem is still ill-posed since small errors in the input data lead to a drastic amount of errors in the output coefficients. The finite difference method with the Crank-Nicolson scheme is adopted as a direct solver of the problem in a fixed domain. The inverse problem is solved subjected to both exact and noisy measurements by using the MATLAB optimization toolbox routine lsqnonlin, which is also applied to minimize the nonlinear Tikhonov regularization functional. The thermal conductivity and heat source coefficients are reconstructed using heat flux measurements. The root mean squares error is used to assess the accuracy of the approximate solutions of the problem. A couple of numerical examples are presented to verify the accuracy and stability of the solutions.

**Keywords:** Inverse problem; Heat equation; Heat flux; Tikhonov regularization; Nonlinear optimization.

## 1 Introduction

Inverse problems for the parabolic heat equation consisting of determining the unknown coefficients and heat source depending on time or space variable, have recently received some attention. An example of coefficient identification problem is to determine a single unknown time- dependent property, such as heat capacity, thermal conductivity, or diffusivity, from additional local or non-local measurements of the main dependent variable at the boundary or inside the domain [1, 2, 3]. The knowledge to this physical property is important to understand the heat transfer in biological tissues, finance, groundwater flow, and oil recovery. In previous papers [4, 5, 6], multiple time-dependent coefficient identifications were considered, while they were recently solved numerically. In these studies, the unknowns were mainly coefficients multiplying the temperature and its partial derivatives. However, in other studies [7, 8, 9], one of the time-dependent unknowns is allowed to be in the free term heat source. Other authors [10] investigated the reconstruction of these coefficients, as well as of the absorption coefficient, using the measurement of the heat moments. The time and space-dependent unknown coefficients from data measurements in the one-dimensional parabolic heat equation were determined elsewhere [11].

In a recent paper [3], the authors investigated the inverse problems of simultaneous numerical

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reconstruction of time-dependent coefficients $b(t)$ (advection/convection coefficient) or *d(t)* (reaction/perfusion coefficient), together with the unknown heat source term f(t), in one-dimensional parabolic equations from integral over-specification conditions. In this paper, we focus on solving numerically the unknown coefficients *a(t)* (thermal conductivity) and *f(t)* together with the unknown temperature satisfying the inverse problem (subjected to both exact and noisy data), using the measurements of heat flux instead of integral conditions. The inverse problems investigated in this paper have already been proved to be locally uniquely solvable by Bereznyts'ka [12], but no numerical reconstruction has been attempted so far. It is the purpose of this paper to undertake the simultaneous numerical solution of these unknowns.

The paper is structured as follows: In Section 2, the mathematical formulation of the inverse problem is stated. The numerical finite-difference with a Crank-Nicolson discretization of the direct problem is described in Section 3. In Section 4, the numerical approach based on the minimization of the nonlinear Tikhonov regularization functional is introduced. In Section 5, numerical results and discussion are illustrated.. Finally, conclusions of the article are given in Section 6.

## 2 Mathematical formulation of inverse problem

In the rectangular domain $\Omega_T = \{(x,t): 0 < x < h, 0 < t < T\}$, we consider the inverse problem of determining the time-dependent coefficients

$$\big(a(t), f(t), u(x,t)\big) \in \left(H^{\frac{\alpha}{2}}[0,T]\right)^2 \times H^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}_T), \qquad 0<\alpha<1, a(t)>0, f(t)\geq 0, t\in[0,T],$$

that satisfies the one-dimensional heat equation

$$u_t = a(t)u_{xx}(x,t) + b(x,t)u_x(x,t) + c(x,t)u(x,t) + f(t)g_0(x,t) + g_1(x,t), \quad (x,t)\in\Omega_T, \tag{1}$$

subject to the initial condition

$$u(x,0) = \phi(x), \quad x\in[0,h], \tag{2}$$

the Dirichlet non-homogeneous boundary conditions

$$u(0,t) = \mu_1(t), \quad u(h,t) = \mu_2(t), \quad t\in[0,T], \tag{3}$$

and the heat flux overdetermination condition

$$u_x(0,t) = \mu_3(t), \quad u_x(h,t) = \mu_4(t), \quad t\in[0,T], \tag{4}$$

where $\phi(x), \mu_i(t)$ for i=1,2,3,4 are given functions satisfying compatibility conditions. The uniqueness of the solution of the inverse problem (1)-(4) was established earlier in [12] and reads as follows.

**Theorem 1.** (Uniqueness of the solution)

Assume that the following conditions are satisfied:

(i) $b(x,t), c(x,t), g_0(x,t) \in H^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}_T)$;

(ii) $W(t) := \big(\mu_1'(t) - b(0,t)\mu_3(t) - c(0,t)\mu_1 - g_1(0,t)\big)g_0(h,t) - \big(\mu_2'(t) - b(h,t)\mu_4(t) - c(h,t)\mu_2 - g_1(h,t)\big)g_0(0,t) \neq 0, \qquad t\in[0,T]$ (5)

Then the inverse problem (1)-(4) cannot have more than one solution in the domain $\Omega_T$.

In this problem, the Holder space $H^{\frac{\alpha}{2}}[0,T]$ denotes the space of differentiable functions on [0,T] with the derivative being Holder continuous with the exponents $\frac{\alpha}{2}$. Also, $H^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}_T)$ denotes the space of continuous functions $u$ along with their partial derivatives $u_x, u_{xx}, u_t$ in $\overline{\Omega}_T$, with $u_{xx}$ being Holder continuous with exponent $\frac{\alpha}{2}$ in $t\in[0,T]$ uniformly with respect to $x\in[0,h]$.

## 3 Direct problem

When $a(t), b(x,t), c(x,t), g_0(x,t), g_1(x,t), f(t), \mu_1(t)$, and $\mu_2(t)$ are known, the direct problem is given by equation (1)-(3). We subdivide the solution domain $\Omega_T = (0,h)\times(0,T)$ into M and N subintervals of equal step lengths, $\Delta x$ and $\Delta t$, where $\Delta x = \frac{h}{M}$ and $\Delta t = \frac{T}{N}$. We denote $u(x_i,t_j) = u_{i,j}$, where $x_i = i\Delta x$, $t_j = j\Delta t$, $a(t_j) = a_j, f(t_j) = f_j, b(x_i,t_j) = b_{ij}\ c(x_i,t_j) = c_{ij}, g_0(x_i,t_j) = g_{0_{ij}}$, $g_1(x_i,t_j) = g_{1_{ij}}$, for $i = \overline{0,M}$, and $j = \overline{0,N}$. We use the finite difference method (FDM) with Cranck-Nicolson scheme [13], which is unconditionally stable and second order accurate in space and time. Based on FDM with C-N, equation (1) can be approximated as:

$$-A_{i,j+1}u_{i-1,j+1} + \big(1+B_{i,j+1}\big)u_{i,j+1} - C_{i,j+1}u_{i+1,j+1} = A_{i,j}u_{i-1,j} + \big(1-B_{i,j}\big)u_{i,j} - C_{i,j}u_{i+1,j} +$$

$$\frac{\Delta t}{2}\left(\left(f_j g_{0_{i,j}} + g_{1_{i,j}}\right) + \left(f_{j+1} g_{0_{i,j+1}} + g_{1_{i,j+1}}\right)\right) \quad (6)$$

for $i = \overline{1,(M-1)}$ and $j = \overline{0,(N-1)}$, where

$$A_{i,j} = \frac{(\Delta t)a_j}{2(\Delta x)^2} - \frac{(\Delta t)b_{ij}}{4(\Delta x)}, \quad B_{i,j} = \frac{(\Delta t)a_j}{(\Delta x)^2} - \frac{(\Delta t)c_{ij}}{2}, \quad C_{i,j} = \frac{(\Delta t)a_j}{2(\Delta x)^2} + \frac{(\Delta t)b_{ij}}{4(\Delta x)}.$$

The initial and boundary conditions in equations (2) and (3) are discretized as follows:

$$u_{i,0} = \phi(x_i), \qquad i = \overline{0,M}, \quad (7)$$

$$u_{0,j} = \mu_1(t_j), \quad u_{M,j} = \mu_2(t_j), \qquad j = \overline{0,N}, \quad (8)$$

At each time step $t_{j+1}$ for $j = \overline{0,(N-1)}$, using the Dirichlet boundary conditions (8), the above difference equation (6) can be reformulated as a (M $-1$) $\times$(M $-1$) system of linear equations of the form

$$D\,u = k, \quad (9)$$

where $u = \left(u_{1j+1}, u_{2j+1}, \dots, u_{M-2j+1}, u_{M-1j+1}\right)^T$, $k = (k_1, k_2, \dots, k_{M-2}, k_{M-1})^T$,

$$D = \begin{bmatrix} 1+B_{1,j+1} & -C_{1,j+1} & 0 & \cdots & & 0 & 0 & 0 \\ A_{2,j+1} & 1+B_{2,j+1} & -B_{2,j+1} & \cdots & & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -A_{M-2,j+1} & 1+B_{M-2,j+1} & -C_{M-2,j+1} & \\ 0 & 0 & 0 & \cdots & 0 & -A_{M-1,j+1} & 1+B_{M-1,j+1} & \end{bmatrix}$$

$k_1 = A_{1,j}u_{0,j} + (1-B_{1,j})u_{1,j} + C_{1,j}u_{2,j} + A_{1,j+1}u_{0,j+1} + \frac{\Delta t}{2}\left(\left(f_j g_{0_{1,j}} + g_{1_{1,j}}\right) + \left(f_{j+1} g_{0_{1,j+1}} + g_{1_{1,j+1}}\right)\right)$,

$k_i = A_{i,j}u_{i-1,j} + (1-B_{i,j})u_{i,j} + C_{i,j}u_{i,j} + \frac{\Delta t}{2}\left(\left(f_j g_{0_{i,j}} + g_{1_{i,j}}\right) + \left(f_{j+1} g_{0_{i,j+1}} + g_{1_{i,j+1}}\right)\right)$, $i = \overline{2,(M-2)}$,

$k_{M-1} = A_{M-1,j}u_{M-2,j} + (1-B_{M-1,j})u_{M-1,j} + C_{M-1,j}u_{M,j} + C_{M-1,j+1}u_{M,j+1} + \frac{\Delta t}{2}\left(\left(f_j g_{0_{M-1,j}} + g_{1_{M-1,j}}\right) + \left(f_{j+1} g_{0_{M-1,j+1}} + g_{1_{M-1,j+1}}\right)\right)$.

The expressions in (4) are calculated using the following finite difference approximation formulas:

$$\mu_3(t_j) = \frac{4u_{1j} - u_{2j} - 3u_{0j}}{2(\Delta x)}, \quad j = \overline{1,N}, \quad (10)$$

$$\mu_4(t_j) = \frac{4u_{M-1j} - u_{M-2j} - 3u_{Mj}}{2(\Delta x)}, \quad j = \overline{0,N}. \quad (11)$$

4 Inverse problem

The numerical solution of the inverse problem (1)–(4) is obtained by minimizing the nonlinear Tikhonov regularization function

$$F(\boldsymbol{a},\boldsymbol{f}) = \sum_{j=1}^{N}\left[u_x(0,t_j) - \mu_3(t_j)\right]^2 + \sum_{j=1}^{N}\left[u_x(h,t_j) - \mu_4(t_j)\right]^2 + \beta_1 \sum_{j=1}^{N} a_j^2 + \beta_2 \sum_{j=1}^{N} f_j^2 \quad (12)$$

where $u$ solves the direct problem (1)–(3) for given *a(t)* and $f(t)$, respectively. It is worth mentioning that in (12) at the first time step, i.e. $j = 0$, the derivatives $u_x(0,0)$ and $u_x(h,0)$ are obtained from the initial condition (2) via (10) and (11), as

$$u_x(0,0) = \frac{4\phi(1) - \phi(2) - 3\phi(0)}{2(\Delta x)}, \quad (13)$$

$$u_x(h,0) = \frac{4\phi(M-1) - \phi(M-2) - 3\phi(M)}{2(\Delta x)}, \quad (14)$$

where $\phi_i = \phi(x_i)$ for $i = \overline{0,M}$. Also, one can remark that at initial time t = 0, the values *a(0)* and $f(0)$ are obtained from equations (25) and (26). The minimization of (12) is performed using the MATLAB toolbox routine *lsqnonlin*, which does not require supplying by the user the gradient of the objective function [14,15]. This routine attempts to find the minimum of a sum of squares by starting from the initial guesses. The inverse problems given by (1)–(3) are solved subject to both exact and noisy measurements (4). The noisy data are numerically simulated as follows

$$\mu_3^{\epsilon 1}(t_j) = \mu_3(t_j) + \epsilon 1_j, \quad \mu_4^{\epsilon 2}(t_j) = \mu_4(t_j) + \epsilon 2_j, \quad j = \overline{0, N}, \tag{15}$$

where $\epsilon 1_j$ and $\epsilon 2_j$ are random variables generated from a Gaussian normal distribution with a mean of zero and standard deviations $\sigma_1$ and $\sigma_2$, respectively, given by

$$\sigma_1 = p \times max_{t\in[0,T]}|\mu_3(t)|, \quad \sigma_2 = p \times max_{t\in[0,T]}|\mu_4(t)|, \tag{16}$$

where $p$ represents the percentage of noise.

5 Results and discussion

To assess the accuracy of the approximate solutions, we introduce the root mean squares error (*RMSE*), defined as follows:

$$rmse(a) = \left[\frac{T}{N}\sum_{j=1}^{N}\left(a^{Numerical}(t_j) - a^{Exact}(t_j)\right)^2\right]^{\frac{1}{2}}, \tag{17}$$

$$rmse(f) = \left[\frac{T}{N}\sum_{j=1}^{N}\left(f^{Numerical}(t_j) - f^{Exact}(t_j)\right)^2\right]^{\frac{1}{2}}. \tag{18}$$

Remark. During the computation processing, we need the values of *a(0)* and $f(0)$. One can derive these values from the governing equation (1) with the help of the boundary and overdetermination conditions (3) and (4), as follows.

We can write equation (1) at $x = 0$ and $x = h$ as

$$u_t(0,t) = a(t)u_{xx}(0,t) + b(0,t)u_x(0,t) + c(0,t)u(0,t) + f(t)g_0(0,t) + g_1(0,t), \tag{19}$$

$$u_t(h,t) = a(t)u_{xx}(h,t) + b(h,t)u_x(h,t) + c(h,t)u(h,t) + f(t)g_0(h,t) + g_1(h,t). \tag{20}$$

Also, by differentiating equation (3) with respect to t, we obtain

$$u_t(0,t) = \mu_1'(t), \quad u_t(h,t) = \mu_2'(t). \tag{21}$$

By using equation (21) and solving the system of equations (19) and (20) for *a(t)* and $f(t)$, we obtain

$$a(t) = \frac{1}{D(t)}\{(b(h,t)\mu_4(t) - \mu_2'(t) + c(h,t)\mu_2(t) + g_1(h,t))g_0(0,t) + (\mu_1'(t) - b(0,t)\mu_3(t) - c(0,t)\mu_1(t) - g_1(0,t))g_0(h,t)\}, \quad t \in [0,T], \tag{22}$$

$$f(t) = \frac{1}{D(t)}\{(\mu_2'(t) - b(h,t)\mu_3(t) - c(h,t)\mu_1(t) - g_1(h,t))w(0,t) + (b(0,t)\mu_3(t) - \mu_1'(t) + c(0,t)\mu_1(t) + g_1(0,t))w(h,t)\}, \quad t \in [0,T], \tag{23}$$

$$D(t) := w(0,t)g_0(h,t) - w(h,t)g_0(0,t), \qquad t \in [0,T], \tag{24}$$

where $w(x,t) = u_{xx}(x,t)$.

One can observe that $D(0) \neq 0$, and since the functions in (24) are continuous in $[0, T]$, we obtain that $D(t) \neq 0$. So, the denominator in (22) and (23) does not vanish over the time interval $t \in [0,T]$.

By letting $t = 0$ in the analogue of expressions (22) and (23), we obtain values of a(0) and f(0) explicitly, as follows:

$$a(0) = \frac{1}{D(0)}\{(b(h,0)\mu_4(0) - \mu_2'(0) + c(h,0)\mu_2(0) + g_1(h,0))g_0(0,0) + (\mu_1'(0) - b(0,0)\mu_3(0) - c(0,0)\mu_1(0) - g_1(0,0))g_0(h,0)\}, \tag{25}$$

$$f(0) = \frac{1}{D(0)}\{(\mu_2'(0) - b(h,0)\mu_3(0) - c(h,0)\mu_1(0) - g_1(h,0))\phi''(0) + (b(0,0)\mu_3(0) - \mu_1'(0) + c(0,0)\mu_1(0) + g_1(0,0))\phi''(h)\}. \tag{26}$$

For simplicity, we take $h = T = 1$ in all Examples.

5.1 Example 1

We consider the inverse problem (1)–(4) and solve this problem with the input data

$$\phi(x) = e^x, \ \mu_1(t) = e^t, \mu_1(t) = e^{1+t}, b(x,t) = x + t,$$

$$c(x,t) = -x - t, \ g_0(x,t) = 2x - t, g_1(x,t) = t(-e^{t+x} + t - 2x), \tag{27}$$

The exact solutions for the heat flux $\mu_3(t)$, $\mu_4(t)$ and the temperature $u(x,t)$ are given by

$$\mu_3(t) = e^t, \quad \mu_4(t) = e^{1+t}, \tag{28}$$

$$u(x,t) = e^{x+t}, \tag{29}$$

The analytical solution of the unknown coefficients $a(t)$ and $f(t)$ is given by

$$a(t) = 1 + t, \quad f(t) = t. \tag{30}$$

The graph of the function $w(t)$ given by equation (5) for Examples 1 and 2 is shown in Figure 1. From this figure, it can be seen that this function never vanishes over the time interval $t \in [0,T]$ and, hence, condition (5) is satisfied. Consequently, according to Theorem 1, the solution to the inverse problem given by equations (1)–(4) with data (27)– (30) is unique. Figure 2 illustrates the absolute error between the analytical solution (29) and the numerical solution for the temperature $u(x,t)$. It can be observed that the accuracy of the numerical solution improves as the mesh size decreases, as shown clearly in Table 1.

**Table 1-**The RMSE values for μ3(t) and μ4(t) with various mesh sizes, M = N ∈ {10, 20, 40}, for the direct problem

| M = N | 10 | 20 | 40 |
|---|---|---|---|
| rmse(μ3) | 0.0037 | 8.4E-4 | 2.0E-4 |
| rmse(μ4) | 0.0198 | 0.0050 | 0.0012 |

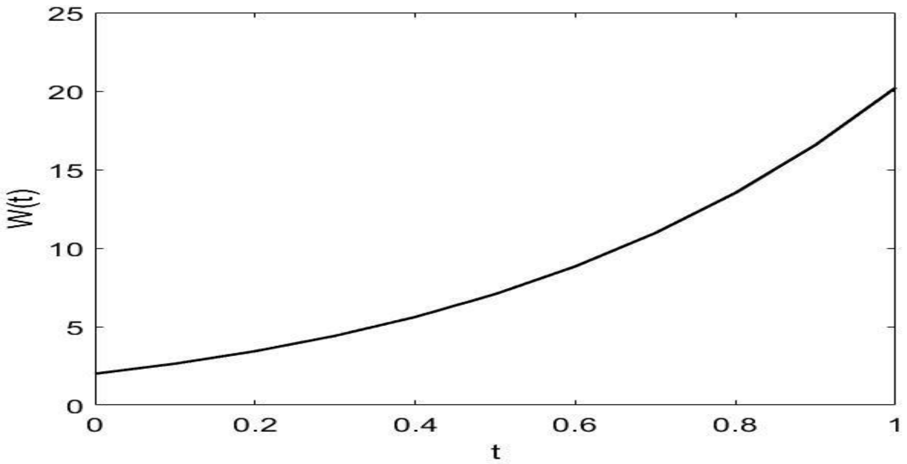


**Figure 1-** The graph of the function W (t) as a function of t, given by (5) for Examples 1 and 2.

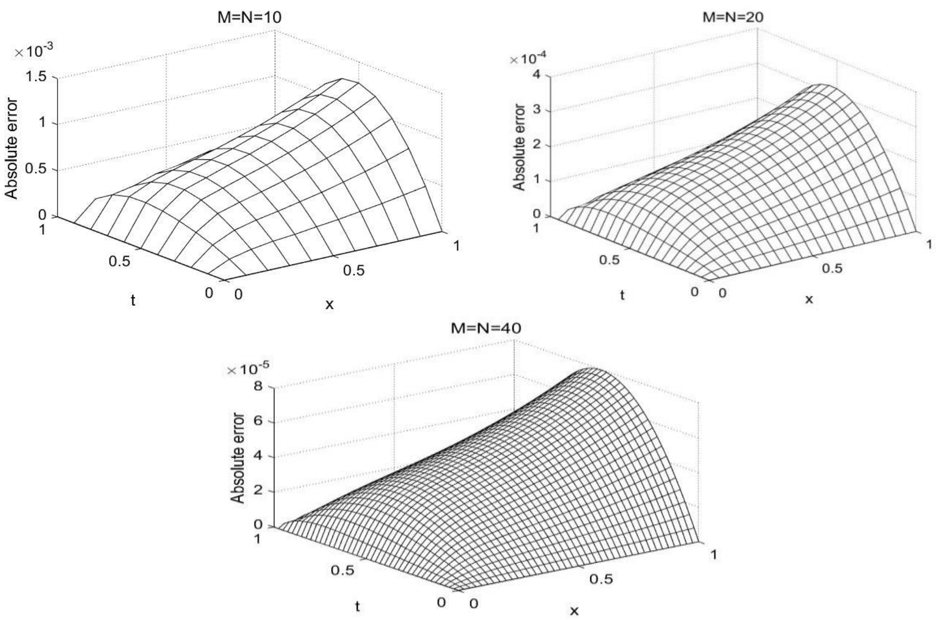

**Figure 2-** The absolute error between the numerical and analytical (29) solutions for u(x, t) with various mesh sizes $M = N \in \{10, 20, 40\}$, for direct problem.

First, we start the investigation for reconstructing the thermal conductivity and heat source coefficients *a (t)* and *f(t)*, respectively, in the case when there is no noise in the input data in (4). The objective function (12), as a function of the number of iterations, is plotted in Figure 3. From this figure, it can be seen that a rapid decrease to a very low value of $O(10^{-25})$ is achieved in about 7 to 11 iterations for Examples 1 and 2. The corresponding exact and numerical solutions for *a(t)* and *f(t)* are presented in Figure 4. From this figure, it can be seen that there is an excellent agreement between the exact and numerical solutions with $\mathrm{rmse}(a) = 1.8 \times 10^{-3}$ and $\mathrm{rmse}(f) = 8.5 \times 10^{-3}$, respectively.

Next, we investigate the stability of the solution with respect to noise. We include $p \in \{0.1\%, 1\%\}$ noise to the formulated data $\mu_3$ and $\mu_4$, as in (15). The objective function (12), as a function of the number of iterations with and without regularization is plotted in Figure 5. From this figure, it can be observed that a monotonic decreasing convergence is achieved and the minimization process stops when the allowed tolerance is reached.

(a) (b)

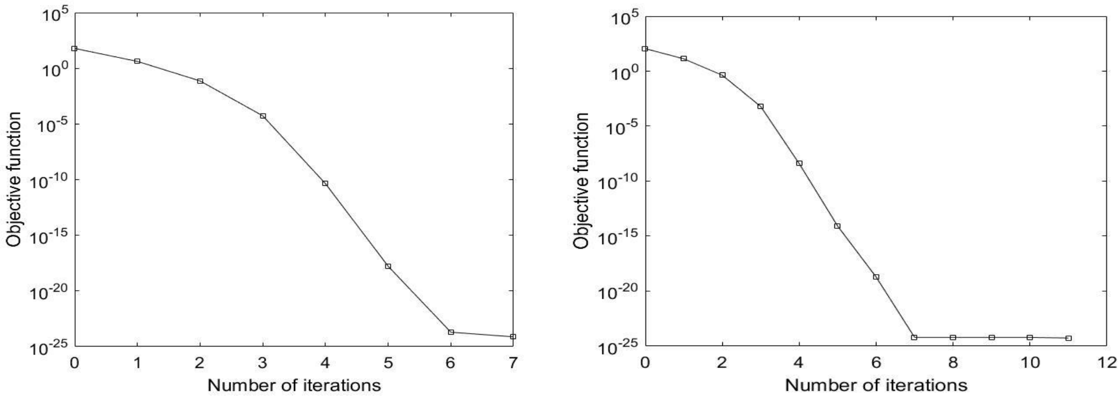


**Figure 3-**The objective function (12), as a function of a number of iterations, with no noise and no regularization for: (a) Example 1 and (b) Example 2.

The numerical solutions for the unknown coefficients plotted in Figure 6 are oscillatory and highly unstable. Therefore, the regularization is needed in order to restore the stability of the solution in the unknown coefficients. Figure 7 displays the associated numerical results for the coefficients *a (t)* and *f (t)*. From this figure and Table 2, it can be seen that accurate and stable results are obtained for $\beta_1 = \beta_2 = 10^{-3}$ for $p = 0.1\%$ and $\beta_1 = \beta_2 = 10^{-2}$ for $p = 1\%$, respectively.

(a) (b)

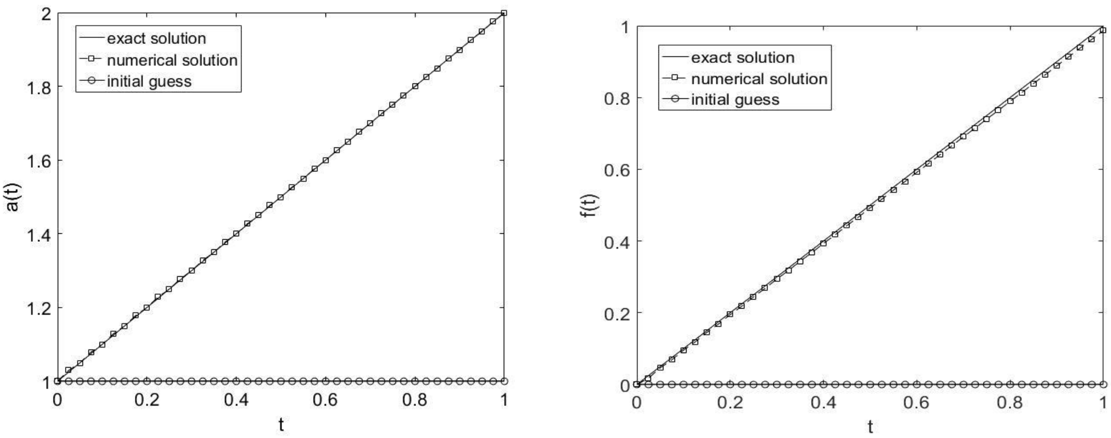

**Figure 4**-The exact (30) and approximate solutions for: (a) thermal conductivity a(t) and (b) heat source f(t), no noise and no regularization, for Example 1.

(a) (b)

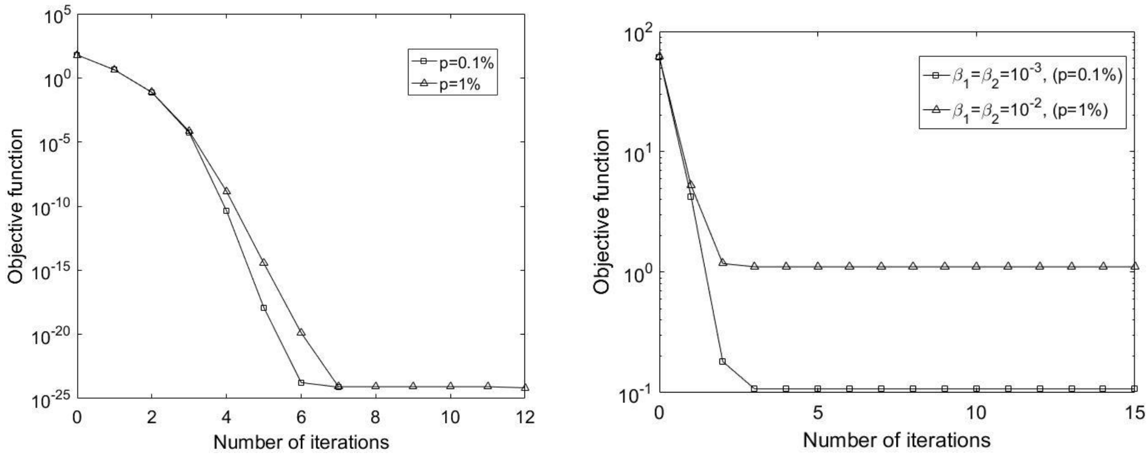


**Figure 5-**The objective function (12), as a function of a number of iterations, with p ∈ {0.1%, 1%} noise for : (a) without regularization and (b) with regularization, for Example 1.

(a) (b)

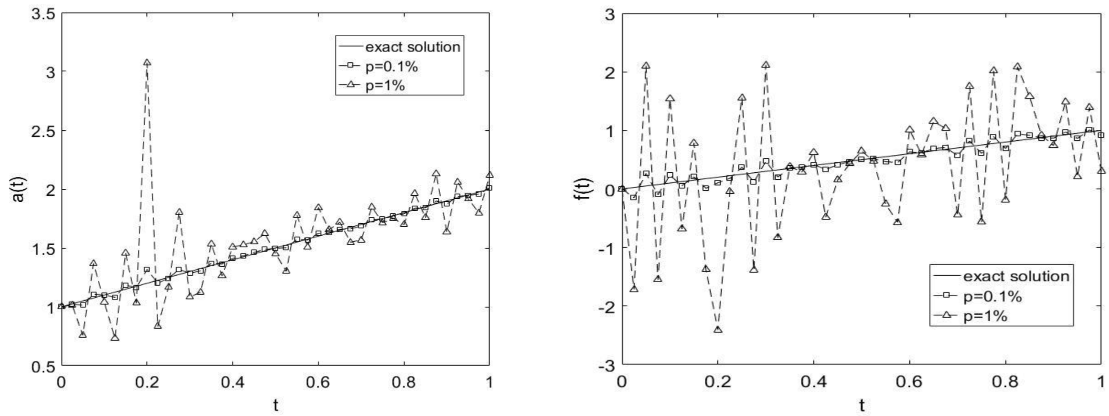


**Figure 6-** The exact (30) and numerical solutions for: (a) a(t) and (b) f (t), with p ∈ {0.1%, 1%} noise and with no regularization for Example 1.

(a) (b)

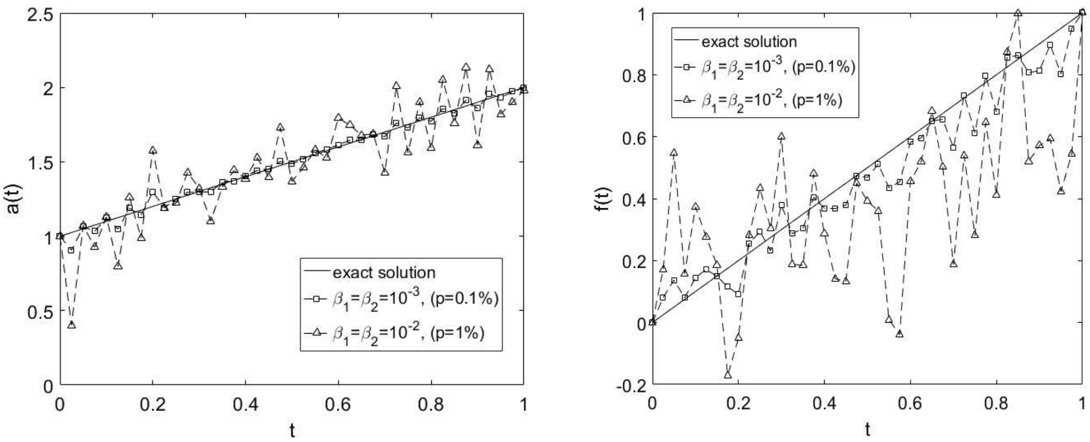


**Figure 7-**The exact (12) and numerical solutions for: (a) a(t) and (b) f (t) with, p ∈ {0.1%, 1%} noise and with regularization for Example 1.

### 5.2 Example 2

In the previous example, we inverted the unknowns of thermal conductivity $a(t) = 1 + t$ and heat source $f(t) = t$, which are smooth functions. In this example, we consider a non-smooth test functions, as in equation (31). We consider the inverse problem (1)–(4) with the following input data

$$b(x,t) = -t - x - 1, \qquad c(x,t) = -t - x - 1$$

$$g_1(x,t) = e^{t+x} - 2e^{t+x}(-t-x-1) - (-t+2x)\left(\begin{cases} 0, & 0 \le t < \frac{1}{4} \\ 1, & \frac{1}{4} \le t < \frac{1}{2} \\ 0, & \frac{1}{2} \le t < \frac{3}{4} \\ 1, & \frac{3}{4} \le t < 1 \end{cases}\right)$$

$$- e^{t+x}\left(\begin{cases} 1, & 0 \le t < \frac{1}{4} \\ 2, & \frac{1}{4} \le t < \frac{1}{2} \\ 1, & \frac{1}{2} \le t < \frac{3}{4} \\ 2, & \frac{3}{4} \le t < 1 \end{cases}\right)$$

One can notice that the conditions of Theorem 1 are satisfied and, hence, the uniqueness of the solution holds. With this data, the analytical solution of the inverse problem is given by

$$a(t) = \begin{cases} 1, & 0 \le t < \frac{1}{4} \\ 2, & \frac{1}{4} \le t < \frac{1}{2} \\ 1, & \frac{1}{2} \le t < \frac{3}{4} \\ 2, & \frac{3}{4} \le t < 1 \end{cases}, \qquad f(t) = \begin{cases} 0, & 0 \le t < \frac{1}{4} \\ 1, & \frac{1}{4} \le t < \frac{1}{2} \\ 0, & \frac{1}{2} \le t < \frac{3}{4} \\ 1, & \frac{3}{4} \le t < 1 \end{cases}$$

As we did in Example 1, we start with the case of exact input data (4), i.e. $p = 0$ in (16). The corresponding numerical results of the timewise thermal conductivity a(t) and heat source **f**(t) are displayed in Figure 8. From this figure, it can be seen that there is an excellent agreement between the analytical and numerical solutions with

$rmse(a) = 1.3 \times 10^{-2}$ and $rmse(f) = 9.3 \times 10^{-3}$.

We add $p \in \{0.1\%, 1\%\}$ noise to the simulated data $\mu_3$ and $\mu_4$, as in (14). Although not illustrated, it is reported that the decreasing monotonic convergence of the objective function (12), as a function of the number of iterations, without and with regularization, is achieved in about 8 to 15 iterations. The resulting thermal conductivity and heat source are plotted in Figure-9 for various levels of noise. With no regularization, the numerical results for the unknown coefficients *a(t)* and *f(t)* presented in Figures 9(a) and 9(b) are quite inaccurate with the values of $rmse(a) = 0.0583$ and $rmse(f) = 0.0918$ for $p = 0.1\%$ and $rmse(a) = 0.3928$ and $rmse(f) = 1.0143$ for $p = 1\%$, respectively. However, when we apply the regularization with the parameters $\beta_1 = \beta_2 = 10^{-3}$ for $p = 0.1\%$ and $\beta_1 = \beta_2 = 10^{-1}$ for $p = 1\%$ to (12), we obtain more accurate reconstructions for a(t) and **f**(t) (Figures 10), with *rmse(a)* and *rmse(f)* values decreasing to $rmse(a,f) = \{0.0381, 0.0855\}$ and $rmse(a,f) = \{0.1640, 0.4903\}$, respectively. The same conclusions as those obtained for Example 1 can be drawn about the stable reconstructions for the unknown coefficients.

(a) (b)

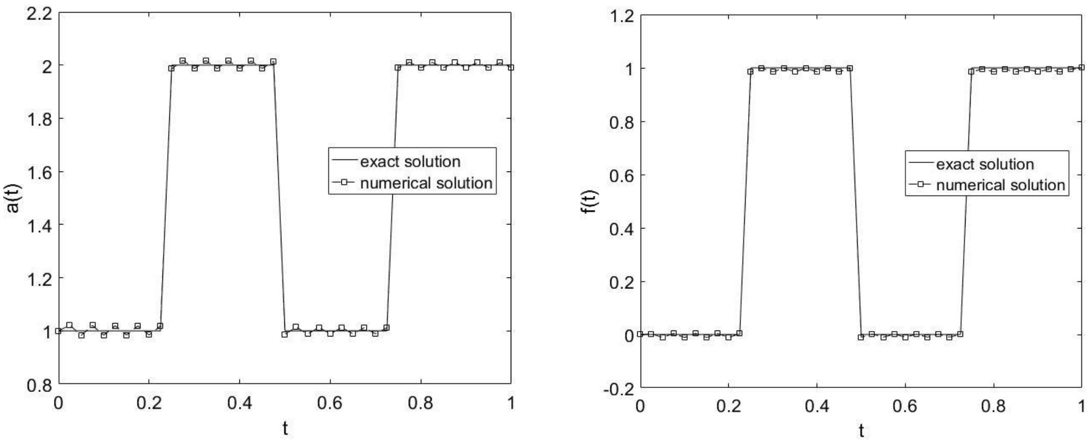


**Figure 8**-The exact (31) and approximate solutions for: (a) thermal conductivity a(t) and (b) heat source f(t), with no noise and no regularization, for Example 2.

(a) (b)

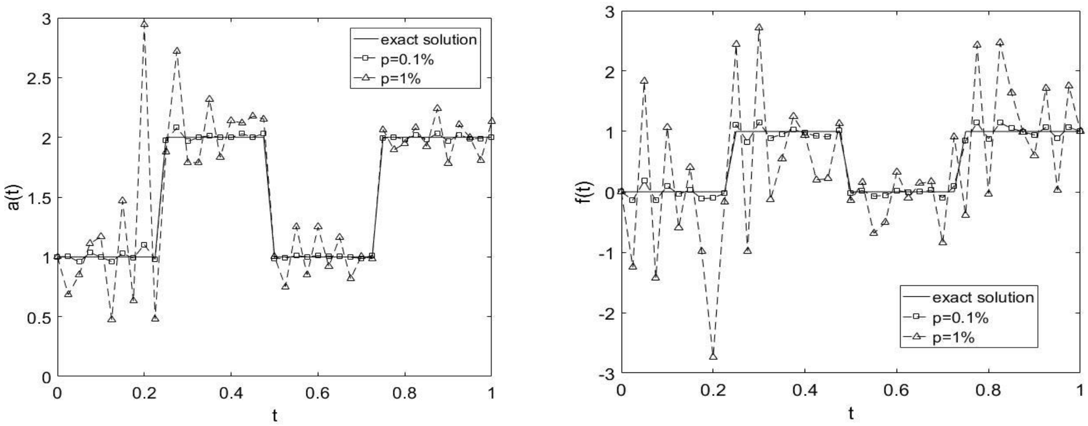


**Figure 9-** The exact (31) and numerical solutions for: (a) a(t) and (b) f (t) with p ∈ {0.1%, 1%} noise, and with no regularization for Example 2.

(a) (b)

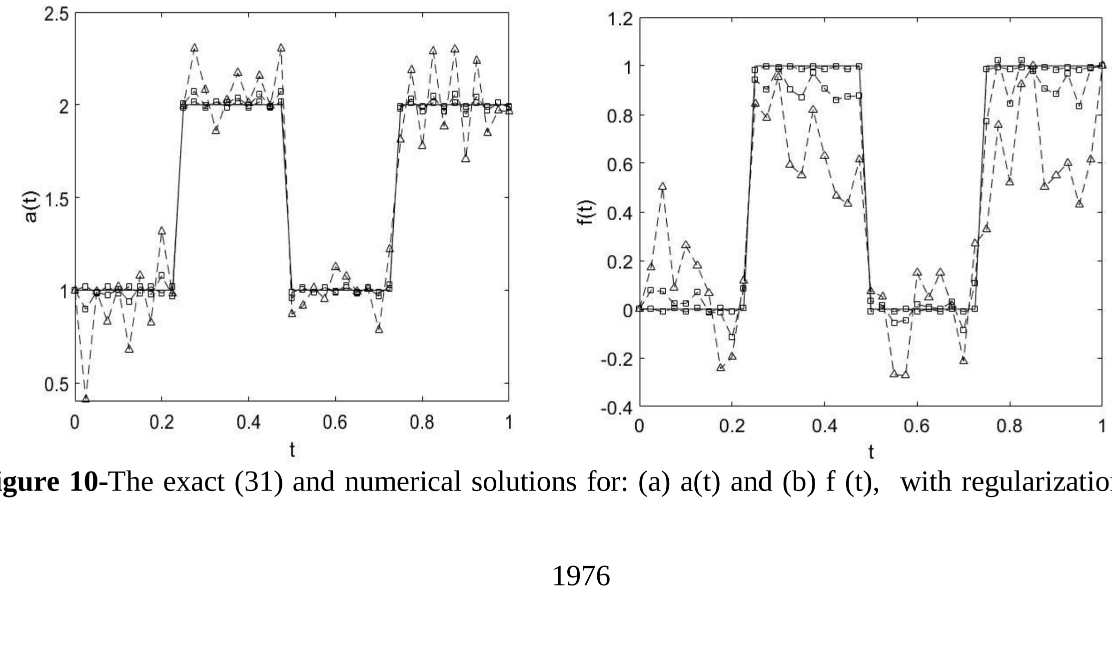


**Figure 10-**The exact (31) and numerical solutions for: (a) a(t) and (b) f (t), with regularization and

noise p = 0.1% (- □ -) and p=1% (-Δ-), for Example 2.

**Table 2-**The rmse values for $p \in \{0, 0.1\%, 1\%\}$ noise, with and without regularization, for Examples 1 and 2.

| Example 1 | *rmse(a)* | *rmse(f)* |
|---|---|---|
| $p = 0, \beta_1 = \beta_2 = 0$ | 1.8E-3 | 8.5E-3 |
| $p = 0.1\%, \beta_1 = \beta_2 = 0$ | 0.0268 | 0.1003 |
| $p = 0.1\%, \beta_1 = \beta_2 = 10^{-3}$ | 0.0351 | 0.0660 |
| $p = 0.1\%, \beta_1 = \beta_2 = 10^{-2}$ | 0.0605 | 0.2162 |
| $p = 1\%, \beta_1 = \beta_2 = 0$ | 0.3551 | 1.0631 |
| $p = 1\%, \beta_1 = \beta_2 = 10^{-3}$ | 0.2796 | 0.5312 |
| $p = 1\%, \beta_1 = \beta_2 = 10^{-2}$ | 0.1915 | 0.3025 |
| | 0.1555 | 0.4219 |
| **Example 2** | *rmse(a)* | *rmse(f)* |
| $p = 0, \beta_1 = \beta_2 = 0$ | 0.0013 | 9.3E-3 |
| $p = 0.1\%, \beta_1 = \beta_2 = 0$ | 0.0583 | 0.0918 |
| $p = 0.1\%, \beta_1 = \beta_2 = 10^{-3}$ | 0.0381 | 0.0855 |
| $p = 0.1\%, \beta_1 = \beta_2 = 10^{-2}$ | 0.0675 | 0.2799 |
| $p = 1\%, \beta_1 = \beta_2 = 0$ | 0.3928 | 1.0143 |
| $p = 1\%, \beta_1 = \beta_2 = 10^{-3}$ | 0.2807 | 0.4997 |
| $p = 1\%, \beta_1 = \beta_2 = 10^{-2}$ | 0.1935 | 0.3170 |
| | 0.1640 | 0.4903 |

## 6 Conclusions

The inverse problem relating to the determination of the time-dependent coefficients a(t) and f(t) along with the temperature u(x, t) in a one-dimensional parabolic equation from over-specification conditions has been investigated for the first time numerically. The direct solver based on the Crank-Nicolson FDM was applied. The resulting nonlinear minimization objective function problem was solved computationally using the MATLAB subroutine lsqnonlin. The Tikhonov regularization was employed in order to obtain stable and accurate solutions since the inverse problem is ill-posed and sensitive to noise. The numerical results for the inverse problem show that stable and accurate approximate results have been obtained. Finally, the generalization of the proposed method for determining the time-dependent coefficients in a two-dimensional heat equation is an interesting topic for future research.

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