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## Compactness Property of Fuzzy Soft Metric Space and Fuzzy Soft Continuous Function

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### Abstract

The theories of metric spaces and fuzzy metric spaces are crucial topics in mathematics.

Compactness is one of the most important and fundamental properties that have been widely used in Functional Analysis. In this paper, the definition of compact fuzzy soft metric space is introduced and some of its important theorems are investigated. Also, sequentially compact fuzzy soft metric space and locally compact fuzzy soft metric space are defined and the relationships between them are studied. Moreover, the relationships between each of the previous two concepts and several other known concepts are investigated separately. Besides, the compact fuzzy soft continuous functions are studied and some essential theorems are proved.

**Keywords:** Fuzzy Soft Metric Space, Compact fuzzy soft metric space, Sequentially compact fuzzy soft metric space, Fuzzy soft continuous function

### خاصية التراص للفضاء المتري الناعم الضبابي والدالة المستمرة الناعمة الضبابية

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### الخلاصة

نظرية الفضاءات المترية والفضاءات المترية الضبابية مواضيع اساسية في الرياضيات. التراص هي واحدة من اكثر الصفات المهمة والاساسية التي لها استخدام واسع في التحليل الدالي. في هذا البحث، تم تقديم لأول مرة تعريف الفضاء المتري الناعم الضبابي المتراص وتم تحري بعض المبرهنات المهمة على الفضاء المتري الناعم الضبابي المتراص. ايضا تم تعريف الفضاء المتري الناعم المتراص بالتتابع و الفضاء المتري الناعم المتراص محليا وتم دراسة العلاقات بين هذين المفهومين. بالإضافة الى ذلك، تم دراسة العلاقة بين كل من المفهومين السابقين وعدد من المفاهيم الأخرى المعروفة بشكل منفصل. الى جانب ذلك، تم دراسة الدوال المستمرة الناعمة الضبابية المتراصة وتم برهان بعض المبرهنات الاساسية.

### 1. Introduction

Most of the problems in medical science, engineering, economics, and so on, have different doubts. The problems in method identification include properties that are basically non-probabilistic in nature. To respond to such a situation, a new trend of mathematics was created by Zadeh [1], that is named the fuzzy set theory, to process concepts from the fuzzy perspective using a specific membership. The soft set theory is one of the branches of mathematics, which aims to describe phenomena and concepts of ambiguous, vague, undefined, and imprecise meaning. It has rich potential for applications in several directions. The soft set theory is applicable where there is no clearly defined

mathematical model. In 1999, Molodtsov [2] introduced the definition of soft sets in the following way: a pair  $(F, A)$  is said to be a soft set defined on a universe set  $U$  such that  $F: A \rightarrow \mathcal{P}(U)$  is a mapping,  $A$  is a subset of the parameter  $E$ , and  $\mathcal{P}(U)$  is the power set of  $U$ .

After Molodtsov's study, many authors studied soft set theory and its applications in various fields [3-18].

A fuzzy generalization of the soft sets was initiated for them to be fuzzy soft sets. Soft sets and fuzzy sets were combined to construct the fuzzy soft sets notion [19]. A pair  $(F, A)$  is said to be a fuzzy soft set defined on a universe set  $U$  such that  $F: A \rightarrow I^U$  is a mapping,  $A$  is a subset of the parameter set  $E$ , and  $I^U$  is the family of all fuzzy subsets of  $U$ .

Roy and Samanta [20] applied the definition of the fuzzy soft set given in [19] on topological spaces, i.e. depending on a fuzzy soft set, a topology will be built. Also, several significant theorems were proved. Depending on basic information of the fuzzy soft set presented in [19] and [20], the concept of fuzzy soft metric space was constructed by Beaula and Raja [21] and many concepts were given, such as the Cauchy sequence, fuzzy soft (open and closed) balls, convergence, and boundedness. Furthermore, some fundamental theorems were proved.

Analogously to the ideas in [2] and [19], Varol and Aygun [22] were able to present a new definition of the fuzzy soft set. A pair  $(f, \check{A})$  is called a fuzzy soft set over a universe set  $\mathcal{S}$ , where  $f: \check{A} \rightarrow I^{\mathcal{S}}$ ,  $\check{A}$  is a fuzzy set and  $I^{\mathcal{S}}$  is the collection of all fuzzy sets on  $\mathcal{S}$  such that for each  $a \in \check{A}$  implies  $f(a) = f_a \in I^{\mathcal{S}}$ . As an abbreviation, a fuzzy soft set is denoted by  $f_{\check{A}} = \{f_a: a \in \check{A}\}$ . A fuzzy soft set  $f_{\check{A}}$  on a universe set  $\mathcal{S}$  is a mapping from the parameter set  $\mathcal{P}$  to  $I^{\mathcal{S}}$ ; that is  $f_{\check{A}}: \mathcal{P} \rightarrow I^{\mathcal{S}}$ , where  $f_{\check{A}}(x) \neq 0_{\mathcal{S}}$  if  $x \in \check{A} \subseteq \mathcal{P}$ , and  $f_{\check{A}}(x) = 0_{\mathcal{S}}$  if  $x \notin \check{A}$ , where  $0_{\mathcal{S}}$  the empty is fuzzy set on  $\mathcal{S}$ . The family of all fuzzy soft sets over  $\mathcal{S}$  will be denoted by  $F(\mathcal{S}, \mathcal{P})$ . Kider [23] relied on the notions of the fuzzy soft set presented in [22] to provide a new definition of fuzzy soft metric space.

It is worth mentioning that the fuzzy soft set definition given in [19] is different from the fuzzy soft set given in [22]. Consequently, the notion of the fuzzy soft metric space given in [23] is different from the fuzzy soft metric space given in [21] and also different from other definitions of fuzzy soft metric space presented in [24] and [25].

The first aim of this paper is to define the compactness property of the fuzzy soft metric space given in [23] and to investigate some fundamental theorems about compact fuzzy soft metric space. The second aim is to introduce the concepts of sequentially compact and locally compact fuzzy soft metric spaces and to establish their properties. Finally, compact fuzzy soft continuous functions are studied and some of their properties are examined in the fuzzy soft metric space.

The structure of the paper is as in the following. In Section 2, some properties and basic concepts of the fuzzy soft metric space are given. Section 3 is devoted to introducing the concept of compact, sequentially compact, and locally compact fuzzy soft metric spaces and establishing main properties related to these concepts in the fuzzy soft metric space. The compact fuzzy soft continuous functions are introduced in Section 4, where some important properties of the given subject are also investigated. The conclusion of this work is given in Section 5.

## 2. Preliminaries

This section gives some main important properties of fuzzy length space on a fuzzy set.

In 2012, Varol and Aygun [22] defined the fuzzy product between two fuzzy soft sets as below.

**Definition 2.1 [22]:** Suppose that  $f_{\check{A}} \in F(\mathcal{S}, \mathcal{P})$  and  $h_{\check{B}} \in F(\mathbb{V}, E)$  be two fuzzy soft sets, then the fuzzy product  $f_{\check{A}} \times h_{\check{B}}$  is denoted by  $(f \times h)_{\check{A} \times \check{B}}$ , where  $(f \times h)_{\check{A} \times \check{B}}(a, b) = f_{\check{A}}(a) \times h_{\check{B}}(b) \in I^{\mathbb{U} \times \mathbb{V}}$  for all  $(a, b) \in \check{A} \times \check{B}$  and for all  $(s, v) \in \mathcal{S} \times \mathbb{V}$ ,  $[(f_{\check{A}}(a) \times h_{\check{B}}(b))](s, v) = f_{\check{A}}(a)(s) \wedge h_{\check{B}}(b)(v)$ . According to this definition, the fuzzy soft set  $f_{\check{A}} \times h_{\check{B}}$  is a fuzzy soft set over  $\mathcal{S} \times \mathbb{V}$  and  $\mathcal{P} \times E$  is its universal parameter.

**Definition 2.2 [26]:** Suppose that  $\widetilde{\odot}: [0,1] \times [0,1] \rightarrow [0,1]$  is a binary operation. Then  $\widetilde{\odot}$  is called continuous t-norm if  $\widetilde{\odot}$  satisfies:

- (1)  $a \widetilde{\odot} b = b \widetilde{\odot} a$ , for each  $a, b \in [0,1]$ .
- (2)  $(a \widetilde{\odot} b) \widetilde{\odot} c = a \widetilde{\odot} (b \widetilde{\odot} c)$ ,  $c \in [0,1]$ .
- (3)  $a \widetilde{\odot} 1 = a$ .
- (4)  $a \widetilde{\odot} b \leq c \widetilde{\odot} d$  whenever  $a \leq c$  and  $b \leq d$ , for each  $a, b, c, d \in [0,1]$ .

**Remark 2.3 [26]**

(1) For any  $a, b \in [0,1]$  with  $a > b$ , there is  $c \in [0,1]$ , such that  $a \widetilde{\odot} c \geq b$ .

(2) For some  $d \in [0,1]$ , there is  $q \in [0,1]$ , such that  $q \widetilde{\odot} q \geq d$ .

Kider [23] presented the notion of fuzzy soft metric space as follows.

**Definition 2.4 [23]:** Let  $\mathcal{S}$  be a universe set and let  $F(\mathcal{S}, \mathcal{P})$  be the family of all soft fuzzy sets over  $\mathcal{S}$ .

Let  $f_{\check{A}}, h_{\check{B}} \in F(\mathcal{S}, \mathcal{P})$  and put  $g_{\check{A} \times \check{B}} = (f \times h)_{\check{A} \times \check{B}} = f_{\check{A}} \times h_{\check{B}}$ . Let  $\widetilde{\odot}$  be a continuous t-norm, then  $g_{\check{A} \times \check{B}}$  is a fuzzy soft metric

on  $\mathcal{S} \times \mathcal{S}$  if for all  $a \in \check{A}, b \in \check{B}$ .

(S1)  $g_{\check{A} \times \check{B}}(a, b)[s, w] > 0$  for all  $s, w \in \mathcal{S}$ .

(S2)  $g_{\check{A} \times \check{B}}(a, b)[s, w] = 1 \iff s = w$ .

(S3)  $g_{\check{A} \times \check{B}}(a, b)[s, w] = g_{\check{A} \times \check{B}}(a, b)[w, s]$  for all  $s, w \in \mathcal{S}$ .

(S4)  $g_{\check{A} \times \check{B}}(a, b)[s, z] \geq g_{\check{A} \times \check{B}}(a, b)[s, w] \widetilde{\odot} g_{\check{A} \times \check{B}}(a, b)[w, z]$  for all  $s, w, z \in \mathcal{S}$ .

(S5)  $g_{\check{A} \times \check{B}}(a, b)[s, w]$  is a continuous function for all  $s, w \in \mathcal{S}$ .

Then, the triple  $(\mathcal{S}, g_{\check{A} \times \check{B}}(a, b), \widetilde{\odot})$  is called a fuzzy soft metric space.

**Definition 2.5 [23]:** Let  $(\mathcal{S}, g_{\check{A} \times \check{B}}(a, b), \widetilde{\odot})$  be a fuzzy soft metric space for all  $a \in \check{A}, b \in \check{B}$ . The soft fuzzy open ball  $\mathcal{SB}(s, r)$  with center  $s \in \mathcal{S}$  and radius  $r \in [0,1]$  is defined by  $\mathcal{SB}(s, r) = \{s \in \mathcal{S} : g_{\check{A} \times \check{B}}(a, b)[s, w] > (1 - r)\}$ .

**3. Compact Fuzzy Soft Metric Space**

In this section, the notion of compact fuzzy soft metric space is introduced and some important theorems about it are proved. Furthermore, the concepts of sequentially compact and locally compact fuzzy soft metric spaces are introduced and some properties about them in the fuzzy soft metric space will be investigated.

**Definition 3.1:** Let  $(\mathcal{S}, g_{\check{A} \times \check{B}}(a, b), \widetilde{\odot})$  be fuzzy soft metric space with  $\mathcal{X} \subseteq \mathcal{S}$ . Suppose that the family of all fuzzy soft open sets in  $\mathcal{S}$  is denoted by  $\hat{\mathcal{C}}$  in the case that  $\mathcal{X} \subseteq \bigcup_{C \in \hat{\mathcal{C}}} C$ , (i.e. there exists a set  $C \in \hat{\mathcal{C}}$  for any  $x \in \mathcal{X}$  in which  $x \in C$ ), then  $\hat{\mathcal{C}}$  is called open covering (or an open cover) of  $\mathcal{X}$ . A finite sub-covering (or a finite subcover) of  $\mathcal{X}$  is a finite subfamily of  $\hat{\mathcal{C}}$  that itself represents a cover.

**Definition 3.2:** Let  $(\mathcal{S}, g_{\check{A} \times \check{B}}(a, b), \widetilde{\odot})$  be a fuzzy soft metric space. If there exists a finite sub-covering for every open covering  $\hat{\mathcal{C}}$  of  $\mathcal{S}$ , this means that there exists a finite subfamily  $\{C_1, C_2, C_3, \dots, C_n\} \subseteq \hat{\mathcal{C}}$  such that  $\mathcal{S} \subseteq \bigcup_{i=1}^n C_i$ , then  $(\mathcal{S}, g_{\check{A} \times \check{B}}(a, b), \widetilde{\odot})$  is called a compact space.

**Definition 3.3:** A compact subset  $\mathcal{X}$  of fuzzy soft metric space  $(\mathcal{S}, g_{\check{A} \times \check{B}}(a, b), \widetilde{\odot})$  is a compact subset with the fuzzy soft metric generated by the function  $g_{\check{A} \times \check{B}}$ .

The following example explains the above definition.

**Example 3.4:**

Let  $(0, 1)$  be an interval in the fuzzy soft metric space  $(\mathbb{R}, g_{\check{A} \times \check{B}}(a, b), \widetilde{\odot})$  where  $g_{\check{A} \times \check{B}}(a, b)[s, w] = \frac{1}{\exp[|s-w|]}$  and  $\alpha \widetilde{\odot} \beta = \alpha\beta$  for all  $\alpha, \beta \in [0,1]$  is not compact, since the collection  $\hat{\mathcal{C}}_n = \left\{ \left( \frac{1}{n}, 1 \right) : n = 2, 3, \dots \right\}$  is an open covering for  $(0, 1)$  but has no finite sub covering for  $(0, 1)$ .

**Remark 3.5:** If  $\mathcal{X}$  is a finite subset of a fuzzy soft metric space  $(\mathcal{S}, g_{\check{A} \times \check{B}}(a, b), \widetilde{\odot})$ , ( $\mathcal{S}$  is finite), then  $\mathcal{X}$  is a compact subset.

The following two definitions of the  $\mu$ -fuzzy net set and fuzzy totally bounded set concepts in a fuzzy soft metric space are introduced.

**Definition 3.6:** Let  $(\mathcal{S}, g_{\check{A} \times \check{B}}(a, b), \widetilde{\odot})$  be a fuzzy soft metric space. A subset  $\mathcal{X}$  of  $\mathcal{S}$  is called an  $\mu$ -fuzzy net for  $\mathcal{S}$  where  $0 < \mu < 1$ , if  $\mathcal{X}$  is finite and if the soft fuzzy open ball  $\mathcal{SB}(s, r)$  with center  $s \in \mathcal{X}$  and radius  $r \in [0,1]$  covers  $\mathcal{S}$ .

**Definition 3.7:** Let  $(\mathcal{S}, g_{\check{A} \times \check{B}}(a, b), \widetilde{\odot})$  be a fuzzy soft metric space and  $\mathcal{X}$  be a subset of  $\mathcal{S}$ , then  $\mathcal{X}$  is called fuzzy totally bounded (or fuzzy pre-compact) if for each  $0 < \mu < 1$ , there exist points  $\omega_1, \omega_2, \omega_3, \dots, \omega_n$  such that the set  $\{\omega_1, \omega_2, \omega_3, \dots, \omega_n\} \subset \mathcal{X}$  whenever  $s$  in  $\mathcal{S}$ ,  $g_{\check{A} \times \check{B}}(a, b)[s, \omega_i] > (1 - \mu)$  for some  $\omega_i \in \{\omega_1, \omega_2, \omega_3, \dots, \omega_n\}$ . This set of points  $\{\omega_1, \omega_2, \omega_3, \dots, \omega_n\}$  is called  $\mu$ -fuzzy net.

The following proposition gives the first direction of the relationship between fuzzy bounded and fuzzy totally bounded fuzzy soft metric space.

**Proposition 3.8:** If a fuzzy soft metric space  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L}), \widetilde{\odot})$  is fuzzy totally bounded then  $\mathcal{S}$  is fuzzy bounded.

**Proof:**

Since  $\mathcal{S}$  is a fuzzy totally bounded, then for each  $0 < \mu < 1$ , there is a finite  $\mu$ -fuzzy net, say  $W$ . Since  $W$  is a finite set of points, then  $g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L})[W] = \sup\{g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L})[\omega_1, \omega_2]: \omega_1, \omega_2 \in W\}$ . Let  $s_1$  and  $s_2$  be any two points in  $\mathcal{S}$ . There exist points  $\omega_1, \omega_2 \in W$  such that  $g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L})[s_1, \omega_1] > 1 - \mu$  and  $g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L})[s_2, \omega_2] > 1 - \mu$ . Now for  $g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L})[W]$  and  $0 < \mu < 1$ , there is  $1 - \gamma$ , where  $0 < \gamma < 1$  such that  $g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L})[W] \widetilde{\odot} (1 - \mu) \widetilde{\odot} (1 - \mu) \geq (1 - \gamma)$ . It follows that

$$\begin{aligned} &g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L})[s_1, s_2] \\ &\geq g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L})[s_2, \omega_2] \widetilde{\odot} g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L})[\omega_1, \omega_2] \widetilde{\odot} g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L})[\omega_1, s_1] \\ &\geq (1 - \mu) \widetilde{\odot} g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L})[W] \widetilde{\odot} (1 - \mu) \geq (1 - \gamma). \end{aligned}$$

So,  $g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L})[\mathcal{S}] = \sup\{g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L})[s_1, s_2]: s_1, s_2 \in \mathcal{S}\}$ . This shows that  $\mathcal{S}$  is fuzzy bounded.

Recall that a fuzzy soft Cauchy sequence is defined as a sequence  $(s_n)$  contained in  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L}), \widetilde{\odot})$ , such that for each  $0 < (1 - r) < 1$  there exists a positive number  $N$  satisfying  $g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L})[s_m, s_n] > (1 - r)$  for all  $m, n \geq N$  [23].

**Theorem 3.9:** Let  $\mathcal{X}$  be a subset of a fuzzy soft metric space  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L}), \widetilde{\odot})$ . Every sequence in  $\mathcal{X}$  includes a Cauchy subsequence if and only if  $\mathcal{X}$  is fuzzy totally bounded.

**Proof:**

Suppose that every sequence  $(s_n)$  in  $\mathcal{X}$  contains a subsequence  $(s_{n_k})$  which is a Cauchy. Let  $0 < r < 1$  and let  $x_1 \in \mathcal{X}$ . If  $\mathcal{X} - SB(x_1, r) = \emptyset$ , it will be obtained a  $\mu$ -fuzzy net, say  $\{x_1\}$ . Now, choose  $x_2 \in SB(x_1, r)$ . If  $\mathcal{X} - [SB(x_1, r) \cup SB(x_2, r)] = \emptyset$  it will be obtained a  $\mu$ -fuzzy net, say  $\{x_1, x_2\}$ . After finite steps, this process will stop. If not, an infinite sequence  $(s_n)$  with property that  $g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L})[s_n, s_m] \leq (1 - r), n \neq m$  is obtained. That is,  $(s_n)$  has no Cauchy subsequence, which is a contradiction.

For the converse, let  $\mathcal{X}$  be a fuzzy totally bounded, hence  $\mathcal{X}$  has a finite  $\mu$ -fuzzy net for any  $0 < \mu < 1$ . We consider  $(s_n)$  to be a sequence in  $\mathcal{X}$ . We choose a finite 0.5-fuzzy net in  $\mathcal{X}$ . Therefore, infinitely many elements, say the subsequence  $(s_{n_1})$ , of the sequence  $(s_n)$  are contained in one of the soft fuzzy open balls of radius 0.5, with the center in the 0.5-fuzzy net. We choose a finite 0.25-fuzzy net in  $\mathcal{X}$ . Therefore, infinitely many elements, say the subsequence  $(s_{n_2})$ , of the sequence  $(s_{n_1})$  are contained in one of the soft fuzzy open balls of radius 0.25, with the center in the 0.25-fuzzy net. By continuing this path, a sequence which its elements are also sequences are obtained, and each sequence is a subsequence of the previous one. Thus, the soft fuzzy open ball of radius  $\frac{1}{2^k}$  with the center in the  $\frac{1}{2^k}$ -fuzzy net contains  $(s_{n_k})$ . Now,  $(s_{n_n})$  is a subsequence of the sequence  $(s_n)$ . We choose  $M$  to be very large that  $(1 - \frac{1}{2^n}) \widetilde{\odot} (1 - \frac{1}{2^{n+1}}) \widetilde{\odot} \dots \widetilde{\odot} (1 - \frac{1}{2^{k-1}}) > (1 - \mu)$ . Then, for  $n > k > M$ :

$$\begin{aligned} g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L})[s_{n_n}, s_{k_k}] &\geq g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L})[s_{n_n}, s_{n_{n+1}}] \widetilde{\odot} g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L})[s_{n_{n+1}}, s_{n_{n+2}}] \widetilde{\odot} \\ &\quad g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L})[s_{k_{k-1}}, s_{k_k}] \\ &\geq (1 - \frac{1}{2^n}) \widetilde{\odot} (1 - \frac{1}{2^{n+1}}) \widetilde{\odot} \dots \widetilde{\odot} (1 - \frac{1}{2^{k-1}}) \\ &> (1 - \mu) \end{aligned}$$

Hence  $(s_{n_n})$  is Cauchy.

The following result gives a condition such that the space is fuzzy totally bounded and complete.

**Theorem 3.10:** If  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L}), \widetilde{\odot})$  is a compact fuzzy soft metric space, then  $\mathcal{S}$  is fuzzy totally bounded and complete.

**Proof:**

Let  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L}), \widetilde{\odot})$  be compact fuzzy soft metric space. For any given  $0 < \mu < 1$ ,  $\mathcal{S}$  is covered by the collection of all soft fuzzy open balls  $SB(s, \mu), s \in \mathcal{S}$ . Since  $\mathcal{S}$  is compact, then  $SB(s, \mu)$  includes a finite subcover, which implies that a finite number of soft fuzzy open balls of radius  $r$  covers  $\mathcal{S}$  for  $0 < \mu < 1$ , i.e. there is a finite  $\mu$ -fuzzy net of points. So,  $\mathcal{S}$  is fuzzy totally bounded.

Now, to prove the completeness of  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\mathcal{O}})$ , suppose that it is not compact, which means that there exists a Cauchy sequence  $(s_n)$  in  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\mathcal{O}})$  that does not have a fuzzy soft limit point in  $\mathcal{S}$ . Let  $s \in \mathcal{S}$ . Since  $(s_n)$  does not converge to  $s$ , there exists  $0 < \sigma < 1$  such that  $g_{\tilde{A} \times \tilde{B}}(a, b)[s_n, s] \leq (1 - \sigma)$ . Since  $(s_n)$  is Cauchy, there exists an integer  $N$  such that  $n, m \geq N$ , which implies that  $g_{\tilde{A} \times \tilde{B}}(a, b)[s_n, s_m] > (1 - \sigma)$ . We take  $m \geq N$ , for which  $g_{\tilde{A} \times \tilde{B}}(a, b)[s_n, s] > (1 - \sigma)$ . So, for only finitely many values of  $n$ , the soft fuzzy open ball  $\mathcal{SB}(s, \mu)$  contains  $s_n$ . In this way, for each  $s \in \mathcal{S}$ , it can be associated with a soft fuzzy open ball  $\mathcal{SB}(s, \mu(s))$ , where  $0 < \mu(s) < 1$  depends on  $s$ , and the soft fuzzy open ball  $\mathcal{SB}(s, \mu(s))$  contains  $s_n$  for only finitely many values of  $n$ . It is clear that  $\mathcal{S} = \bigcup_{s \in \mathcal{S}} \mathcal{SB}(s, \mu(s))$ , which means that  $\mathcal{S}$  is covered by  $\{\mathcal{SB}(s, \mu(s)) : s \in \mathcal{S}\}$ . Now, there exists a finite sub-covering  $\mathcal{SB}(s_i, \mu(s_i)), i = 1, 2, 3, \dots, n$  (because  $\mathcal{S}$  is compact). Consequently, the soft fuzzy open balls are in the finite sub-covering because, for only a finite number of values of  $n$ , each soft fuzzy open ball contains  $s_n$ . Thus, the points  $s_n$  must be contained in  $\mathcal{S}$  for only a finite number of values of  $n$ , which is a contradiction. Therefore,  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\mathcal{O}})$  must be complete.

For the fuzzy soft metric space, the sequential compactness property is introduced in the following definition.

**Definition 3.11:** A fuzzy soft metric space  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\mathcal{O}})$  is called sequentially compact fuzzy soft metric space if every sequence in  $\mathcal{S}$  has a convergent subsequence.

Consider a fuzzy soft metric space  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\mathcal{O}})$ . A sequence  $(s_n)$  in  $\mathcal{S}$  is said to be fuzzy soft converges to  $s \in \mathcal{S}$  if for each  $\sigma > 0$  there exist  $N$  such that  $g_{\tilde{A} \times \tilde{B}}(a, b)[s_n, s] > (1 - \sigma)$  for each  $n \geq N$ . This is written as  $\lim_{n \rightarrow \infty} s_n = s$  or simply written as  $s_n \rightarrow^s s$ . The limit point  $s$  is called the fuzzy soft limit of  $(s_n)$  [23].

The next characterization is provided for the fuzzy soft metric space.

**Theorem 3.12:** A fuzzy soft metric space  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\mathcal{O}})$  is a sequentially compact fuzzy soft metric space if and only if every infinite subset of  $\mathcal{S}$  has at least one fuzzy soft limit point.

**Proof:**

Let  $\mathcal{X}$  be an infinite subset of  $\mathcal{S}$  and let  $(s_n)$  be a sequence in  $\mathcal{X}$ . Thus,  $(s_n)$  includes a convergent subsequence and the limit is a fuzzy soft limit point of  $\mathcal{X}$  because  $\mathcal{S}$  is sequentially compact.

Conversely, consider  $(s_n)$  to be a sequence in  $\mathcal{S}$ . Consider that the set  $\{s_1, s_2, \dots\}$  will be finite which implies that one of the set's points, named  $s_{i_0}$ , satisfies  $s_{i_0} = s_j, j \in N$ . Hence the sequence  $(s_{i_0})$  is a subsequence of  $(s_n)$  that converges to the same point  $s_{i_0}$ . Now, consider that the set  $\{s_1, s_2, \dots\}$  is infinite. Then by assumption, it has at least one fuzzy soft limit point  $s \in \mathcal{S}$ . Let  $n_1 \in N$  with  $g_{\tilde{A} \times \tilde{B}}(a, b)[s_{n_1}, s] > 0$ . Let  $n_{k+1} \in N$  with  $n_{k+1} > n_k$  and  $g_{\tilde{A} \times \tilde{B}}(a, b)[s_{n_{k+1}}, s] > 1 - \frac{1}{k+1}$ . Then, the sequence  $(s_{n_k})$  converges to  $s$ .

The following theorem shows the relationship between compact and sequentially compact fuzzy soft metric spaces.

**Theorem 3.13:** The followings are equivalent for a fuzzy soft metric space  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\mathcal{O}})$ :

- (i)  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\mathcal{O}})$  is compact.
- (ii)  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\mathcal{O}})$  is sequentially compact.

**Proof(i)  $\rightarrow$  (ii):**

Let  $\mathcal{S}$  be compact, then by Theorem 3.10,  $\mathcal{S}$  is fuzzy totally bounded and complete. Suppose that  $(s_n)$  be any sequence in  $\mathcal{S}$ . Since  $\mathcal{S}$  is totally fuzzy bounded, then by using Theorem 3.9, the sequence  $(s_n)$  contains a Cauchy subsequence, say  $(s_{n_k})$ . So,  $(s_{n_k})$  converges to  $s \in \mathcal{S}$ , since  $\mathcal{S}$  is complete. Consequently, each sequence  $(s_n)$  in  $\mathcal{S}$  includes a convergent subsequence  $(s_{n_k})$ .

**proof(ii)  $\rightarrow$  (i):**

Suppose that  $\mathcal{S}$  is sequentially compact, then this means that each sequence  $(s_n)$  in  $\mathcal{S}$  contains a convergent subsequence  $(s_{n_k})$ . Now, by using Theorem 3.9, we obtain that  $\mathcal{S}$  is totally fuzzy bounded. It remains to prove that  $\mathcal{S}$  is complete. Let  $(s_n)$  be a Cauchy sequence in  $\mathcal{S}$ . By assumption,  $(s_n)$  includes a subsequence  $(s_{n_k})$  that converges to a point  $s \in \mathcal{S}$ . Now, to prove that  $(s_n)$  converges to  $s$ . Let  $0 < \gamma < 1$  be given by Remark (2.3), then there is  $0 < \mu < 1$  such that  $(1 - \mu) \widetilde{\mathcal{O}}(1 -$

$\mu) > (1 - \gamma)$ . Now,  $(s_{n_k})$  converges to  $s$ ; that is, there exists  $N1$  such that  $g_{\tilde{A} \times \tilde{B}}(a, b)[s_{n_k}, s] > (1 - \mu)$  for all  $n_k \geq N1$ . Since the sequence  $(s_n)$  is Cauchy, then there exists  $N2$  such that  $g_{\tilde{A} \times \tilde{B}}(a, b)[s_n, s_m] > (1 - \mu)$  for all  $m, n \geq N2$ . Let  $N = \min\{N1, N2\}$ . Then,

$$\begin{aligned} g_{\tilde{A} \times \tilde{B}}(a, b)[s_n, s] &\geq g_{\tilde{A} \times \tilde{B}}(a, b)[s_n, s_{n_k}] \widetilde{\odot} g_{\tilde{A} \times \tilde{B}}(a, b)[s_{n_k}, s] \\ &> (1 - \mu) \widetilde{\odot} (1 - \mu) > (1 - \gamma) \end{aligned}$$

for all  $n \geq N$ . This completes the proof.

In a fuzzy soft metric space  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\odot})$ , a set  $\mathcal{X}$  is a fuzzy soft neighbourhood of a point  $s$  if there exists a soft fuzzy open ball  $SB(s, r)$  with centre  $r \in [0, 1]$  and radius  $s$ , such that  $SB(s, r) = \{s \in \mathcal{S} : g_{\tilde{A} \times \tilde{B}}(a, b)[s, w] > (1 - r)\}$  is contained in  $\mathcal{X}$ .

**Definition 3.14.** A fuzzy soft metric space  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\odot})$  is said to be locally compact fuzzy soft metric space if each  $s \in \mathcal{S}$  has a compact fuzzy soft neighborhood.

An immediate relationship between compact fuzzy soft metric space and locally compact fuzzy soft metric space is given in the next result.

**Theorem 3.15:** Every compact fuzzy soft metric space  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\odot})$  is a locally compact fuzzy soft metric space.

**Proof:**

Let  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\odot})$  be a compact fuzzy soft metric space. Then by Definition 3.2, every open cover  $\hat{C}_i$  has finite subcover, which means that  $\mathcal{S} \subseteq \bigcup_{i=1}^n \hat{C}_i$ . Now, for each  $s \in \mathcal{S}$  implies  $SB(s, r) \subseteq \bigcup_{i=1}^n \hat{C}_i$ , which means that each  $s \in \mathcal{S}$  has a compact  $SB(s, r)$ . Hence  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\odot})$  is a locally compact fuzzy soft metric space.

The previous result is used to prove the following proposition.

**Proposition 3.16:** If  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\odot})$  is sequentially compact fuzzy soft metric space, then  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\odot})$  is a locally compact fuzzy soft metric space.

**Proof:**

Since  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\odot})$  is sequentially compact fuzzy soft metric space, then by Proposition 3.13,  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\odot})$  is compact and by Theorem 3.15, it will be inferred that  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\odot})$  is a locally compact fuzzy soft metric space.

#### 4. Compact Fuzzy Soft Continuous Function

In this section, the compactness of fuzzy soft continuous function is discussed. The definition of the fuzzy soft uniformly continuous function is introduced initially.

The function  $T$  from a fuzzy soft metric space  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\odot})$  and another fuzzy soft metric space  $(\mathbb{V}, k_{\tilde{L} \times \tilde{M}}(\ell, m), \widetilde{\odot})$  is known as fuzzy soft continuous at a point  $s_0 \in \mathcal{S}$ , if for every  $0 < (1 - \mu) < 1$  there exists  $0 < (1 - \gamma) < 1$ , such that  $g_{\tilde{A} \times \tilde{B}}(a, b)[s, s_0] > (1 - \gamma)$ , whenever  $k_{\tilde{L} \times \tilde{M}}(\ell, m)[T(s), T(s_0)] > (1 - \mu)$  for all  $s \in \mathcal{S}$ .

In the following, the concept of a fuzzy soft uniformly continuous function is introduced.

**Definition 4.1:**

Let  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\odot})$  and  $(\mathbb{V}, k_{\tilde{L} \times \tilde{M}}(\ell, m), \widetilde{\odot})$  be two fuzzy soft metric spaces. The function  $T: \mathcal{S} \rightarrow \mathbb{V}$  is known as fuzzy soft uniformly continuous on  $\mathcal{S}$ , if for every  $0 < (1 - \mu) < 1$  there is some  $0 < (1 - \gamma) < 1$  such that  $k_{\tilde{L} \times \tilde{M}}(\ell, m)[T(s_1), T(s_2)] > (1 - \mu)$  whenever  $g_{\tilde{A} \times \tilde{B}}(a, b)[s_1, s_2] > (1 - \gamma)$ .

The following theorem proves the compactness of the image of a function  $T$  when  $T$  is fuzzy soft continuous under some conditions. First, the notion of the fuzzy soft open set is needed in a fuzzy soft metric space  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\odot})$ . A subset  $\mathcal{X}$  of  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\odot})$  is fuzzy soft open, if for all  $x \in \mathcal{X}$  there exists  $r \in [0, 1]$  such that  $SB(x, r) \subseteq \mathcal{X}$  [23].

**Theorem 4.2:** Let  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, b), \widetilde{\odot})$  be a compact fuzzy soft metric space and  $(\mathbb{V}, k_{\tilde{L} \times \tilde{M}}(\ell, m), \widetilde{\odot})$  be a fuzzy soft metric space. If  $T: \mathcal{S} \rightarrow \mathbb{V}$  is a fuzzy soft continuous function, then  $T(\mathcal{S})$  is compact.

**Proof:**

Let  $\{\hat{C}_\alpha : \alpha \in \Lambda\}$  be an open covering of  $T(\mathcal{S})$ . It is required to prove that  $\{\hat{C}_\alpha : \alpha \in \Lambda\}$  contains a finite sub-covering. Since  $T$  is fuzzy soft continuous then by [15, Theorem 4.4],  $T^{-1}(\hat{C}_\alpha)$  is open in  $\mathcal{S}$ . Moreover  $\{T^{-1}(\hat{C}_\alpha) : \alpha \in \Lambda\}$  is an open covering of  $\mathcal{S}$ . Since  $\mathcal{S}$  is compact, then there exists  $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$  in  $\Lambda$  such that  $\mathcal{S} = \bigcup_{j=1}^n T^{-1}(\hat{C}_{\alpha_j})$ .

Now,  $T(\mathcal{S}) = T\left(\bigcup_{j=1}^n T^{-1}(\hat{C}_{\alpha_j})\right) = \bigcup_{j=1}^n T(T^{-1}(\hat{C}_{\alpha_j})) \subseteq \bigcup_{j=1}^n \hat{C}_{\alpha_j}$ . So,  $\{\hat{C}_{\alpha_j}: j = 1, 2, \dots, n\}$  is a finite sub covering of  $T(\mathcal{S})$ . Consequently,  $T(\mathcal{S})$  is compact.

A Lebesgue number of the covering in the fuzzy soft metric space is given as follows.

**Definition 4.3:** Let  $\{\hat{C}_{\alpha}: \alpha \in \Lambda\}$  be an open covering of the fuzzy soft metric space  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L}), \widetilde{\odot})$ . Any number  $0 < \rho < 1$ , such that for each  $s_2 \in \mathcal{S}$  there exists  $\alpha \in \Lambda$  (dependent on  $s$ ), for which  $\mathcal{SB}(s, \rho) \subset \hat{C}_{\alpha}$  is called a Lebesgue number of the covering  $\{\hat{C}_{\alpha}: \alpha \in \Lambda\}$ .

A fuzzy soft continuous function is given to be fuzzy soft uniformly continuous in the following.

**Theorem 4.4:** Let  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L}), \widetilde{\odot})$  be a sequentially compact fuzzy soft metric space and assume that  $(\mathbb{V}, k_{\tilde{L} \times \tilde{M}}(\ell, \mathcal{M}), \widetilde{\odot})$  is a fuzzy soft metric space. If  $T: \mathcal{S} \rightarrow \mathbb{V}$  is a fuzzy soft continuous function, then  $T$  is fuzzy soft uniformly continuous.

**Proof:**

Let  $0 < (1 - \sigma) < 1$ . For each  $s_2 \in \mathcal{S}$ , there exists  $0 < (1 - \gamma) < 1$ , such that  $k_{\tilde{L} \times \tilde{M}}(\ell, \mathcal{M})[T(s_1), T(s_2)] > (1 - \sigma)$  for all  $s_1 \in \mathcal{S}$ , with  $g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L})[s_1, s_2] > (1 - \gamma)$ . Then,  $\{\mathcal{SB}(s_2, \gamma(s_2)): s_2 \in \mathcal{S}\}$  is an open cover of  $\mathcal{S}$ , so by Definition 3.5, there exists  $0 < \rho < 1$ , such that for every  $s_1 \in \mathcal{S}$  there exists  $s_2 \in \mathcal{S}$  with  $\mathcal{SB}(s_1, \rho) \subset \mathcal{SB}(s_2, \gamma(s_2))$ . Hence,  $g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L})[s_1, s] > (1 - \rho)$ , which implies that  $s_1, s \in \mathcal{SB}(s_2, \gamma(s_2))$ . So,

$$k_{\tilde{L} \times \tilde{M}}(\ell, \mathcal{M})[T(s_1), T(s)] \geq k_{\tilde{L} \times \tilde{M}}(\ell, \mathcal{M})[T(s_1), T(s_2)] \widetilde{\odot} k_{\tilde{L} \times \tilde{M}}(\ell, \mathcal{M})[T(s_2), T(s)] \\ k_{\tilde{L} \times \tilde{M}}(\ell, \mathcal{M})[T(s_1), T(s)] > (1 - \sigma) \widetilde{\odot} (1 - \sigma)$$

By Remark 2.3,  $(1 - \sigma) \widetilde{\odot} (1 - \sigma) > (1 - \mu)$ , then  $k_{\tilde{L} \times \tilde{M}}(\ell, \mathcal{M})[T(s_1), T(s)] > (1 - \mu)$ , which shows that  $T$  is fuzzy soft uniformly continuous.

## 5. Conclusions

This paper introduced many concepts of the fuzzy soft metric space, such as compactness, total boundedness, sequential compactness, and local compactness. The basic properties of the given concepts of the fuzzy soft metric space are examined and essential theorems are proved. Every compact fuzzy soft metric space  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L}), \widetilde{\odot})$  is proved to be a locally compact fuzzy soft metric space. Furthermore, a fuzzy soft metric space  $(\mathcal{S}, g_{\tilde{A} \times \tilde{B}}(a, \mathcal{L}), \widetilde{\odot})$  is proved to be locally compact fuzzy soft metric space if it's sequentially compact fuzzy soft metric space. Finally, the concept of uniformly continuous function in a fuzzy soft metric space is presented to study the main properties of compact fuzzy soft continuous function.

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