Multipliers of an AT-algebra

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Received: 9/8/2020 Accepted: 15/2/2021

Abstract
In this work, we present the notion of a multiplier on AT-algebra and investigate several properties. Also, some theorems and examples are discussed. The notions of the kernel and the image of multipliers are defined. After that, some propositions related to isotone and regular multipliers are proved. Finally, the Left and the Right derivations of the multiplier are obtained.

Keywords: AT-algebra; multiplier; isotone multiplier; regular multiplier; Left and the Right derivations.

1. Introduction
Prabpayak and Leerawat introduced a new algebraic structure named KU-algebra. They studied a homomorphism of KU-algebra and discussed some ideals of this structure [1,2]. The notion of AT-algebra was introduced as a generalization of KU-algebra. Several properties and many types of ideals on AT-algebras were discussed[3]. Investigations on multipliers were published by various researchers in the context of rings and semigroups [4]. Some authors studied the properties of multipliers and algebraic structures on rings and semirings see,[5-12].
The concept of derivation plays a significant role in analysis, algebraic geometry, and algebra. In 1957, Posner introduced the notion of derivation in the ring and the near-ring[13]. In addition, the notion of derivation was applied in BCI-algebras [14]. Furthermore, the notion of derivation in B-algebras was introduced and some of their properties were investigated [15]. In 2014, Mostafa and Kareem studied the notion of derivation in KU-algebras[16].
In this study, the multiplier of an AT-algebra and some of its properties are introduced. The kernel and the image of multipliers on AT-algebras are discussed. Also, some properties of the isotone and regular multipliers are given. In addition, the notion of the Left and the Right derivations of the multiplier in AT-algebra is studied.

2. Preliminaries
In this part, we review some basic definitions and theories of AT-algebra.

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Definition 2.1[3]. An AT-algebra is a nonempty set \( \mathbb{N} \) with a constant 0 and a binary operation \( * \), satisfying the following condition: for all \( \varepsilon, \sigma, \tau \in \mathbb{N} \),

(i) \( (\varepsilon * \sigma) * ((\sigma * \tau) * (\varepsilon * \tau)) = 0 \),

(ii) \( 0 * \varepsilon = \varepsilon \), \( \varepsilon * 0 = 0 \).

Then the following properties are satisfied in AT-algebra \( (\mathbb{N}, *, 0) \):

(1) \( ((\sigma * \tau) * (\varepsilon * \tau)) \leq (\varepsilon * \sigma) \),

(2) \( 0 \leq \varepsilon \).

where \( \leq \sigma \) if and only if \( \sigma * \varepsilon = 0 \) and \( \varepsilon, \sigma, \tau \in \mathbb{N} \).

Proposition 2.2 [3]. Let \( (\mathbb{N}, *, 0) \) be an AT-algebra. Then the following axioms hold:

i. \( \tau * \tau = 0 \),

ii. \( \tau * (\varepsilon * \tau) = 0 \),

iii. \( \sigma * ((\sigma * \tau) * \tau) = 0 \),

iv. \( \varepsilon * \sigma = 0 \) implies that \( \varepsilon * 0 = \sigma * 0 \),

v. \( \varepsilon = 0 * (0 * \varepsilon) \),

vi. \( 0 * \varepsilon = 0 * \sigma \) implies that \( \varepsilon = \sigma \).

where \( \varepsilon, \sigma, \tau \in \mathbb{N} \).

Example 2.3[3]. Let \( \mathbb{N} = \{0, a, b, c, d, e\} \) be a set with the operation \( * \) defined by the following table

<table>
<thead>
<tr>
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<th>0</th>
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Then, \( (\mathbb{N}, *, 0) \) is an AT-algebra.

Proposition 2.4[3]. In any AT-algebra \( (\mathbb{N}, *, 0) \), the following properties hold for all \( \varepsilon, \sigma, \tau \in \mathbb{N} \):

a) \( \varepsilon \leq \sigma \) implies that \( \sigma * \tau \leq \varepsilon * \tau \),

b) \( \varepsilon \leq \sigma \) implies that \( \tau * \varepsilon \leq \tau * \sigma \),

c) \( \tau * \varepsilon \leq \tau * \sigma \) implies that \( \varepsilon \leq \sigma \),

d) \( \varepsilon * \sigma \leq \tau \) implies \( \tau * \sigma \leq \varepsilon \).

Definition 2.5[3]. A nonempty subset \( S \) of an AT-algebra \( \mathbb{N} \) is called an AT-subalgebra \( \varepsilon \sigma \in S \), whenever \( \varepsilon, \sigma \in S \).

Definition 2.6[3]. A nonempty subset \( I \) of an AT-algebra \( \mathbb{N} \) is called an AT-ideal if for all \( \varepsilon, \sigma, \tau \in \mathbb{N} \):

(\( AT_1 \)) \( 0 \in I \),

(\( AT_2 \)) \( (\varepsilon * (\sigma * \tau)) \in I \) and \( \sigma \in I \) imply that \( \varepsilon * \tau \in I \).

Definition 2.7. An AT-algebra \( (\mathbb{N}, *, 0) \) is said to be AT-commlerative if:

(\( AT_3 \)) \( \varepsilon ** (\sigma * \varepsilon) = (\sigma ** \varepsilon) * \varepsilon \), for all \( \varepsilon, \sigma \in \mathbb{N} \).

Example 2.8. Let \( \mathbb{N} = \{0, a, b, c\} \) be a set with the operation \( * \), defined by the following table

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By using the definition 2.7, we can prove that \((\mathbb{N}, \cdot, 0)\) is an AT-commutative.

**Theorem 2.9.** For an AT-algebra \((\mathbb{N}, \cdot, 0)\), the followings are equivalent for all \(\varepsilon, \sigma \in \mathbb{N}:
(a) \mathbb{N} is a commutative,
(b) \((\varepsilon \cdot \sigma) \cdot \sigma \leq (\sigma \cdot \varepsilon) \cdot \varepsilon,
(c) \((\sigma \cdot \varepsilon) \cdot \varepsilon = ((\varepsilon \cdot \sigma) \cdot \sigma) = 0.

**Proof.** Clearly by applying the definition 2.7.

3. A self-map \(\rho\)

**Definition 3.1.** Let \((\mathbb{N}, \cdot, 0)\) be an AT-algebra. A self-map \(\rho\) of \(\mathbb{N}\) is called a multiplier if \(\rho(\varepsilon \cdot \sigma) = \varepsilon \cdot \rho(\sigma),\) for all \(\varepsilon, \sigma \in \mathbb{N}.

**Example 3.2.** Let \(\mathbb{N} = \{0, a, b, c, d\}\) be a set with the operation \(\cdot\), defined by the following table:

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</tbody>
</table>

Based on definition 2.1, \((\mathbb{N}, \cdot, 0)\) is an AT-algebra and the self map \(\rho\) of \(\mathbb{N}\) is defined by:

\[
\rho(\varepsilon) = \begin{cases} 
0 & \text{if } \varepsilon = 0, a, c \\
\varepsilon & \text{if } \varepsilon = b, d 
\end{cases}
\]

Based on definition 2.1, \(\rho\) is a multiplier of \(\mathbb{N}.

**Example 3.3.** Based on Example 2.3, \(\rho: \mathbb{N} \to \mathbb{N}\) is defined as follows:

\[
\rho(0) = 0, \quad \rho(a) = 0, \quad \rho(b) = b, \quad \rho(c) = c, \quad \rho(d) = d, \quad \rho(e) = d \quad \text{and } c = \rho(a \cdot e) = a \cdot \rho(e) = b.
\]

Then, \(\rho\) is not a multiplier of \(\mathbb{N}.

**Lemma 3.4.** If \(\rho_1\) and \(\rho_2\) are two multipliers of an AT-algebra \(\mathbb{N}\), then \(\rho_1 \circ \rho_2\) is a multiplier of \(\mathbb{N}.

**Proof.** Let \(\rho_1\) and \(\rho_2\) be two multipliers of \(\mathbb{N}\). Then, \(\rho_1 \circ \rho_2(\varepsilon \cdot \sigma) = \rho_1(\rho_2(\varepsilon \cdot \sigma)) = \rho_1((\varepsilon \cdot \rho_2(\sigma)) = \varepsilon \cdot \rho_1(\rho_2(\sigma)) = \varepsilon \cdot (\rho_1 \circ \rho_2)(\sigma), \) for all \(\varepsilon, \sigma \in \mathbb{N}.

**Proposition 3.5.** Let \(\mathbb{N}\) be an AT-algebra and \(\rho\) be a multiplier of \(\mathbb{N}\). Then, for all \(\varepsilon, \sigma \in \mathbb{N},\) we have

(i) \(\rho(0) = 0,
(ii) \(\rho(\varepsilon \cdot 0) = 0,
(iii) \(\rho(\varepsilon) \leq \varepsilon,
(iv) \varepsilon \leq \sigma \Rightarrow \rho(\varepsilon) \leq \sigma,
(v) \rho(\rho(\varepsilon) \cdot \varepsilon) = 0,
(vi) \rho(\varepsilon \cdot \sigma) \leq \rho(\varepsilon) \cdot \rho(\sigma).

**Proof.**

(i) Based on definition 3.1, \(\rho(0) = \rho(\rho(0) \cdot 0) = \rho(0) \cdot \rho(0) = 0.
(ii) Based on (i), \(\rho(\varepsilon \cdot 0) = \rho(0) = 0,\) where \(\varepsilon \in \mathbb{N}.
(iii) Based on definition 3.1, \(0 = \rho(0) = \rho(\varepsilon \cdot \varepsilon) = \varepsilon \cdot \rho(\varepsilon),\) hence \(\rho(\varepsilon) \leq \varepsilon,\) where \(\varepsilon \in \mathbb{N}.
(iv) Suppose that \(\varepsilon \leq \sigma\) for every \(\varepsilon, \sigma \in \mathbb{N},\) then \(\sigma \cdot \varepsilon = 0.\) Thus, \(0 = \rho(0) = \rho(\sigma \cdot \varepsilon) = \sigma \cdot \rho(\varepsilon),\) hence \(\rho(\varepsilon) \leq \sigma.
(v) Based on definition 3.1, \(\rho(\rho(\varepsilon) \cdot \varepsilon) = \rho(\varepsilon) \cdot \rho(\varepsilon) = 0,\) where \(\varepsilon \in \mathbb{N}.

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Based on (iii), \( \rho(\epsilon) \leq \epsilon \) for all \( \epsilon \in \mathbb{N} \), and based on proposition 2.4, we have \( \epsilon * \rho(\sigma) \leq \epsilon * \rho(\sigma) \), thus \( \rho(\epsilon * \sigma) \leq \rho(\epsilon) * \rho(\sigma) \).

**Definition 3.6.** Let \( \mathbb{N} \) be an AT-algebra and \( \rho \) be a multiplier of \( \mathbb{N} \). Then, \( \rho \) is said to be an isotone if \( \epsilon \leq \sigma \Rightarrow \rho(\epsilon) \leq \rho(\sigma) \) for all \( \epsilon, \sigma \in \mathbb{N} \).

**Lemma 3.7.** Let \( \mathbb{N} \) be an AT-algebra and \( \rho \) be a multiplier of \( \mathbb{N} \). For all \( \epsilon, \sigma \in \mathbb{N} \), if \( \rho(\epsilon * \sigma) = \rho(\epsilon) * \rho(\sigma) \), then \( \rho \) is an isotone.

**Proof.** Let \( \rho(\epsilon * \sigma) = \rho(\epsilon) * \rho(\sigma) \). If \( \epsilon \leq \sigma \Rightarrow \sigma * \epsilon = 0 \), for all \( \epsilon, \sigma \in \mathbb{N} \). Then, we have

\[
\rho(\epsilon) = \rho(0 * \epsilon) \text{ by definition 2.1 (iii),}
\]

\[
= \rho((\epsilon * \epsilon) * \epsilon) \text{ by hypothesis above,}
\]

\[
= \rho(\epsilon * \epsilon) * \rho(\epsilon) \text{ by hypothesis above,}
\]

\[
= [\rho(\epsilon) * \rho(\epsilon)] * \rho(\epsilon) \text{ by hypothesis above,}
\]

and \( \leq \rho(\sigma) \) by definition 2.1 (3).

Thus, \( \rho(\epsilon) \leq \rho(\sigma) \) and then \( \rho \) is an isotone.

**Proposition 3.8.** Let \( \rho \) be a multiplier of \( \mathbb{N} \). If \( \rho \) is an endomorphism on \( \mathbb{N} \), then \( \rho \) is an isotone.

**Proof.** Let \( \epsilon \leq \sigma \Rightarrow \sigma * \epsilon = 0 \) and \( 0 = \rho(0) = \rho(\sigma * \epsilon) = \rho(\sigma) * \rho(\epsilon) \). Hence, \( \rho(\epsilon) \leq \rho(\sigma) \), then \( \rho \) is an isotone.

### 4. The kernel and the image of the multiplier \( \rho \)

**Definition 4.1.** Let \( \mathbb{N} \) be an AT-algebra and \( \rho \) be a multiplier of \( \mathbb{N} \). Then, the kernel of \( \rho \), denoted \( \text{ker}(\rho) \), is defined by \( \text{ker}(\rho) = \{ \epsilon \in \mathbb{N}; \rho(\epsilon) = 0 \} \).

**Lemma 4.2.** Let \( \mathbb{N} \) be an AT-algebra and \( \rho \) be a multiplier of \( \mathbb{N} \). Then, \( \text{ker}(\rho) \) is an AT-subalgebra of \( \mathbb{N} \).

**Proof.** Since \( 0 \in \text{ker}(\rho) \), then \( \text{ker}(\rho) \neq \emptyset \). Let \( \epsilon, \sigma \in \text{ker}(\rho) \), it follows that \( \rho(\epsilon) = 0 \) and \( \rho(\sigma) = 0 \). Now, \( \rho(\epsilon * \sigma) = \epsilon * \rho(\sigma) = \epsilon * 0 = 0 \), hence \( \epsilon * \sigma \in \text{ker}(\rho) \). Thus, \( \text{ker}(\rho) \) is an AT-subalgebra of \( \mathbb{N} \).

**Remark 4.3.** Let \( \mathbb{N} \) be an AT-algebra and \( \rho \) be a multiplier of \( \mathbb{N} \). If \( \rho \) is one to one map, then \( \text{ker}(\rho) = \{0\} \).

**Proof.** Suppose that \( \rho \) is one to one and \( \epsilon \in \text{ker}(\rho) \). Then, \( \rho(\epsilon) = 0 = \rho(0) \) and thus \( \epsilon = 0 \), it follows that \( \text{ker}(\rho) = \{0\} \).

The reverse of Remark 4.3 is incorrect. Example 4.4− shows the reverse.

**Example 4.4.** Let \( \mathbb{N} = \{0, a, b\} \) be a set with the operation \( * \), defined by the following table:

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<th>0</th>
<th>a</th>
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<tbody>
<tr>
<td>0</td>
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<td>b</td>
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</tr>
</tbody>
</table>

It is easy to check that \( (\mathbb{N}, *, 0) \) is an AT-algebra and the multiplier \( \rho \) of \( \mathbb{N} \) is defined by

\[
\rho(\epsilon) = \begin{cases} 0 & \text{if } \epsilon = 0 \\ a & \text{otherwise} \end{cases}
\]

Now, it is obvious that \( \text{ker}(\rho) = \{0\} \) and \( \rho(a) = \rho(b) = a \). Therefore, \( \rho \) is not one to one map.

**Lemma 4.5.** Let \( \mathbb{N} \) be an AT-commutative and \( \rho \) be a multiplier of \( \mathbb{N} \). If \( \epsilon \in \text{ker}(\rho) \) and \( \sigma \leq \epsilon \), then \( \sigma \in \text{ker}(\rho) \).

**Proof.** Let \( \epsilon \in \text{ker}(\rho) \) and \( \sigma \leq \epsilon \). Then, \( \rho(\epsilon) = 0 \) and \( \epsilon * \sigma = 0 \).
\[ \rho(\sigma) = \rho(0 \ast \sigma) = \rho((\varepsilon \ast \sigma) \ast \sigma) = \rho((\sigma \ast \varepsilon) \ast \varepsilon) = (\sigma \ast \varepsilon) \ast \rho(\varepsilon) = (\sigma \ast \varepsilon) \ast 0 = 0, \]

thus \( \sigma \in \ker(\rho) \).

**Remark 4.6.** Let \( \mathbb{N} \) be an AT-algebra and \( \rho \) be endomorphism map of \( \mathbb{N} \). Then \( \ker(\rho) \) is an AT-ideal of \( \mathbb{N} \).

**Proof.** Clearly, \( 0 \in \ker(\rho) \). Let \( \sigma \in \ker(\rho) \) and \((\varepsilon \ast (\sigma \ast \tau)) \in \ker(\rho) \). Then, we have \( \rho(\sigma) = 0 \) and \( \rho(\varepsilon \ast (\sigma \ast \tau)) = 0 \), thus

\[ 0 = \rho(\varepsilon \ast (\sigma \ast \tau)) = (\rho(\varepsilon) \ast ((\rho(\sigma) \ast (\rho(\tau)))) = (\rho(\varepsilon) \ast \rho(\tau)) = \rho(\varepsilon \ast \tau). \]

This implies that \( \ast \tau \in \ker(\rho) \). Hence, \( \ker(\rho) \) is an AT-ideal of \( \mathbb{N} \).

**Definition 4.7.** Let \( \mathbb{N} \) be an AT-algebra and \( \rho \) be a multiplier of \( \mathbb{N} \). \( \rho \) is named idempotent if \( \rho(\rho(\varepsilon)) = \rho(\varepsilon) \), for all \( \varepsilon \in \mathbb{N} \).

**Example 4.8.** Let \( \mathbb{N} = \{0, a, b\} \) be a set with the operation \( \ast \), defined by the following table:

<table>
<thead>
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<th>0</th>
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<tbody>
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<td>0</td>
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</table>

It is easy to check that \( (\mathbb{N}, \ast, 0) \) is an AT-algebra and the multiplier \( \rho \) of \( \mathbb{N} \) is defined by \( \rho(\varepsilon) = \varepsilon \).

Then, \( \rho \) is idempotent.

**Remark 4.9.** Let \( \mathbb{N} \) be an AT-algebra and \( \rho \) be a multiplier of \( \mathbb{N} \). If \( \rho \) is idempotent, then

(i) \( \text{Im}(\rho) \cap \ker(\rho) = \{0\} \).

(ii) \( \varepsilon \in \text{Im}(\rho) \iff \rho(\varepsilon) = \varepsilon \), for all \( \varepsilon \in \mathbb{N} \).

**Proof.**

(i) \( \varepsilon \in \text{Im}(\rho) \cap \ker(\rho) \), then \( \rho(\sigma) = \varepsilon \) for some \( \sigma \in \mathbb{N} \) and \( \rho(\varepsilon) = 0 \). It follows that \( 0 = \rho(\varepsilon) = \rho(\rho(\sigma)) = \rho(\sigma) = \varepsilon \), thus \( \text{Im}(\rho) \cap \ker(\rho) = \{0\} \).

(ii) Sufficiency is obvious. If \( \varepsilon \in \text{Im}(\rho) \), then \( \rho(\sigma) = \varepsilon \) for some \( \sigma \in \mathbb{N} \). Thus

\[ \rho(\varepsilon) = \rho(\rho(\sigma)) = \rho(\sigma) = \varepsilon. \]

**Remark 4.10.** Let \( M(\mathbb{N}) \) be the set of all multiplier idempotent maps of AT-algebra \( \mathbb{N} \) and \( \bullet \) be a binary operation on \( M(\mathbb{N}) \) defined by \( (\rho \bullet \delta)(\varepsilon) = \rho(\varepsilon) \ast \delta(\varepsilon) \), for all \( \rho, \delta \in M(\mathbb{N}) \) and \( \varepsilon \in \mathbb{N} \). It is easy to show that \( (M(\mathbb{N}), \bullet, 0) \) is an AT-algebra.

**Theorem 4.11.** Let \( \rho, \delta \in M(\mathbb{N}) \), then

(i) \( \rho(\varepsilon) = \delta(\varepsilon) \) for all \( \varepsilon \in \mathbb{N} \), then \( \rho \bullet \delta \in M(\mathbb{N}) \).

(ii) For all \( \varepsilon \in \mathbb{N} \), \( \text{Im}(\delta) \subseteq \text{Im}(\rho) \) and \( \rho(\delta(\varepsilon)) = \delta(\rho(\varepsilon)) \), then \( \rho \bullet \delta = 0 \).

(iii) \( \text{Im}(\delta) \cap \ker(\rho) \subseteq \text{Im}(\rho \bullet \delta) \).

**Proof.**

(i) Assume that \( \rho(\delta(\varepsilon)) = \delta(\rho(\varepsilon)) \) for all \( \varepsilon \in \mathbb{N} \). Then

\[
(\rho \bullet \delta)(\rho(\delta(\varepsilon))) = (\rho \bullet \delta)(\rho(\delta(\varepsilon))) = \rho(\rho(\delta(\varepsilon))) = \rho(\delta(\rho(\varepsilon))) = \delta(\rho(\delta(\varepsilon))) = \delta(\rho(\delta(\varepsilon)) = \delta(\rho(\delta(\varepsilon))) = \delta(0) \ast (\rho \bullet \delta)(\varepsilon) = (\rho \bullet \delta)(\varepsilon).
\]

Then \( \rho \bullet \delta \) is idempotent, hence \( \rho \bullet \delta \in M(\mathbb{N}) \).

(ii) Suppose that \( \text{Im}(\delta) \subseteq \text{Im}(\rho) \) and \( \rho(\delta(\varepsilon)) = \delta(\rho(\varepsilon)) \), for all \( \varepsilon \in \mathbb{N} \).

Since \( \delta(\varepsilon) \in \text{Im}(\delta) \subseteq \text{Im}(\rho) \) for all \( \varepsilon \in \mathbb{N} \), it follows from lemma 3.7 that

\[
(\rho \bullet \delta)(\varepsilon) = \rho(\varepsilon) \ast \delta(\varepsilon) = \rho(\varepsilon) \ast \delta(\rho(\varepsilon)) = \rho(\varepsilon) \ast \delta(\rho(\varepsilon)) = \delta(\rho(\varepsilon) \ast \rho(\varepsilon)) = \delta(0) = \delta(\rho(\varepsilon) \ast \rho(\varepsilon)) = \delta(0)
\]

for all \( \varepsilon \in \mathbb{N} \), hence \( \rho \bullet \delta = 0 \).

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(iii) If \( \sigma \in \text{Im}(\delta) \cap \ker(\rho) \), then \( \delta(\varepsilon) = \sigma \text{ and } \rho(\sigma) = 0 \) for some \( \varepsilon \in \mathbb{K} \). It follows that 
\[
\sigma = \delta(\varepsilon) = 0 \ast \delta(\varepsilon) = \rho(\sigma) \ast \delta(\sigma) = (\rho \ast \delta)(\sigma) \in \text{Im}(\rho \ast \delta).
\]
Hence, the proof is completed.

5. The Left and The Right Derivations of multiplier maps in AT-algebra
The left and the right derivations of multiplier maps in AT-algebra are introduce in this section.

Definition 5.1. Let \( \rho \) be a multiplier map and \( D_\rho \) be a self map of \( \mathbb{K} \). Then, \( D_\rho \) is called \( L_\rho \)-derivation if 
\[
D_\rho(\varepsilon \ast \sigma) = D_\rho(\varepsilon) \ast \rho(\sigma), \text{ for all } \varepsilon, \sigma \in \mathbb{K}, \text{ and it is called } R_\rho \text{-derivation if } D_\rho(\varepsilon \ast \sigma) = \rho(\varepsilon) \ast D_\rho(\sigma), \text{ for all } \varepsilon, \sigma \in \mathbb{K}. \text{ If } D_\rho \text{ is both } L_\rho \text{-derivation and } R_\rho \text{-derivation of } \mathbb{K}, \text{ we say that } D_\rho \text{ is } \text{ap-derivation of } \mathbb{K}.
\]

Definition 5.2. A self map \( D_\rho \) of \( \mathbb{K} \) is called a regular if \( D_\rho(0) = 0 \).

Proposition 5.3. A \( L_\rho \)-derivation of an AT-algebra \( \mathbb{K} \) is a regular.

Proof. Let \( D_\rho \) be a \( L_\rho \)-derivation of \( \mathbb{K} \), then for all \( \varepsilon \in \mathbb{K} \)
\[
D_\rho(0) = D_\rho(\varepsilon \ast 0) = D_\rho(\varepsilon) \ast \rho(0) = D_\rho(\varepsilon) \ast 0 = 0.
\]
Thus, \( L_\rho \)-derivation of \( \mathbb{K} \) is a regular.

The reverse of Lemma 5.3 is incorrect. Example 5.4 shows the reverse.

Example 5.4. Let \( \mathbb{K} = \{0, a, b\} \) be a set in Example 4.4. We define a map \( D_\rho : \mathbb{K} \rightarrow \mathbb{K} \) by
\[
D_\rho(\varepsilon) = \begin{cases} 
0 & \varepsilon = 0 \\
b & \text{otherwise}
\end{cases}
\]
Then, it is easy to show that \( D_\rho \) is a regular map but not \( L_\rho \)-derivation of \( \mathbb{K} \), since 
\[
D_\rho(0 \ast a) = b \text{ and } D_\rho(0) \ast \rho(a) = a. \text{ Then, } D_\rho(0 \ast a) \neq D_\rho(0) \ast \rho(a).
\]

Lemma 5.5. Let \( D_\rho \) be a regular map of an AT-algebra \( \mathbb{K} \), then
(i) If \( D_\rho \) is a \( L_\rho \)-derivation of \( \mathbb{K} \), then \( D_\rho(\varepsilon) = \rho(\varepsilon) \) for all \( \varepsilon \in \mathbb{K} \).
(ii) If \( D_\rho \) is a \( L_\rho \)-derivation of \( \mathbb{K} \), then \( D_\rho((0 \ast \varepsilon) \ast \sigma) = \rho(\varepsilon) \ast \rho(\sigma) \) for all \( \varepsilon, \sigma \in \mathbb{K} \).
(iii) If \( D_\rho \) is a \( R_\rho \)-derivation of \( \mathbb{K} \), then \( D_\rho((0 \ast \varepsilon) \ast \sigma) = \rho(\varepsilon) \ast D_\rho(\sigma) \) for all \( \varepsilon, \sigma \in \mathbb{K} \).

Proof. (i) Let \( D_\rho \) be a \( L_\rho \)-derivation of \( \mathbb{K} \), then 
\[
D_\rho(\varepsilon) = D_\rho(0 \ast \varepsilon) = D_\rho(0) \ast \rho(\varepsilon) = 0 \ast \rho(\varepsilon) = \rho(\varepsilon).
\]
(ii) Let \( D_\rho \) be a \( L_\rho \)-derivation of \( \mathbb{K} \), then 
\[
D_\rho((0 \ast \varepsilon) \ast \sigma) = D_\rho(0 \ast \varepsilon) \ast \rho(\sigma) = \left(D_\rho(0) \ast \rho(\varepsilon)\right) \ast \rho(\sigma) = (0 \ast \rho(\varepsilon)) \ast \rho(\sigma) = \rho(\varepsilon) \ast \rho(\sigma).
\]
(iii) Let \( D_\rho \) be a \( R_\rho \)-derivation of \( \mathbb{K} \), then 
\[
D_\rho((0 \ast \varepsilon) \ast \sigma) = \rho(0 \ast \varepsilon) \ast D_\rho(\sigma) = \rho(\varepsilon) \ast D_\rho(\sigma).
\]

Lemma 5.6. Let \( D_\rho \) be a regular and \( L_\rho \)-derivation of \( \mathbb{K} \), then
(i) \( D_\rho(\varepsilon \ast \sigma) = \varepsilon \ast \rho(\sigma) \), for all \( \varepsilon, \sigma \in \mathbb{K} \).
(ii) \( D_\rho(\varepsilon \ast D_\rho(\varepsilon)) = 0 \).

Proof. Let \( D_\rho \) be a regular and \( L_\rho \)-derivation of \( \mathbb{K} \), then we have 
(i) \( D_\rho(\varepsilon \ast \sigma) = D_\rho((0 \ast (\varepsilon \ast \sigma))) = D_\rho(0) \ast \rho(\varepsilon \ast \sigma) = 0 \ast (\varepsilon \ast \sigma) = \rho(\varepsilon \ast \sigma) = \varepsilon \ast \rho(\sigma) \)
(ii) \( D_\rho(\varepsilon \ast D_\rho(\varepsilon)) = D_\rho(\varepsilon) \ast \rho(D_\rho(\varepsilon)) = \rho(D_\rho(\varepsilon) \ast D_\rho(\varepsilon)) = \rho(0) = 0 \).

Definition 5.7. Let \( D_\rho \) be a \( \rho \)-derivation of an AT-algebra \( \mathbb{K} \). Then, \( D_\rho \) is said to be an isotone \( \rho \)-derivation if \( \varepsilon \leq \sigma \Rightarrow D_\rho(\varepsilon) \leq D_\rho(\sigma) \), for all \( \varepsilon, \sigma \in \mathbb{K} \).

Lemma 5.8. Let \((\mathbb{K}, \ast, \sigma)\) be an AT-algebra and \( D_\rho \) be \( \rho \)-derivation on \( \mathbb{K} \). For all \( \varepsilon, \sigma \in \mathbb{K} \), if \( D_\rho(\varepsilon \ast \sigma) = D_\rho(\varepsilon) \ast D_\rho(\sigma) \), then \( D_\rho \) is an isotone \( \rho \)-derivation.

Proof. Let \( D_\rho(\varepsilon \ast \sigma) = D_\rho(\varepsilon) \ast D_\rho(\sigma) \). If \( \varepsilon \leq \sigma \Rightarrow \sigma \ast \varepsilon = 0 \) for all \( \varepsilon, \sigma \in \mathbb{K} \). Then, we have 
\[
D_\rho(\varepsilon) = D_\rho(0 \ast \varepsilon) = D_\rho((\sigma \ast \varepsilon) \ast \varepsilon) = D_\rho(\sigma \ast \varepsilon) \ast D_\rho(\varepsilon) = [D_\rho(\sigma) \ast D_\rho(\varepsilon)] \ast D_\rho(\varepsilon) = D_\rho(\sigma).
\]
Thus, \( D_\rho(\varepsilon) \leq D_\rho(\sigma) \), which implies that \( D_\rho \) is an isotone \( \rho \)-derivation.

Lemma 5.9. Let \((\mathbb{K}, \ast, \sigma)\) be an AT-algebra with a partial order \( \leq \), and \( D_\rho \) be a self map of \( \mathbb{K} \). Then for all \( \varepsilon, \sigma \in \mathbb{K} \), we have
(i) If \( D_\rho \) is \( L_\rho \)-derivation of \( \mathbb{K} \), then \( D_\rho(\varepsilon \ast \sigma) \leq D_\rho(\varepsilon) \ast \rho(\sigma) \),
(ii) If \( D_\rho \) is \( R_\rho \)-derivation of \( \mathbb{K} \), then \( D_\rho(\varepsilon \ast \sigma) \leq \rho(\varepsilon) \ast D_\rho(\sigma) \).
(iii) If $D_\rho$ is a regular and $R_\rho$-derivation of $\mathfrak{N}$, then $\ker(D_\rho) = \{\varepsilon \in \mathfrak{N} : D_\rho(\varepsilon) = 0\}$ is an AT-subalgebra of $\mathfrak{N}$.

**Proof.** (i) If $D_\rho$ is $L_\rho$-derivation of $\mathfrak{N}$, we have

$$D_\rho(\varepsilon) \ast \rho(\sigma) \ast D_\rho(\varepsilon \ast \sigma) = (D_\rho(\varepsilon) \ast \rho(\sigma)) \ast (D_\rho(\varepsilon) \ast \rho(\sigma)) = 0$$

Then, $D_\rho(\varepsilon \ast \sigma) \leq D_\rho(\varepsilon) \ast \rho(\sigma)$.

(ii) If $D_\rho$ is $R_\rho$-derivation of $\mathfrak{N}$, we have

$$\rho(\varepsilon \ast D_\rho(\sigma)) \ast D_\rho(\varepsilon \ast \sigma) = \rho(\varepsilon \ast D_\rho(\sigma)) \ast \rho(\varepsilon \ast D_\rho(\sigma)) = 0$$

Then, $D_\rho(\varepsilon \ast \sigma) \leq \rho(\varepsilon) \ast D_\rho(\sigma)$.

(iv) Since $D_\rho$ is a regular map, then $D_\rho(0) = 0$, it follows that $\ker(D_\rho) \neq \varphi$.

Now, let $\sigma \in \ker(D_\rho)$, then $D_\rho(\varepsilon) = 0, D_\rho(\sigma) = 0$. Since $D_\rho$ is $R_\rho$-derivation of $\mathfrak{N}$, then $D_\rho(\varepsilon \ast \sigma) = \rho(\varepsilon) \ast D_\rho(\sigma) = \rho(\varepsilon) \ast 0 = 0$. Hence, $\varepsilon \ast \sigma \in \ker(D_\rho)$.

Therefore, $\ker(D_\rho)$ is an AT-subalgebra of $\mathfrak{N}$.

**Definition 5.10.** Let $(\mathfrak{N}, *, 0)$ be an AT-algebra and $\rho$ be a multiplier of $\mathfrak{N}$. Then, an AT-ideal $I$ is named an $\rho$-ideal if $\rho(I) \subseteq I$.

**Definition 5.11.** Let $D_\rho$ be a self map of an AT-algebra $\mathfrak{N}$. An $\rho$-ideal $I$ of $\mathfrak{N}$ is said to be $D_\rho$-invariant if $D_\rho(I) \subseteq I$.

**Proposition 5.12.** Let $D_\rho$ be a regular $L_\rho$-derivation of an AT-algebra $\mathfrak{N}$, then every $\rho$-ideal $I$ of $\mathfrak{N}$ is $D_\rho$-invariant.

**Proof.** By Lemma 5.5(i), we have $D_\rho(\varepsilon) = \rho(\varepsilon)$ for all $\varepsilon \in \mathfrak{N}$. Let $\sigma \ast \tau \in D_\rho(I)$. Then, $\sigma \ast \tau = D_\rho(\varepsilon)$, for some $\varepsilon \in I$. It follows that

$$(\sigma \ast \rho(\varepsilon) \ast \tau) = \rho(\varepsilon) \ast (\sigma \ast \tau) = \rho(\varepsilon) \ast D_\rho(\varepsilon) = (\rho(\varepsilon) \ast \rho(\varepsilon)) = 0 \in I.$$ Since $\varepsilon \in I$, then $\rho(\varepsilon) \in \rho(I) \subseteq I$, as $I$ is an $\rho$-ideal. It follows that $\sigma \ast \tau \in I$ and since $I$ is an AT-ideal of $\mathfrak{N}$. Hence $D_\rho(I) \subseteq I$, thus $I$ is $D_\rho$-invariant.

**Corollary 5.13.** Let $D_\rho$ be $\rho$-derivation of an AT-algebra $\mathfrak{N}$, then $D_\rho$ is a regular if and only if every $\rho$-ideal of $\mathfrak{N}$ is $D_\rho$-invariant.

**Proof.** Assume that every $\rho$-ideal of $\mathfrak{N}$ is $D_\rho$-invariant. Then, since the zero ideal $\{0\}$ is $\rho$-ideal and $D_\rho$-invariant, we have $D_\rho(\{0\}) \subseteq \{0\}$, which implies that $D_\rho(0) = 0$. Thus, $D_\rho$ is regular.

Combining this and Theorem 5.12, the proof is complete.

**References**